Journal of Mathematics and Informatics
Vol. 11, 2017, 21-28
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v11a4

Journal of **Mathematics and Informatics**

On the Ternary Quadratic Diophantine Equation $3(x^2+y^2)-5xy+2(x+y)+4=15z^2$

***K.Selva Keerthana*[1] and *S.Mallika*[2]**

Department of Mathematics, Shrimati Indira Gandhi College
Trichy-2, Tamilnadu, India. [1]e-mail: kselvakeerthana273@gmail.com
[2]e-mail: msmallika65@gmail.com

Received 1 November 2017; accepted 8 December 2017

Abstract. The ternary non-homogeneous quadratic equation is analysed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramidal numbers are presented.

Keywords: Ternary quadratic, integer solutions, non-homogeneous quadratic, polygonal numbers, pyramidal numbers.

AMS Mathematics Subject Classification (2010):11D09

1. Introduction

The Diophantine equation offer an unlimited field for research due to their variety $[1-3]$. In particular, one may refer $[4-12]$ for quadratic equation with three unknowns. The communication concerns with yet another interesting equation $3(x^2+y^2)-5xy+2(x+y)+4=15z^2$ representing non-homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. Notations

1. Polygonal number of rank n with side m

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

2. Gnonomic number of rank n

$$G_n=(2n-1)$$

3. Pronic number of rank n

$$PR_n=n(n+1)$$

4. Centered Hexagonal pyramidal number of rank n

$$CP_{n,6}=n^3$$

5. Centered Polygonal number of rank n with m sides

$$Ct_{m,n} = \frac{mn(n-1)+2}{2}$$

6. Stella Octangular number of rank n

$$SO_n = n(2n^2 - 1)$$

3. Method of analysis

The equation representing the ternary quadratic equation to be solved for its non-zero distinct integer solution is

$$3(x^2 + y^2) - 5xy + 2(x + y) + 4 = 15z^2 \tag{1}$$

The substitution of linear transformations

$$x = u + v, y = u - v, u \neq v \neq 0 \tag{2}$$

in (1) leads to

$$U^2 + 11v^2 = 15z^2 \tag{3}$$

where

$$U = u - 2 \tag{4}$$

Assume that

$$z = a^2 + b^2 \tag{5}$$

3.1. Pattern-1

Write 15 as

$$15 = (2 + i\sqrt{11})(2 - i\sqrt{11}) \tag{6}$$

Substituting $(4),(5),(6)$ in (3) and using the method of factorization we get,

$$(U + i\sqrt{11}v)(U - i\sqrt{11}v) = (a + i\sqrt{11}b)^2 (a - i\sqrt{11}b)^2 (2 + i\sqrt{11})(2 - i\sqrt{11}) \tag{7}$$

Equating the positive and negative factors, the resulting equations are

$$(U + i\sqrt{11}v) = (2 + i\sqrt{11})(a + i\sqrt{11}b)^2 \tag{8}$$

$$(U - i\sqrt{11}v) = (2 - i\sqrt{11})(a - i\sqrt{11}b)^2 \tag{9}$$

Equating the real and imaginary parts in (8)

$$\left.\begin{array}{l} U = 2a^2 - 22b^2 - 22ab \\ v = a^2 - 11b^2 + 4ab \end{array}\right\} \tag{10}$$

In view of (4)

$$\left.\begin{array}{l} u = 2a^2 - 22b^2 - 22ab - 2 \\ v = a^2 - 11b^2 + 4ab \end{array}\right\} \tag{11}$$

Substituting (11) in (2),

$$x = x(a,b) = 3a^2 - 33b^2 - 18ab - 2$$

$$y = y(a,b) = a^2 - 11b^2 - 26ab - 2$$

$$z = z(a,b) = a^2 + 11b^2$$

Thus the above equation represents the non-zero distinct integer solutions to (1)

Properties:

1. $x(a,a+1)-3y(a,a+1)-60PR_a=0$
2. $y(a,a)-z(a,a)+48t_{4,a}\equiv 0 \pmod 2$
3. $x(a,1)+z(a,1)-t_{10,a}\equiv 9 \pmod{15}$

3.2. Pattern- 2

Instead of (6), 15 can be written as

$$15=\frac{(7+i\sqrt{11})(7-i\sqrt{11})}{4}$$

Proceeding as in Pattern: 1, the non-zero distinct integral solutions to (1) as

$$x=x(a,b)=4a^2-44b^2-4ab-2$$
$$y=y(a,b)=3a^2-33b^2-18ab-2$$
$$z=z(a,b)=a^2+11b^2$$

Properties:

1. $x(a,1)+4z(a,1)-t_{18,a}\equiv 1 \pmod 3$
2. $x(a,1)-y(a,1)-t_{4,a}-aSO_2\equiv 0 \pmod{11}$
3. $z(a,a)-12t_{4,a}=0$

3.3. Pattern-3

The substitution of linear transformation

$$z=X+11T, V=X+15T, U=2\sigma \qquad (12)$$

in (3) leads to

$$X^2=165T^2+\sigma^2$$
$$X^2-\sigma^2=165T^2 \qquad (13)$$

Write (13) as

$$(X+\sigma)(X-\sigma)=165T^2 \qquad (14)$$

The equation (14) is written as the system of two equations as follows

System	1	2	3	4	5	6
$X+\sigma$	$15T$	15	165	$55T$	55	$55T^2$
$X-\sigma$	$11T$	$11T^2$	T^2	$3T$	$3T^2$	3

System 1:

Consider

$$X+\sigma=15T$$
$$X-\sigma=11T$$

And solving we get

$$\left.\begin{array}{l} X = 26k \\ \sigma = 4k \\ T = 2k \end{array}\right\} \quad (15)$$

Substituting (15) in (12) and (2), we get the corresponding non-zero distinct integer solutions to (1) as

$x(k) = 64k - 2$

$y(k) = -48k - 2$

$z(k) = -48k$

Properties:

1. $6(x(1) + y(1))$ is a nasty number
2. $z(k) - G_k - 1 \equiv 0 \pmod{46}$
3. $x(k^2) + y(k^2) + z(k^2) - 62t_{4,k} \equiv 0 \pmod{2}$

System 2:

Consider

$X + \sigma = 15$

$X - \sigma = 11T^2$

And solving we get

$$\left.\begin{array}{l} X = 22k^2 + 8k + 30 \\ \sigma = -22k^2 - 22k + 2 \\ T = 2k + 1 \end{array}\right\} \quad (16)$$

Substituting (16) in (12) and (2), we get the corresponding distinct non-zero integral solutions to (1) as

$x(k) = -22k^2 + 8k + 30$

$y(k) = -66k^2 - 96k - 26$

$z(k) = 22k^2 + 44k + 24$

Properties:

1. $x(k) - y(k) - t_{90,k} \equiv 56 \pmod{147}$
2. $x(k) + y(k) + 176t_{3,k} \equiv 0 \pmod{4}$
3. $x(k) + z(k) - G_k \equiv 5 \pmod{50}$

System 3:

Consider

$X + \sigma = 165$

$X - \sigma = T^2$

And solving we get,

$$\left.\begin{aligned} X &= 2k^2-2k+83 \\ \sigma &= -2k^2+2k+82 \\ T &= 2k-1 \end{aligned}\right\} \qquad (17)$$

Substituting (17) in (12) and (2), we get the corresponding non-zero distinct integer solution to (1) as

$x(k) = -2k^2+32k+230$

$y(k) = -6k^2-24k+94$

$z(k) = 2k^2+20k+72$

Properties:

1. $y(1) - CP_{4,6} = 0$
2. $x(k)+y(k)+16t_{3,k} \equiv 4(\mathrm{mod}\,16)$
3. $x(k)-y(k)-z(k)-t_{6,k} \equiv 27(\mathrm{mod}\,37)$

System 4:

Consider

$X+\sigma = 55T$

$X-\sigma = 3T$

And solving we get,

$$\left.\begin{aligned} X &= 58k \\ \sigma &= 52k \\ T &= 2k \end{aligned}\right\} \qquad (18)$$

Substituting (18) in (12) and (2), we get the corresponding distinct non-zero integral solutions to (1) as

$x(k) = 192k-2$

$y(k) = 16k-2$

$z(k) = 80k$

Properties:

1. $y(k) - G_k + 1 \equiv 0(\mathrm{mod}\,14)$
2. $x(k)-y(k)+2Ct_{176,k} - 176PR_k \equiv 2(\mathrm{mod}\,176)$
3. $z(k^2)-y(k^2)-64t_{4,k} \equiv 0(\mathrm{mod}\,2)$

System 5:

Consider

$X+\sigma = 55$

$X-\sigma = 3T^2$

And solving we get,

$$\left.\begin{aligned} X &= 6k^2+6k+29 \\ \sigma &= -6k^2-6k+26 \\ T &= 2k+1 \end{aligned}\right\} \qquad (19)$$

Substituting (19) in (12) and (2), we get the corresponding non-zero distinct integer solutions to (1) as

$x(k) = -6k^2+24k+94$
$y(k) = -18k^2-48k+6$
$z(k) = 6k^2+28k+40$

Properties:
1. $x(k)-y(k)-24t_{3,k} \equiv 28(\mathrm{mod}\,60)$
2. $y(1)+z(1)-SO_2 = 0$
3. $x(2k-1)+z(2k-1)-52G_k \equiv 0(\mathrm{mod}\,134)$

System 6:
Consider
$X+\sigma = 55T^2$
$X-\sigma = 3$
And solving we get,

$$\left.\begin{aligned} X &= 110k^2+110k+29 \\ \sigma &= 110k^2+110k+26 \\ T &= 2k+1 \end{aligned}\right\} \qquad (20)$$

Substituting (20) in (12) and (2), we get the corresponding distinct non-zero distinct integral solutions to (1) as

$x(k) = 330k^2+330k+94$
$y(k) = 110k^2+80k+6$
$z(k) = 110k^2+132k+40$

Properties:
1. $z(k(2k-1))-y(k(2k-1))-52t_{6,k} \equiv 0(\mathrm{mod}\,2)$
2. $y(1)-t_{4,14} = 0$
3. $x(k)-y(k)-z(k)-110PR_k \equiv 10(\mathrm{mod}\,38)$

3.4. Pattern- 4
(3) can be rewritten as

$$U^2-4v^2 = 15(z^2-v^2) \qquad (21)$$

Write (21) in the form of ratio as

$$\frac{U+2v}{z+v}=\frac{15(z-v)}{U-2v}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the following two equations,

$$U\beta+(2\beta-\alpha)v-\alpha z=0$$
$$U\alpha+(15\beta-2\alpha)v-15z\beta=0$$

On employing the method of cross multiplication, we get

$$U=2\alpha^2-30\alpha\beta+30\beta^2$$
$$v=\alpha^2-15\beta^2$$
$$z=\alpha^2-4\alpha\beta+15\beta^2$$

In view of (4),

$$u=2\alpha^2-30\alpha\beta+30\beta^2-2$$
$$v=\alpha^2-15\beta^2 \qquad (22)$$
$$z=\alpha^2-4\alpha\beta+15\beta^2$$

Substituting (22) in (2), we get

$$\left.\begin{aligned} x&=x(\alpha,\beta)=3\alpha^2-30\alpha\beta+15\beta^2-2\\ y&=y(\alpha,\beta)=\alpha^2-30\alpha\beta+45\beta^2-2\\ z&=z(\alpha,\beta)=\alpha^2-4\alpha\beta+15\beta^2\end{aligned}\right\} \qquad (23)$$

Thus (23) represent non-zero distinct integral solutions of (1)

Properties:

1. $x(\alpha,1)-y(\alpha,1)-2t_{4,\alpha}\equiv 0 \pmod 3$
2. $z(\alpha,\alpha)-12t_{4,\alpha}=0$
3. $3x(\alpha,1)-y(\alpha,1)-t_{18,\alpha}\equiv 49 \pmod{53}$

4. Conclusion

In this paper, we have made an attempt to obtain all integer solutions to ternary quadratic equation $3(x^2+y^2)-5xy+2(x+y)+4=15z^2$.One may search for other patterns of solutions and their corresponding properties.

REFERENCES

1. L.E.Dickson, *History of Theory of Numbers and Diophantine Analysis*, Dover Publications, New York 2005.
2. L.J.Mordell, *Diophantine Equations*, Academic Press, New York 1970.
3. R.D Carmicheal, *The Theory of Numbers and Diophantine Analysis*, Dover Publications, New York 1959.
4. M.A.Gopalan and D.Geetha , Lattice Points on the Hyperboloid of Two Sheets $x^2-6xy+y^2+6x-2y+5=z^2+4$, *Impact J Sci Tech*, 4 (2010) 23-32.
5. 5. M.A.Gopalan , S.Vidhyalakshmi, A. Kavitha, Integral points on the homogeneous cone $z^2=2x^2-7y^2$, *The Diophantus J Math* 1(2) (2012) 127-136.

6. M.A.Gopalan, S. Vidhyalakshmi and G. Sumathi, Lattice points on the hyperboloid of one sheet $4z^2 = 2x^2 + 3y^2 - 4$. *The Diophantus J Math* 1(2) 2012 109-115.
7. M.A.Gopalan , S.Vidhyalakshmi.,K. Lakshmi, Integral points on the hyperboloid of two sheets $3y^2 = 7x^2 - z^2 + 21$. *The Diophantus J Math* 1(2) (2012) 99-107.
8. M.A.Gopalan, S.Vidhyalakshmi and S.Mallika, Observations on hyperboloid of one sheet $x^2 + 2y^2 - z^2 = 2$. *Bessel J Math.,* 2(3) (2012) 221-226.
9. M.A.Gopalan , S.Vidhyalakshmi,T.R. Usha Rani ,S. Mallika, Integral points on the homogeneous cone $6z^2 + 3y^2 - 2x^2 = 0$, *The Impact J Sci Tech.,* 6(1) (2012) 7-13.
10. M.A.Gopalan, S.Vidhyalakshmi and G.Sumathi, Lattice points on the elliptic paraboloid $z = 9x^2 + 4y^2$, *Advances in theoretical and Applied Mathematics,* 7(4) (2012) 379-385.
11. M.A.Gopalan, S.Vidhyalakshm and T.R.Usha Rani, Integral points on the non-homogeneous $2z^2 + 4xy + 8x - 4z = 0$. *Global Journal of Mathematics and Mathematical sciences,* (1) (2012) 61-67.
12. M.A.Gopalan, S.Vidhyalakshmi and K. Lakshmi, Lattice points on the elliptic paraboloid $16y^2 + 9z^2 = 4x$, *Bessel J of Math,* 3(2) (2013) 137-145.