Journal of Mathematics and Informatics
Vol. 11, 2017, 95-102
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v11a12

Journal of Mathematics and Informatics

On Negative Pellian Equation $y^2=40x^2-15$

***P.Abinaya*[1] and *S.Mallika*[2]**

Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India.
Tamilnadu, India.
[1]e-mail: abinayapalani01@gmail.com; [2]e-mail: smallika65@gmail.com

Received 3 November 2017; accepted 9 December 2017

Abstract. The binary quadratic equation represents by negative Pellian $y^2 = 40x^2 - 15$ is analyzed for its distinct integer solutions.A few interesting relations among the solution are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The binary quadratic Diophantine equations (both homogeneous and non-homogeneous) are rich in variety. In [1-16] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. This communication concerns with yet another binary quadratic equation given by $y^2 = 40x^2 - 15$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. Method of analysis

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 40x^2 - 15 \tag{1}$$

whose smallest positive integer solution is

$x_0 = 1$ and $y_0 = 5$

To obtain the other solution of (1), consider the Pell equation $y^2 = 40x^2 + 1$ whose solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{40}} g_n$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (19 + 3\sqrt{40})^{n+1} + (19 - 3\sqrt{40})^{n+1}$$

$g_n = (19+3\sqrt{40})^{n+1} - (19-3\sqrt{40})^{n+1}$

Applying Brahmagupta lemma, the general solution of (1) is found to be

$$x_{n+1} = \frac{1}{2} f_n + \frac{5}{2\sqrt{40}} g_n$$

$$y_{n+1} = \frac{5}{2} f_n + \frac{40}{2\sqrt{40}} g_n$$

The recurrence relation satisfied by the solution x and y are given by

$$x_{n+3} - 38x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 38y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
-1	1	5
0	34	215
1	1291	8165
2	49024	310055

From the above table, we observe some interesting relations among the solutions which are presented below.

- x_n values are alternatively odd and even.
- y_n values are odd.

- Each of the following expression is a nasty number:
  - $\frac{1}{45}\left[540 + 2580x_{2n+2} - 60x_{2n+3}\right]$
  - $\frac{1}{1710}\left[20520 + 97980x_{2n+2} - 60x_{2n+4}\right]$
  - $\frac{1}{285}\left[3420 + 16320x_{2n+2} - 60y_{2n+3}\right]$
  - $\frac{1}{10815}\left[129780 + 619680x_{2n+2} - 60y_{2n+4}\right]$
  - $\frac{1}{45}\left[540 + 97980x_{2n+3} - 2580x_{2n+4}\right]$
  - $\frac{1}{285}\left[3420 + 480x_{2n+3} - 2580y_{2n+2}\right]$
  - $\frac{1}{15}\left[180 + 16320x_{2n+3} - 2580y_{2n+3}\right]$

- $\frac{1}{285}\left[3420+619680x_{2n+3}-2580y_{2n+4}\right]$
- $\frac{1}{10815}\left[129780+480x_{2n+4}-97980y_{2n+2}\right]$
- $\frac{1}{285}\left[3420+16320x_{2n+4}-97980y_{2n+3}\right]$
- $\frac{1}{15}\left[180+619680x_{2n+4}-97980y_{2n+4}\right]$
- $\frac{1}{45}\left[540+12Y_{2n+3}-408y_{2n+2}\right]$
- $\frac{1}{1710}\left[20520+12y_{2n+4}-15492y_{2n+2}\right]$
- $\frac{1}{45}\left[540+408y_{2n+4}-15492y_{2n+3}\right]$

➢ Each of the following expressions is a cubical integer

- $2025[(430x_{3n+3}-10x_{3n+4})+6075(430x_{n+1}-10x_{n+2})]$
- $2924100[(16330x_{3n+3}-10x_{3n+5})+8772300(16330x_{n+1}-10x_{n+3})]$
- $81225[(2720x_{3n+3}-10y_{3n+4})+243675(2720x_{n+1}-10y_{n+2})]$
- $116964225[(103280x_{3n+3}-10y_{3n+5})+350892675(103280x_{n+1}-10y_{n+3})]$
- $2025[(16330x_{3n+4}-430y_{3n+5})+6075(16330x_{n+2}-430x_{n+3})]$
- $81225[(80x_{3n+4}-430y_{3n+3})+243675(80x_{n+2}-430y_{n+1})]$
- $225[(2720x_{3n+4}-430y_{3n+4})+675(2720x_{n+2}-430y_{n+2})]$
- $81225[(103280x_{3n+4}-430y_{3n+5})+243675(103280x_{n+2}-430y_{n+3})]$
- $116964225[(80x_{3n+5}-16330y_{3n+3})+350892675(80x_{n+3}-16330y_{n+1})]$
- $81225[(2720x_{3n+5}-16330y_{3n+4})+243675(2720x_{n+3}-16330y_{n+2})]$
- $225[(103280x_{3n+5}-16330y_{3n+5})+675(103280x_{n+3}-16330y_{n+3})]$
- $2025[(2y_{3n+4}-68y_{3n+3})+6075(2y_{n+2}-204y_{n+1})]$
- $2924100[(2y_{3n+5}-2582y_{3n+3})+8772300(2y_{n+3}-2582y_{n+1})]$
- $2025[(68y_{3n+5}-2582y_{3n+4})+6075(68y_{n+3}-2582y_{n+2})]$

➢ Relations among the solutions are given below:

- $3y_{n+1}-x_{n+1}+19x_{n+2}=0$
- $38x_{n+2}-x_{n+1}-x_{n+3}=0$

- $114y_{n+1} + 721x_{n+1} - x_{n+3} = 0$
- $1710y_{n+2} - 1711581x_{n+1} + 1041x_{n+3} = 0$
- $1710y_{n+3} - 10000076x_{n+1} - 3069x_{n+3} = 0$
- $x_{n+3} - x_{n+1} - 6y_{n+2} = 0$
- $19y_{n+1} + 120x_{n+1} - y_{n+2} = 0$
- $721x_{n+3} - x_{n+1} - 114y_{n+3} = 0$
- $721y_{n+1} + 4560 - y_{n+3} = 0$
- $3y_{n+2} + 19x_{n+2} - x_{n+3} = 0$
- $19y_{n+2} - 120x_{n+2} - y_{n+1} = 0$
- $y_{n+3} - 240x_{n+2} - y_{n+1} = 0$
- $19y_{n+1} - 721x_{n+2} + 3y_{n+3} = 0$
- $19y_{n+2} + 120x_{n+2} - y_{n+3} = 0$
- $21630x_{n+1} - 90x_{n+3} + 18912y_{n+1} = 0$
- $865200x_{n+2} - 194800x_{n+3} + 44414000y_{n+1} = 0$
- $4326y_{n+2} + 14044944y_{n+1} + 6160x_{n+3} = 0$
- $21630y_{n+3} - 1169600x_{n+3} + 266668930y_{n+1} = 0$
- $22800x_{n+1} - 9`160x_{n+3} + 549920y_{n+2} = 0$
- $40x_{n+1} - 159480x_{n+3} + 25216y_{n+3} = 0$
- $3x_{n+2} - 702247x_{n+3} + 111035y_{n+3} = 0$
- $3y_{n+1} + 18493y_{n+3} - 116960x_{n+3} = 0$
- $3y_{n+2} - 4441400x_{n+3} + 702247y_{n+3} = 0$
- $1800x_{n+1} - 35y_{n+2} + 1145y_{n+1} = 0$
- $y_{n+3} + y_{n+1} - 38y_{n+2} = 0$
- $13680x_{n+1} - 7y_{n+3} + 8695y_{n+1} = 0$
- $45610x_{n+3} - 721y_{n+3} + y_{n+1} = 0$

3. Remarkable observation

3.1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

Sl.no	Hyperbola	(X_n, Y_n)
1	$40X_n^2 - Y_n^2 = 324000$	$[(430x_{n+1} - 10x_{n+2}), (80x_{n+2} - 2720x_{n+1})]$

2	$40X_n^2 - Y_n^2 = 11696400$	$[(16330x_{n+1} - 10x_{n+3}),(80x_{n+3} - 103280x_{n+1})]$
3	$40X_n^2 - Y_n^2 = 12996000$	$[(2720x_{n+1} - 10y_{n+2}),(80y_{n+2} - 17200x_{n+1})]$
4	$40X_n^2 - Y_n^2 = 18714276000$	$[(103280x_{n+1} - 10y_{n+3}),(80y_{n+3} - 653200x_{n+1})]$
5	$40X_n^2 - Y_n^2 = 324000$	$[(16330x_{n+2} - 430x_{n+3}),(2720x_{n+3} - 103280x_{n+2}$
6	$40X_n^2 - Y_n^2 = 12996000$	$[(80x_{n+2} - 430y_{n+1}),(2720y_{n+1} - 400x_{n+2})]$
7	$40X_n^2 - Y_n^2 = 36000$	$[(2720x_{n+2} - 430y_{n+2}),(2720y_{n+2} - 17200x_{n+2})]$
8	$40X_n^2 - Y_n^2 = 12996000$	$[(103280x_{n+2} - 430y_{n+3}),(2720y_{n+3} - 653200x_{n+}$
9	$40X_n^2 - Y_n^2 = 18714276000$	$[(80x_{n+3} - 16330y_{n+1}),(400x_{n+3} - 103280y_{n+1})]$
10	$40X_n^2 - Y_n^2 = 12996000$	$[(2720x_{n+3} - 16330y_{n+2}),(103280y_{n+2} - 17200x$
11	$40X_n^2 - Y_n^2 = 36000$	$[(103280x_{n+3} - 16330y_{n+3}),(653200x_{n+3} - 10328$
12	$40X_n^2 - Y_n^2 = 324000$	$[(2y_{n+2} - 68y_{n+1}),(430y_{n+1} - 10y_{n+2})]$
13	$40X_n^2 - Y_n^2 = 467856000$	$[(2y_{n+3} - 258y_{n+1}),(16330y_{n+1} - 10y_{n+3})]$
14	$40X_n^2 - Y_n^2 = 324000$	$[(68y_{n+3} - 2582y_{n+2}),(16330y_{n+2} - 430y_{n+3})]$

3.2. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table 3 below.

Table 3: Parabola

Sl.no	Parabola	(X_n, Y_n)
1	$1800X_n - Y_n^2 = 324000$	$[(90 + 430x_{2n+2} - 10x_{2n+3}),(80x_{n+2} - 2720x_{n+1})]$
2	$68400X_n - Y_n^2 = 467856000$	$[(3420 + 16330x_{2n+2} - 10x_{2n+4}),(80x_{n+3} - 103280x_{n+}$
3	$11400X_n - Y_n^2 = 12996000$	$[(570 + 2720x_{2n+2} - 10y_{2n+3}),(80y_{n+2} - 34400x_{n+1})]$

4	$432600X_n - Y_n^2 = 1871427$	$[(21630 + 103280x_{2n+2} - 10y_{2n+4}),(80y_{n+3} - 653200$
5	$1800X_n - Y_n^2 = 324000$	$[(90 + 16330x_{2n+3} - 430x_{2n+4}),(2720x_{n+3} - 103280x$
6	$40X_n - Y_n^2 = 12996000$	$[(570 + 80x_{2n+3} - 430y_{2n+2}),(2720y_{n+1} - 400x_{n+2})]$
7	$600X_n - Y_n^2 = 36000$	$[(30 + 2720x_{2n+3} - 430y_{2n+3}),(2720y_{n+2} - 17200x_{n+}$
8	$11400X_n - Y_n^2 = 12996000$	$[(570 + 103280x_{2n+3} - 430y_{2n+4}),(2720y_{n+3} - 65320$
9	$432600X_n - Y_n^2 = 1871427$	$[(21630 + 80x_{2n+4} - 16330y_{2n+2}),(400x_{n+3} - 103280$
10	$11400X_n - Y_n^2 = 12996000$	$[(570 + 2720x_{2n+4} - 16330y_{2n+3}),(103280y_{n+2} - 172$
11	$600X_n - Y_n^2 = 36000$	$[(30 + 103280x_{2n+4} - 16330y_{2n+4}),(653200x_{n+3} - 10$
12	$1800X_n - Y_n^2 = 324000$	$[(90 - 68y_{2n+2} + 2y_{2n+3}),(430y_{n+1} - 10y_{n+2})]$
13	$68400X_n - Y_n^2 = 46785600$	$[3420 + (2y_{2n+4} - 2582y_{2n+2}),(16330y_{n+1} - 10y_{n+3})]$
14	$1800X_n - Y_n^2 = 324000$	$[(90 + 68y_{2n+4} - 2582y_{2n+3}),(16330y_{n+2} - 430y_{n+3})$

3.3. Consider $m = x_{n+1} + y_{n+1}$, $n = x_{n+1}$, observe that m>n>0
Treat m, n as the generators of the Pythogorean triangle T(α,β,γ)

$\alpha = 2mn$, $\beta = m^2 - n^2$, $\gamma = m^2 + n^2$

Then the following interesting relations are observed.

1. $\alpha - 20\beta + 19\gamma = 15$
2. $21\alpha - \gamma - \frac{80A}{P} = 15$
3. $11\alpha - 10\beta + 9\gamma - \frac{40A}{P} = 15$
4. $\frac{2A}{P} = x_{n+1}y_{n+1}$

4. Conclusion

In this paper, we have presented infinitely many integer solutions for all hyperbola represented by the negative Pell Equations $y^2 = 40x^2 - 15$.As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative Pell Equations and determine their integer solutions along with suitable properties.

REFERENCES

1. R.A.Mollin and Anithasrinivasan, A note on the negative pell equation, *International Journal of Algebra*, 4(19) (2010) 919-922.
2. E.E.Whitford, Some solutions of the Pellian Equation $x^2 - Ay^2 = \pm 4$ *JSTOR:Annals of Mathematics*, Second series, 1 (1913-1914) 157-160.
3. S.AhmetTekcan, B.W.Gezer and O. Bizim. On the Integer solutions of the pell Equations $x^2 - dy^2 = 2^t$, *World Academy of Science, Engineering and Technology,* 1 (2007) 522-526.
4. S.Ahmet Tekcan, The Pell Equation $x^2 - (k^2 - k)y^2 = 2^t$, *World Academy of Science, Engineering and Technology*, 19 (2008) 697-701.
5. V. Sangeetha, M.A.Gopalan and M.Somanath. On the Integral Solutions of the Pell equation $x^2 = 13y^2 - 3^t$, *International Journal of Applied Mathematical Research*, 3(1) (2014) 58-61.
6. M.A.Gopalan, G.Sumathi and S.Vidhyalakshmi, Observations on the hyperbola $x^2 = 19y^2 - 3^t$, *Journal of the Engineering and Technology,* 2(2A) (2014) 152-155.
7. M.A.Gopalan, S.Vidhyalakshmi and A. Kavitha, On the Integral solutions of the binary quadratic equation $x^2 = 15y^2 - 11^t$, *Scholars Journal of the Engineering and Technology*, 2(2A) (2014) 156-158.
8. S.Vidhyalakshmi,V.Krithika and K.Agalya, On the negative pell equation $y^2 = 80x^2 - 31$,*Proceeding of the National Conference on MATAM, PSNA College of Enineering and Technology*,Dindugal, (2015) 4-10.
9. K.Meena, M.A.Gopalan and R.Karthika, On the Negative pell equation $y^2 = 10x^2 - 6$, *Integral Journal of Multidisciptinary Research and Developmen,* 2(12) (2015) 390-392.
10. M.A.Gopalan, S.Vidhyalakshmi, V.Pandichelvi, P.Sivakamasundari and C.Priyadharsini, On the Negative pellequation $y^2 = 45x^2 - 11$ *International Journal of Pure Mathematical Science*, 16 (2016) 30-36.
11. K.Meena, S.Vidhyalakshmi and A.Rukmani. On the negative pell equation $y^2 = 31x^2 - 6$, *Universe of Engineering Technologies and Science*, 11(X11) (2015) 1-4.
12. K.Meena, M.A.Gopalan and E.Bhuvaneswari, On the Negative pell equation $y^2 = 60x^2 - 15$, *Scholars Bulletin*, 1(11) (2015) 310-316.
13. M.A.Gopalan, S.Vidhyalakshmi, R.Presenna and M.Vanitha, Observation On the Negative pell equation $y^2 = 180x^2 - 11$, *Universal Journal of Mathematics*, 2(1) (2016) 41-45.

14. S.Vidhyalakshmi, M.A.Gopalan, E.Premalatha and S.SofiaChristinal, On the Negative Pell Equation $y^2 = 72x^2 - 8$. *International Journal of Engineering Research*, 4(2) (2016) 25-28.
15. M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi. A study on the Hyperbola $y^2 = 8x^2 - 31$. *International Journal of latest Research in Science and Technology*, 2(1) (2013) 454-456.
16. M.A.Gopalan, S.Vidhyalakshmi, T.R.Usha Rani, S.Mallika. Observation on $y^2 = 12x^2 - 3$. *Bessel Journal of math*., 2(3) (2012) 153-158.