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Observations on the Hyperbola $y^2=182x^2+14$

***R.Divyabharathi*[1] and *G.Sumathi*[2]**

Department of mathematics, Shrimati Indira Gandhi College
Trichy-2,Tamil Nadu, India.
[1]e-mail: rdivyabharathi013@gmail.com; [2]e-mail: b.deepacharan@gmail.com

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Abstract. The binary quadratic equation $y^2 = 182x^2 + 14$ representing hyperbola is considered for finding its integer solutions. A few interesting properties among the solutions are presented. Also, we present infinitely many positive integer solutions in terms of Generalized Fibonacci sequences of numbers, Generalized Lucas sequences of numbers.

Keywords: Binary quadratic integral solutions, generalized Fibonacci Sequences of numbers, generalized Lucas Sequences of numbers.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions. when D takes different integral values [1, 2]. In [3] infinitely many Pythagorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2 = 3x^2 + 14$.In [4] a, special Pythagorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [5] different patterns of infinitely many Pythagorean triangles are obtained by employing the non-integral solutions of $y^2 = 12x^2 + 1$. In this context one may also refer [6-11].These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 182x^2 + 14$ representing a hyperbola. A few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration a few patterns of Pythagorean triangles are obtained.

2. Notations

$GF_n(k,s)$: Generalized Fibonacci Sequences of rank n.

$GL_n(k,s)$: Generalized Lucas Sequences of rank n.

3. Methods of analysis

Consider the binary quadratic equation

$$y^2 = 182x^2 + 14 \quad (1)$$

with least positive integer solutions is

$x_0 = 1$, $y_0 = 1$

To obtain the other solutions of (1),

Consider the Pellian equation

$$y^2 = 182x^2 + 1 \quad (2)$$

whose general solution ($\tilde{x}_n, \tilde{y}_n$) is represented by

$$\tilde{x}_n = \frac{g_n}{2\sqrt{182}} \ , \quad \tilde{y}_n = \frac{f_n}{2}$$

in which,

$$f_n = \left(27 + 2\sqrt{182}\right)^{n+1} + \left(27 - 2\sqrt{182}\right)^{n+1}$$

$$g_n = \left(27 + 2\sqrt{182}\right)^{n+1} + \left(27 - 2\sqrt{182}\right)^{n+1}$$

where ,n=-1,0,1,2,3,4,5……………

Applying Brahmagupta lemma between the solutions of x_0, y_0 and $\left(\tilde{x}_n, \tilde{y}_n\right)$ the general solutions of equation (1) are found to be

$$x_{n+1} = \frac{f_n}{2} + \frac{14g_n}{2\sqrt{182}} \quad (3)$$

$$y_{n+1} = 7f_n + \frac{182g_n}{2\sqrt{182}} \quad (4)$$

Thus (3) and (4) represent the non-zero distinct integer solutions of (1)

The recurrence relations satisfied by the values of x_{n+1} and y_{n+1} are respectively.

$$x_{n+3} - 54x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 54y_{n+2} + y_{n+1} = 0$$

A few numerical examples are presented in the table1 below:

Table 1: Examples

N	x_n	y_n
-1	1	14
0	55	742
1	2969	40054
2	160271	2162538
3	8651665	8614580
4	467029639	465027049
5	2521094884	25102846066

3.1. A few interesting relations between the solutions are given below

a) $54x_{n+2} - x_{n+1} - x_{n+3} = 0$

b) $x_{n+2} - 27x_{n+1} - 2y_{n+1} = 0$

c) $27x_{n+2} - x_{n+1} - 2y_{n+2} = 0$

d) $928109x_{n+2} - 17199x_{n+1} - 1247y_{n+3} = 0$

e) $x_{n+3} + x_{n+1} - 54x_{n+2} = 0$

f) $x_{n+3} - 1457x_{n+1} - 108y_{n+1} = 0$

g) $x_{n+3} - x_{n+1} - 4y_{n+2} = 0$

h) $1457x_{n+3} - x_{n+1} - 108y_{n+3} = 0$

i) $108y_{n+1} + 1457x_{n+1} - x_{n+3} = 0$

j) $27y_{n+1} + 364x_{n+1} - y_{n+2} = 0$

k) $1457y_{n+1} + 19656x_{n+1} - y_{n+3} = 0$

l) $928109y_{n+2} + 231868x_{n+1} - 17199y_{n+3} = 0$

m) $17199x_{n+3} - 928109x_{n+2} - 1247y_{n+1} = 0$

n) $x_{n+3} - 27x_{n+2} - 2y_{n+2} = 0$

o) $27x_{n+3} - x_{n+2} - 2y_{n+3} = 0$

p) $x_{n+2} - 2y_{n+1} - 27x_{n+1} = 0$

q) $108y_{n+1} + 1457x_{n+1} - x_{n+3} = 0$

r) $27y_{n+1} + 364x_{n+1} - y_{n+2} = 0$

s) $1457y_{n+1} + 19656x_{n+1} - y_{n+3} = 0$

t) $928109y_{n+2} + 231868x_{n+1} - 17199y_{n+3} = 0$

u) $17199x_{n+3} - 928109x_{n+2} - 1247y_{n+1} = 0$

v) $x_{n+3} - 27x_{n+2} - 2y_{n+2} = 0$

w) $27x_{n+3} - x_{n+2} - 2y_{n+3} = 0$

x) $x_{n+2} - 2y_{n+1} - 27x_{n+1} = 0$

3.2. Each of the following expression is a Nasty number

a) $\frac{6}{28}\left[56 + 28x_{2n+3} - 1484x_{2n+2}\right]$

b) $\frac{6}{1512}\left[3024 + 28x_{2n+4} - 80108x_{2n+2}\right]$

c) $\frac{6}{14}\left[28 + 28y_{2n+2} - 364x_{2n+2}\right]$

d) $\frac{6}{378}\left[756 + 28y_{2n+3} - 20020x_{2n+2}\right]$

e) $\frac{6}{20398}\left[40796+28y_{2n+4}-1080716x_{2n+2}\right]$

f) $\frac{6}{28}\left[56+1484x_{2n+4}-80108x_{2n+3}\right]$

g) $\frac{6}{378}\left[756+1484y_{2n+2}-364x_{2n+3}\right]$

h) $\frac{6}{14}\left[28+1484y_{2n+3}-20020x_{2n+3}\right]$

i) $\frac{6}{378}\left[756+1484y_{2n+4}-1080716_{2n+3}\right]$

j) $\frac{6}{20398}\left[40796+80108y_{2n+2}-364x_{2n+4}\right]$

k) $\frac{6}{378}\left[756+80108y_{2n+3}-20020x_{2n+4}\right]$

l) $\frac{6}{14}\left[28+80108y_{2n+4}-1080716x_{2n+4}\right]$

m) $\frac{6}{5096}\left[10192+20020y_{n+2}-364y_{2n+3}\right]$

n) $\frac{6}{275184}\left[550368+1080716y_{2n+2}-364y_{2n+4}\right]$

o) $\frac{6}{5096}\left[10192+1080716y_{2n+3}-20020y_{2n+4}\right]$

3.3. Each of the following expression is a cubical integer

a) $\frac{1}{28}\left[\left(28x_{3n+4}-1484x_{3n+3}\right)+3\left(28x_{n+2}-1484x_{n+1}\right)\right]$

b) $\frac{1}{1512}\left[\left(28x_{3n+5}-80108x_{3n+3}\right)+3\left(28x_{n+3}-80108x_{n+1}\right)\right]$

c) $\frac{1}{14}\left[\left(28y_{3n+3}-364x_{3n+3}\right)+3\left(28y_{n+1}-364x_{n+1}\right)\right]$

d) $\frac{1}{378}\left[\left(28y_{3n+4}-20020x_{3n+3}\right)+3\left(28y_{n+2}-20020x_{n+1}\right)\right]$

e) $\frac{1}{20398}\left[\left(28y_{3n+5}-1080716x_{3n+3}\right)+3\left(28y_{n+3}-1080716x_{n+1}\right)\right]$

f) $\frac{1}{28}\left[\left(1484x_{3n+5}-80108x_{3n+4}\right)+3\left(1484x_{n+3}-80108x_{n+2}\right)\right]$

g) $\frac{1}{378}\left[\left(1484y_{3n+3}-364x_{3n+4}\right)+3\left(1484y_{n+1}-364x_{n+2}\right)\right]$

h) $\frac{1}{14}\left[(1484y_{3n+4}-20020x_{3n+4})+3(1484y_{n+2}-20020x_{n+2})\right]$

i) $\frac{1}{378}\left[(1484y_{3n+5}-1080716x_{3n+4})+3(1484y_{n+3}-1080716x_{n+2})\right]$

j) $\frac{1}{20398}\left[(80108y_{3n+3}-364x_{3n+5})+3(80108y_{n+1}-364x_{n+3})\right]$

k) $\frac{1}{378}\left[(80108y_{3n+4}-20020x_{3n+5})+3(80108y_{n+2}-20020x_{n+3})\right]$

l) $\frac{1}{14}\left[(80108y_{3n+5}-1080716x_{3n+5})+3(80108y_{n+3}-1080716x_{n+3})\right]$

m) $\frac{1}{5096}\left[(20020y_{3n+3}-364y_{3n+4})+3(20020y_{n+1}-364y_{n+2})\right]$

n) $\frac{1}{275184}\left[(1080716y_{3n+3}-364y_{3n+5})+3(1080716y_{3n+3}-364y_{n+3})\right]$

o) $\frac{1}{5096}\left[(1080716y_{3n+4}-20020y_{3n+5})+3(1080716y_{n+2}-20020y_{n+3})\right]$

3.4. Each of the following expression is a bi-quadratic integer

a) $\frac{1}{28^2}\left[(784x_{4n+5}-41552x_{4n+4})+4(28x_{n+2}-1484x_{n+1})^2-1568\right]$

b) $\frac{1}{1512^2}\left[(42336x_{4n+6}-121123296x_{4n+4})+4(28x_{n+3}-80108x_{n+1})^2-4572288\right]$

c) $\frac{1}{14^2}\left[(392y_{4n+4}-5096x_{4n+4})+4(28y_{n+1}-364x_{n+1})^2-392\right]$

d) $\frac{1}{378^2}\left[(10584y_{4n+5}-7567560x_{4n+4})+4(28y_{n+2}-20020x_{n+1})^2-285768\right]$

e) $\frac{1}{20398^2}\left[(571144y_{4n+6}-22044444968x_{4n+4})+4(28y_{n+3}-1080716x_{n+1})^2-832156808\right]$

f) $\frac{1}{28^2}\left[(41552x_{4n+6}-2243024x_{4n+5})+4(1484x_{n+3}-80108x_{n+2})^2-1568\right]$

g) $\frac{1}{378^2}\left[(560952y_{4n+4}-13608x_{4n+5})+4(1484y_{n+1}-36x_{n+2})^2-285768\right]$

h) $\frac{1}{14^2}\left[(20776y_{4n+5}-280280x_{4n+5})+4(1484y_{n+2}-20020x_{n+2})^2-392\right]$

i) $\frac{1}{378^2}\left[(560952y_{4n+6}-408510648x_{4n+5})+4(1484y_{n+3}-1080716x_{n+2})^2-285768\right]$

j) $\frac{1}{20398^2}\left[(1634042984y_{4n+4}-7424872x_{4n+6})+4(80108y_{n+1}-364x_{n+3})^2-83215680\right]$

k) $\frac{1}{378^2}\left[(30280824y_{4n+5}-7567560x_{4n+6})+4(80108y_{n+2}-20020x_{n+3})^2-285768\right]$

l) $\frac{1}{14^2}\left[(1121512y_{4n+6}-15130024x_{4n+6})+4(80108y_{n+3}-1080716x_{n+3})^2-392\right]$

m) $\frac{1}{5096^2}\left[(102021920y_{4n+4}-7287280y_{4n+3})+4(20020y_{n+1}-364y_{n+2})^2-51938432\right]$

n) $\frac{1}{275184^2}\left[(297395751744y_{4n+4}-100166976y_{4n+6})+4(1080716y_{n+1}-364y_{n+3})^2 -1514524677\right]$

o) $\frac{1}{5096^2}\left[(5507328736y_{4n+5}-102021920y_{4n+6})+4(1080716y_{n+2}-20020y_{n+3})^2 -51938432\right]$

3.5. The solutions of (1) in terms of special integers namely, generalized Fibonacci GF_n and Lucas GL_n are exhibited below

$$x_{n+1}=\frac{GL_{n+1}}{2}(54,-1)+28GF_{n+1}(54,-1)$$
$$y_{n+1}=7GL_{n+1(54,-1)}+364GF_{n+1}(54,-1)$$

4. Remarkable observations

4.1. Employing the linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in table 2 below:

Table 2: Hyperbola

S.No	Hyperbola	(X_n, Y_n)
1	$182X_n^2-Y_n^2=570752$	$X_n=28x_{n+2}-1484x_{n+1}$ $Y_n=20020x_{n+1}-364x_{n+2}$
2	$182X_n^2-Y_n^2=1664312832$	$X_n=28x_{n+3}-80108x_{n+1}$ $Y_n=1080716x_{n+1}-364x_{n+3}$
3	$182X_n^2-Y_n^2=142688$	$X_n=28y_{n+1}-364x_{n+1}$ $Y_n=5096x_{n+1}-364y_{n+1}$
4	$182X_n^2-Y_n^2=104019552$	$X_n=28y_{n+2}-20020x_{n+1}$ $Y_n=270088x_{n+1}-364y_{n+2}$
5	$182X_n^2-Y_n^2=302905078112$	$X_n=28y_{n+3}-1080716x_{n+1}$ $Y_n=14579656x_{n+1}-364y_{n+2}$

6	$182X_n^2 - Y_n^2 = 570752$	$X_n = 1484x_{n+3} - 80108x_{n+2}$ $Y_n = 1080716x_{n+2} - 20020x_{n+3}$
7	$182X_n^2 - Y_n^2 = 104019552$	$X_n = 1484y_{n+1} - 364x_{n+2}$ $Y_n = 5096x_{n+2} - 20020y_{n+1}$
8	$182X_n^2 - Y_n^2 = 142688$	$X_n = 1484y_{n+3} - 20020x_{n+2}$ $Y_n = 270088x_{n+2} - 20020y_{n+2}$
9	$182X_n^2 - Y_n^2 = 104019552$	$X_n = 1484y_{n+3} - 1080716x_{n+2}$ $Y_n = 14579656x_{n+2} - 20020y_{n+3}$
10	$182X_n^2 - Y_n^2 = 302905078112$	$X_n = 80108y_{n+1} - 364x_{n+3}$ $Y_n = 5096x_{n+3} - 1080716y_{n+1}$
11	$182X_n^2 - Y_n^2 = 104019552$	$X_n = 80108y_{n+2} - 20020x_{n+3}$ $Y_n = 270088x_{n+3} - 1080716y_{n+2}$
12	$182X_n^2 - Y_n^2 = 142688$	$X_n = 80108y_{n+3} - 1080716x_{n+3}$ $Y_n = 14579656x_{n+3} - 1080716y_{n+3}$
13	$182X_n^2 - Y_n^2 = 18905589248$	$X_n = 20020y_{n+1} - 364y_{n+2}$ $Y_n = 5096y_{n+2} - 270088y_{n+1}$
14	$182X_n^2 - Y_n^2 = 55128698247168$	$X_n = 1080716y_{n+1} - 364y_{n+3}$ $Y_n = 5096y_{n+3} - 14579656y_{n+1}$
15	$182X_n^2 - Y_n^2 = 18905589248$	$X_n = 1080716y_{n+2} - 20020y_{n+3}$ $Y_n = 270088y_{n+3} - 14579656y_{n+2}$

4.2. Employing the linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in table 3 below:

Table 3: Parabola

S. No	**Parabola**	(X_n, Y_n)
1	$5096X_n - Y_n^2 = 570752$	$X_n = 28x_{2n+3} - 1484x_{2n+2} + 56$ $Y_n = 20020x_{n+1} - 364x_{n+2}$
2	$275184X_n - Y_n^2 = 1664312832$	$X_n = 28x_{2n+4} - 80108x_{2n+2} + 3024$ $Y_n = 1080716x_{n+1} - 364x_{n+3}$
3	$2548X_n - Y_n^2 = 142688$	$X_n = 28y_{2n+2} - 364x_{2n+2} + 28$ $Y_n = 5096x_{n+1} - 364y_{n+1}$

4	$68796X_n - Y_n^2 = 104019552$	$X_n = 28y_{2n+3} - 20020x_{n+2} + 756$ $Y_n = 270088x_{n+1} - 364y_{n+2}$
5	$3712436X_n - Y_n^2 = 302905078112$	$X_n = 28y_{2n+4} - 1080716x_{2n+2} + 40796$ $Y_n = 14579656x_{n+1} - 364y_{n+2}$
6	$5096X_n - Y_n^2 = 570752$	$X_n = 1484x_{2n+4} - 80108x_{2n+3} + 56$ $Y_n = 1080716x_{n+2} - 20020x_{n+3}$
7	$68796X_n - Y_n^2 = 104019552$	$X_n = 1484y_{2n+2} - 364x_{2n+3} + 756$ $Y_n = 5096x_{n+2} - 20020y_{n+1}$
8	$2548X_n - Y_n^2 = 14268886$	$X_n = 1484y_{2n+3} - 20020x_{2n+3} + 28$ $Y_n = 270088x_{n+2} - 20020y_{n+2}$
9	$68796X_n - Y_n^2 = 104019552$	$X_n = 1484y_{2n+4} - 1080716x_{2n+3} + 756$ $Y_n = 14579656x_{n+2} - 20020y_{n+3}$
10	$3712436X_n - Y_n^2 = 302905078112$	$X_n = 80108y_{2n+2} - 364x_{2n+4} + 40796$ $Y_n = 5096x_{n+3} - 1080716y_{n+1}$
11	$68796X_n - Y_n^2 = 104019552$	$X_n = 80108y_{2n+3} - 20020x_{2n+4} + 756$ $Y_n = 270088x_{n+3} - 1080716y_{n+2}$
12	$2548X_n - Y_n^2 = 142688$	$X_n = 80108y_{2n+4} - 1080716x_{2n+4} + 28$ $Y_n = 14579656x_{n+3} - 1080716y_{n+3}$
13	$927472X_n - Y_n^2 = 18905589248$	$X_n = 20020y_{2n+2} - 364y_{2n+3} + 10192$ $Y_n = 5096y_{n+2} - 270088y_{n+1}$
14	$50083488X_n - Y_n^2 = 55128698247168$	$X_n = 1080716y_{2n+2} - 364y_{2n+4} + 550368$ $Y_n = 5096y_{n+3} - 14579656y_{n+1}$
15	$927472X_n - Y_n^2 = 18905589248$	$X_n = 1080716y_{2n+3} - 20020y_{2n+4} + 1019$ $Y_n = 270088y_{n+3} - 14579656y_{n+2}$

5. Conclusion

To conclude, one may search for other choices of positive Pell equations for finding their integer solutions with suitable properties.

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