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A Ternary Quadratic Diophantine Equation $x^2+y^2=65z^2$

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Abstract. The Quadratic Diophantine equation with three unknowns represented by $x^2+y^2=65z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained. A few interesting properties among the solutions are presented.

Keywords: Ternary quadratic equation with three unknowns, integral solutions, polygonal numbers, and pyramidal numbers.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The quadratic Diophantine equation with three unknowns offers an unlimited field for research because of their variety [1-3]. In particular, one may refer [4-16] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2+y^2=65z^2$ representing homogeneous quadratic Diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given. Also, formulas for generating sequences of integer solutions based on its given solution are presented.

2. Notation

1. $T_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$ - Polygonal Number of Rank n with side m.
2. $T_{3,n}=\frac{n(n+1)}{2}$ - Triangular Number of Rank n.
3. $PR_n=n(n+1)$ - Pronic Number of Rank n.
4. $Cp_{n,6}=n^3$ - Centered Hexagonal Pyramidal Number of Rank n.
5. $T_{4,n}=n^2$ - Square Number of Rank n.
6. $T_{8,n}=3n^2-2n$ - Octogonal number of Rank n.

3. Method of analysis

The ternary quadratic Diophantine equation to be solved for its non-zero distinct integral solution is

$$x^2 + y^2 = 65z^2 \tag{1}$$

Different patterns of solution of (1) are presented below.

3.1. PATTERN- 1

Write 65 as

$$65 = (8+i)(8-i) \tag{2}$$

Assume $z = a^2 + b^2$ (3)

Where a,b are non-zero distinct integers.

Using (2) and (3) in (1) we get

$$x^2 + y^2 = (8+i)(8-i)(a^2+b^2)^2$$

Employing the method of factorization the above equation is written as

$$(x+iy)(x-iy) = (8+i)(8-i)(a+ib)^2(a-ib)^2$$

Equating the positive and negative factors we get,

$$x + iy = (8+i)(a+ib)^2 \tag{4}$$

$$x - iy = (8-i)(a-ib)^2 \tag{5}$$

Equating the real and imaginary part either in (4) or (5) we get

$$\left.\begin{aligned} x(a,b) &= 8a^2 - 8b^2 - 2ab \\ y(a,b) &= a^2 - b^2 + 16ab \end{aligned}\right\} \tag{6}$$

Thus (6) and (3) represents non-zero distinct integral solutions of (1)

Properties :

1. $x(n,1) - t_{18,n} - 10t_{3,n} + 5t_{4,n} + 8 = 0$
2. $y(n,2) + z(n,2) - 2t_{4,n} \equiv 0 (\bmod 2)$

3.2. PATTERN - 2

Write 65 as

$$65 = (7+4i)(7-4i) \tag{7}$$

Where a,b are non-zero distinct integers,

Using (7) and (3) in (1) we get

$$x^2 + y^2 = (7+4i)(7-4i)(a^2+b^2)^2$$

Employing the method of factorization the above equation is written as

$$(x+iy)(x-iy) = (7+4i)(7-4i)(a+ib)^2(a-ib)^2$$

Equating the positive and negative factors we get,

$$x + iy = (7+4i)(a+ib)^2 \tag{8}$$

$$x-iy=(7-4i)(a-ib)^2 \tag{9}$$

Equating the real and imaginary part either in (8) or (9) we get,

$$\left.\begin{aligned} x(a,b) &= 7a^2-7b^2-8ab \\ y(a,b) &= 4a^2-4b^2+14ab \end{aligned}\right\} \tag{10}$$

Thus (10) and (3) represents non-zero distinct integral solutions of (1)

Properties :

1. $x(n.n)+8t_{4,n}=0$
2. $y(n,1)+4z(n,1)+6t_{4,n}-14PR_n=0$
3. $x(n,1)-y(n,1)-t_{8,n}=17(\text{mod } 20)$

3.3. PATTERN -3

(1)can be written in the form of ratio as

$$\frac{x+8z}{z+y}=\frac{z-y}{x-8z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{11}$$

(11) is equivalent to the system of double equations

$$\left.\begin{aligned} \beta x-\alpha y+(8\beta-\alpha)z &= 0 \\ \alpha x+\beta y-(8\alpha+\beta)z &= 0 \end{aligned}\right\} \tag{12}$$

Solving (12) by applying the method of cross multiplication, the corresponding non-zero distinct integral solutions to (1) are obtained by

$$x(\alpha,\beta)=8\alpha^2-8\beta^2+2\alpha\beta$$

$$y(\alpha,\beta)=-\alpha^2+\beta^2+16\alpha\beta$$

$$z(\alpha,\beta)=\alpha^2+\beta^2$$

Properties :

1. $x(n,1)-8t_{4,n}+8\equiv 0(\text{mod } 2)$
2. $y(n,1)+z(n,1)-16PR_n+16t_{4,n}$ is even number
3. $x(n,1)+8z(n,1)-16t_{4,n}\equiv 0(\text{mod } 2)$

Remark: In addition to (11), (1) may also be expressed in the form ratio as

$$\frac{x+7z}{4z+y}=\frac{4z-y}{x-7z}=\frac{\alpha}{\beta},\beta\neq 0$$

Following the procedure as presented above the corresponding non-zero distinct integral solutions to (1) is given by

$$x(\alpha,\beta)=7\alpha^2-7\beta^2+8\alpha\beta$$

$$y(\alpha,\beta)=-4\alpha^2+4\beta^2+14\alpha\beta$$

$$z(\alpha,\beta)=\alpha^2+\beta^2$$

Properties :

1. $x(n,2)+7z(n,2)-14PR_n \equiv 0(\text{mod } 2)$
2. $x(n,2)+y(n,2)-3t_{4,n} \equiv 32(\text{mod } 44)$
3. $y(1,n)+4z(1,n)-8PR_n \equiv 0(\text{mod } 2)$

3.4. PATTERN- 4

Introducing the linear transformations,

$x=3u+v$, $y=u-3v$, $z=2w$ (13)

In (1) it is written as

$$u^2+v^2=26w^2 \quad (14)$$

Assume $w=a^2+b^2$ (15)

Write as $26=(5+i)(5-i)$ (16)

Substituting (15) and (16) in (14) we get,

$$(u+iv)(u-iv)=(5+i)(5-i)(a+ib)^2(a-ib)^2$$

Equating the positive and negative parts we get,

$$u+iv=(5+i)(a+ib)^2 \quad (17)$$

$$u-iv=(5-i)(a-ib)^2 \quad (18)$$

Equating the real and imaginary parts either in (17) and (18) we get,

$$\left.\begin{array}{l} u=5a^2-5b^2-2ab \\ v=a^2-b^2+10ab \end{array}\right\} \quad (19)$$

Substituting (19) and (16) in (14) the corresponding non-zero integral solution to (1) are given by

$$x(a,b)=16a^2-16b^2+4ab$$

$$y(a,b)=2a^2-2b^2-32ab$$

$$z(a,b)=2a^2+2b^2$$

Properties :

1. $y(n,1)+z(n,1)-4t_{4,n} \equiv 0(\text{mod } 2)$
2. $x(n,1)-16t_{4,n} \equiv 0(\text{mod } 4)$
3. $x(n,2)+8z(n,2)-32PR_n \equiv 0(\text{mod } 2)$

4. Generation of solutions

In this section, we obtain general formula for generating sequences of integer solutions to (1) based on its initial solution.

Formula 1. Let (x_0, y_0, z_0) be the initial solution to (1)

Let $x_1=x_0+8h$, $y_1=y_0$, $z_1=h-z_0$ (20)

be the first solution to (1), where h is the non-zero integer to be determined.

Substituting (20) in (1) and simplifying, we get

$h = 130z_0 + 16x_0$ (21)

Therefore, $x_1 = 129x_0 + 1040z_0, z_1 = 16x_0 + 129z_0$,

Expressing the above equations in the matrix form, we have

$\begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$, where $M = \begin{pmatrix} 129 & 1040 \\ 16 & 129 \end{pmatrix}$

Repeating the above process, the general values of x and z are given by

$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$

We know that,

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I)$$

where α and β are the eigen values of M and I is the unit matrix of order two.

For our problem,

$\alpha = 129 + 16\sqrt{65}$ and $\beta = 129 - 16\sqrt{65}$

Therefore,

$$M^n = \frac{1}{32\sqrt{65}} \begin{pmatrix} 16\sqrt{65}Y_n & 1040X_n \\ 16X_n & 16\sqrt{65}Y_n \end{pmatrix}$$

where $Y_n = \alpha^n + \beta^n$

$X_n = \alpha^n - \beta^n$

Thus, the general solution to (1) based on its initial solution is

$$x_n = \frac{1}{32\sqrt{65}}[16\sqrt{65}Y_n x_0 + 1040X_n z_0]$$

$$y_n = y_0$$

$$z_n = \frac{1}{32\sqrt{65}}[16X_n x_0 + 16\sqrt{65}Y_n z_0]$$

Formula 2

Let $x_1 = 7x_0 - h, y_1 = 4y_0 - h, z_1 = z_0$

be the first set of solution to(1). Following the procedure presented above, the corresponding general solution to (1) is given by

$$x_n = \frac{1}{4\sqrt{7}}\left[2\sqrt{7}Y_n x_0 - 4X_n y_0\right]$$

$$y_n = \frac{1}{4\sqrt{7}}\left[-7X_n x_0 + 2\sqrt{7}Y_n y_0\right]$$

$$z_n = z_0$$

where $Y_n = \left(2\sqrt{7}\right)^n + \left(-2\sqrt{7}\right)^n$

$X_n = \left(2\sqrt{7}\right)^n - \left(-2\sqrt{7}\right)^n$

Formula 3

Let $x_1 = x_0$, $y_1 = y_0 - 8h$, $z_1 = z_0 - h$ be the first set of solution to (1). Following the procedure presented above the corresponding general solution to (1) is given by

$$x_n = x_0$$

$$y_n = \frac{1}{32\sqrt{65}}\left[16\sqrt{65}Y_n y_0 - 1040 X_n z_0\right]$$

$$z_n = \frac{1}{32\sqrt{65}}\left[16 X_n y_0 + 258 + 16\sqrt{65}Y_n z_0\right]$$

where $Y_n = \alpha^n + \beta^n$, $X_n = \alpha^n - \beta^n$

5. Conclusion

In this paper, we have made an attempt to obtain infinitely many non-zero distinct integer solutions to the equation given by $x^2 + y^2 = 65z^2$. As ternary quadratic equations are rich in variety, one may search for the other choice of ternary quadratic Diophantine equations and determine their integer solutions along with suitable properties.

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