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On The Homogeneous Cubic Equation With Four Unknowns $(x^3+y^3)=7zw^2$

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Abstract. The homogeneous cubic equation with four unknowns represented by the Diophantine equation $(x^3+y^3)=7zw^2$ is analyzed for its patterns of non-zero distinct integer solutions. A few interesting properties between the solutions and special numbers are exhibited.

Keywords: Homogeneous cubic, cubic with four unknowns, integral solutions.

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

The Diophantine equation offer an unlimited field for research due to their variety $[1-2]$. In particular, one may refer $[3-7]$ for the cubic equation with three and four unknowns. This communication concerns with yet another interesting equation $(x^3+y^3)=7zw^2$ representing homogeneous cubic with four unknowns for determining its infinitely many non-zero integral points, also a few interesting relations among the solutions are presented.

2. Notations

1) Polygonal number of rank ‘n’ with m sides

$$t_{m,n}=n\left(1+\left(\frac{(n-1)(m-2)}{2}\right)\right)$$

2) Jacobsthal-Lucas number of rank n

$$j_n=2^n+(-1)^n$$

3) Pronic number of rank ‘n’

$$PR_n=n(n-1)$$

4) Centered Polygonal number of rank ‘n’ with m sides

$$Ct_{m,n} = \frac{mn(n-1)+2}{2}$$

5) Centered hexagonal Pyramidal number of rank 'n'

$$CP_{n,6} = n^3$$

3. Method of analysis

The equation representing the homogeneous cubic equation to be solved for its non-zero distinct integer solution

$$(x^3 + y^3) = 7zw^2 \quad (1)$$

It is to be noted that, (1) is satisfied by the following two integer quadruples

$(76k, 4k, 80k, 28k)$, $(32k, -16k, 16k, 16k)$

The substitution of linear transformation

$$x = u + v,\ y = u - v,\ z = 2u,\ u \neq v \neq 0 \quad (2)$$

in (1) leads to

$$u^2 + 3v^2 = 7w^2 \quad (3)$$

Assume that $w = a^2 + 3b^2$, where $a, b \succ 0$ (4)

3.1. Pattern-1

Write7 as,

$$7 = (2 + i\sqrt{3})(2 - i\sqrt{3}) \quad (5)$$

Substituting (4) and (5) in (3)

Using the method of factorization, we get

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (a + i\sqrt{3}b)^2 (a - i\sqrt{3}b)^2 (2 + i\sqrt{3})(2 - i\sqrt{3}) \quad (6)$$

Equating the positive and negative factors, the resulting equations are

$$(u + i\sqrt{3}v) = (2 + i\sqrt{3})(a + i\sqrt{3}b)^2 \quad (7)$$

$$(u - i\sqrt{3}v) = (2 - i\sqrt{3})(a - i\sqrt{3}b)^2 \quad (8)$$

Equating the real and imaginary parts, we have

$$\left.\begin{array}{l} u = 2a^2 - 6b^2 - 6ab \\ v = a^2 - 3b^2 + 4ab \end{array}\right\} \quad (9)$$

Hence in view of (2), the non-zero distinct integer values of x, y, z, w of (1) are given by

$$x = 3a^2 - 9b^2 - 2ab$$
$$y = a^2 - 3b^2 - 10ab$$
$$z = 4a^2 - 12ab - 12b^2$$
$$w = a^2 + 3b^2$$

Properties:

1.$x(a,a) - 3y(a,a) - 28t_{4,a} = 0$

2.$y(a, a+1) + w(a, a+1) - 2t_{4,a} + 20t_{3,a} = 0$

3.$6[w(1,1)]$ is a nasty number

3.2. Pattern-2

Instead of (5), 7 can be written as

$$7=\frac{\left(5+i\sqrt{3}\right)\left(5-i\sqrt{3}\right)}{4}$$

Proceeding as in pattern: 1, the non-zero distinct integer solution to (1) are given by

$x=3a^2-9b^2+2ab$
$y=2a^2-6b^2-8ab$
$z=5a^2-15b^2-6ab$
$w=a^2+3b^2$

Properties:

$1.w(a,a,)-4t_{4,a}=0$
$2.x(a,1)+y(a,1)-t_{12,a}\equiv 1(\mathrm{mod}\,8)$
$3.z(1,1)+j_4-1=0$

3.3. Pattern-3

The substitution of linear transformation

$w=X+3T\ ,V=X+7T\ ,\ u=2U$ (10)

in (3) leads to

$U^2=X^2-21T^2$
$X^2-U^2=21T^2$ (11)

write (11) as

$\left(X+U\right)\left(X-U\right)=21T^2$ (12)

The equation (12) is written as the system of two equations as follows:

System	**1**	**2**	**3**
$X+U$	21	$3T^2$	$7T^2$
$X-U$	T^2	7	3

System 1:

Consider

$X+U=21$
$X-U=T^2$

Solving these two equations we get

$$\left.\begin{aligned}X&=2k^2+12k+38\\U&=-2k^2-2k+20\\T&=2k+1\end{aligned}\right\}$$

(13)Substituting (13) in (10) and (2), we get the corresponding non-zero distinct integer solutions to (1) as follows:

$x=-2k^2+12k+38$
$y=-6k^2-20k+2$

$z = 40 - 8k^2 - 8k$
$w = 2k^2 + 8k + 14$

Properties:

1. $x(k) - y(k) - t_{10,k} \equiv 1 \pmod{35}$
2. $x(k) + y(k) + 16t_{3,k} \equiv 0 \pmod{5}$
3. $z + 2w + 4t_{4,k} \equiv 4 \pmod{8}$

System 2:

Consider,

$X + U = 3T^2$
$X - U = 7$

Solving these two equations we get

$$\left.\begin{aligned} X &= 6k^2 - 6k + 5 \\ U &= 6k^2 - 6k - 2 \\ T &= 2k - 1 \end{aligned}\right\} \qquad (14)$$

Substituting equation (14) in (10) and (2), we get the corresponding non-zero distinct integer solutions to (1) as follows

$x = 18k^2 - 4k - 6$
$y = 6k^2 - 20k - 2$
$z = 24k^2 - 24k - 8$
$w = 6k^2 + 2$

Properties:

1. $x(1) - CP_{2,6} = 6$
2. $x(k) + w(k) - t_{50,k} \equiv -4 \pmod{19}$
3. $6[2w(1)]$ is a nasty number

System 3:

Consider,

$X + U = 7T^2$
$X - U = 3$

Solving these two equations we get

$$\left.\begin{aligned} X &= 14k^2 + 14k + 5 \\ U &= 14k^2 + 14k + 2 \\ T &= 2k + 1 \end{aligned}\right\} \qquad (15)$$

Substituting (15) in (10) and (2) ,we get the corresponding non-zero distinct integer solutions to (1) as follows

$x = 42k^2 + 56k + 16$
$y = 14k^2 - 8$
$z = 56k^2 + 56k + 8$
$w = 14k^2 + 20k + 8$

Properties:

1.$6[y(1)-2]$is a nasty number

2.$x(k)+y(k)-56PR_k-8=0$

3.$y(k)+w(k)-28PR_k \equiv 0(\mod 8)$

3.4. Pattern-4

(11) can be re-written as

$$X^2 = 21T^2 + U^2$$

which is satisfied by

$$\left.\begin{aligned} X &= 21m^2 + n^2 \\ U &= 21m^2 - n^2 \\ T &= 2mn \end{aligned}\right\} \qquad (16)$$

Using (16) in (11) we get

$$\left.\begin{aligned} u &= 42m^2 - 2n^2 \\ v &= 21m^2 + 14mn + n^2 \\ w &= 21m^2 + 6mn + n^2 \end{aligned}\right\} \qquad (17)$$

Thus in view (2) and (3) ,the non-zero distinct integer solutions to (1) are obtained by

$$x = 63m^2 - n^2 + 14mn$$
$$y = 21m^2 - 3n^2 + 14mn$$
$$z = 84m^2 - 4n^2$$
$$w = 21m^2 + n^2 + 6mn$$

Properties:

1.$y(1,n)-w(1,n)+4t_{4,n} \equiv 0(\mod 8)$

2.$w(m,1)-t_{44,m}-1 \equiv 0(\mod 26)$

3.$w(m,m+1)-t_{154,m}+1 \equiv 0(\mod 87)$

3.5. Pattern-5

Consider (12) as

$$(X+U)(X-U) = 21T^2 \qquad (18)$$

Case:1

Write (18) in the form of ratio as

$$\frac{(X+U)}{T} = \frac{21T}{(X-U)} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the following two equations

$$\beta X + \beta U - \alpha T = 0$$
$$\alpha X - \alpha U - 21\beta T = 0$$

On employing the method of cross multiplication, we get

$U = \alpha^2 - 21\beta^2$

$X = 21\beta^2 + \alpha^2$

$T = 2\alpha\beta$

Thus, in view of (10) we get

$$\left.\begin{aligned} u &= 2\alpha^2 - 42\beta^2 \\ v &= \alpha^2 + 14\alpha\beta + 21\beta^2 \\ w &= \alpha^2 + 6\alpha\beta + 21\beta^2 \end{aligned}\right\} \quad (19)$$

Substituting in (2), we get

$$\left.\begin{aligned} x &= x(\alpha,\beta) = 3\alpha^2 + 14\alpha\beta - 21\beta^2 \\ y &= y(\alpha,\beta) = \alpha^2 - 14\alpha\beta - 63\beta^2 \\ z &= z(\alpha,\beta) = 4\alpha^2 - 84\beta^2 \\ w &= w(\alpha,\beta) = \alpha^2 + 6\alpha\beta + 21\beta^2 \end{aligned}\right\} \quad (20)$$

Thus (20) represents non-zero distinct integer solution of (1)

Properties:

1. $x(\alpha,1) + y(\alpha,1) - 2t_{4,\alpha} \equiv 0 \pmod 2$
2. $z(\beta+1,\beta) + 80t_{4,\alpha} \equiv 4 \pmod 8$
3. $z(\alpha,1) - y(\alpha,1) - t_{8,\alpha} \equiv 11 \pmod{16}$

Case 2:

Write (18) in the form of ratio as,

$$\frac{(X+U)}{3T} = \frac{7T}{(X-U)} = \frac{\alpha}{\beta}, \beta \neq 0$$

Proceeding as in case (1), we get

$$\left.\begin{aligned} x &= x(\alpha,\beta) = 9\alpha^2 + 14\alpha\beta - 7\beta^2 \\ y &= y(\alpha,\beta) = 3\alpha^2 - 14\alpha\beta - 21\beta^2 \\ z &= z(\alpha,\beta) = 12\alpha^2 - 28\beta^2 \\ w &= w(\alpha,\beta) = 3\alpha^2 + 6\alpha\beta + 7\beta^2 \end{aligned}\right\} \quad (21)$$

Thus (21) represents the non-zero distinct integer solution of (1)

Properties:

1. $x(\alpha,1) - y(\alpha,1) - t_{14,\alpha} \equiv 14 \pmod{33}$
2. $z(\alpha,1) - 12PR_{\alpha} \equiv 8 \pmod{12}$
3. $x(\alpha,\alpha) + y(\alpha,\alpha) + 16t_{4,\alpha} = 0$

4. Remarkable observations

Triple 1:

Let u_0, v_0, w_0 be the initial solution of (3)

$$\left.\begin{aligned} u_1 &= u_0 \\ v_1 &= v_0 + 3h \\ w_1 &= w_0 + 2h \end{aligned}\right\} \qquad (22)$$

be the second solution of (3), where h is a non-zero integer to be determined.

Then from (3), we get

$$h = 18v_0 - 28w_0$$

$$w_1 = -55w_0 + 36v_0$$

$$\therefore\ v_1 = 55v_0 - 84w_0$$

$$u_1 = u_0$$

Hence the matrix representation of the above solution is

$$\begin{bmatrix} w_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -55 & 36 \\ -84 & 55 \end{bmatrix} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix}$$

where, $\mathrm{A} = \begin{bmatrix} -55 & 36 \\ -84 & 55 \end{bmatrix}$

Repeating the above process, the general values for v and w are given by

$$\begin{pmatrix} w_n \\ v_n \end{pmatrix} = A^n \begin{pmatrix} w_0 \\ v_0 \end{pmatrix}$$

$$\begin{pmatrix} w_n \\ v_n \end{pmatrix} = \begin{bmatrix} \frac{1^n}{2}(-54) + \frac{(-1)^n}{-2}(-56) & \frac{1^n}{2}(36) + \frac{(-1)^n}{-2}(36) \\ \frac{1^n}{2}(-84) + \frac{(-1)^n}{-2}(-84) & \frac{1^n}{2}(56) + \frac{(-1)^n}{-2}(54) \end{bmatrix} \begin{pmatrix} w_0 \\ v_0 \end{pmatrix}$$

Then we get the n^{th} solution as

$$w_n = \left[\frac{1^n}{2}(-54) + \frac{(-1)^n}{-2}(-56)\right] w_0 + \left[\frac{1^n}{2}(36) + \frac{(-1)^n}{-2}(36)\right] v_0$$

$$v_n = \left[\frac{1^n}{2}(-84) + \frac{(-1)^n}{2}(-84)\right] w_0 + \left[\frac{1^n}{2}(56) + \frac{(-1)^n}{-2}(54)\right] v_0$$

$$u_n = u_0$$

In view of (2), the general solution of (1) is

$$x_n = u_n + v_n$$

$$= u_0 + \left[\frac{1^n}{2}(-84) + \frac{1}{2}(-84)\right] w_0 + \left[\frac{1^n}{2}(56) + \frac{(-1)^n}{-2}(54)\right] v_0$$

$$y_n = u_n - v_n$$

$$= u_0 - \left[\frac{1^n}{2}(-84) + \frac{1}{2}(-84)\right] w_0 + \left[\frac{1^n}{2}(56) + \frac{(-1)^n}{-2}(54)\right] v_0$$

$$z_n = 2u_n$$

$= 2u_0$

Triple 2:

Let u_0, v_0, w_0 be the initial solution of (3)

$$\left.\begin{aligned} u_1 &= u_0 + 3h \\ v_1 &= v_0 \\ w_1 &= w_0 + h \end{aligned}\right\} \qquad (24)$$

be the second solution of (3),

Following the procedure as above, the corresponding integer solutions to (1) is given by

$$w_n = \left[\frac{1^n}{2}(9) + \frac{(-1)^n}{-2}(7)\right]w_0 + \left[\frac{1^n}{2}(-3) + \frac{(-1)^n}{-2}(-3)\right]u_0$$

$$x_n = \left[\left[\frac{(1)^n}{2}(21) + \frac{(-1)^n}{-2}(21)\right]w_0 + \left[\frac{(1)^n}{2}(-3) + \frac{(-1)^n}{-2}(-3)\right]u_0\right] + v_0$$

$$y_n = \left[\left[\frac{(1)^n}{2}(21) + \frac{(-1)^n}{-2}(21)\right]w_0 + \left[\frac{(1)^n}{2}(-3) + \frac{(-1)^n}{-2}(-3)\right]u_0\right] - v_0$$

$$z_n = \left[\left[\frac{(1)^n}{2}(21) + \frac{(-1)^n}{-2}(21)\right]w_0 + \left[\frac{(1)^n}{2}(-3) + \frac{(-1)^n}{-2}(-3)\right]u_0\right]$$

Triple 3:

Let u_0, v_0, w_0 be the initial solution of (3)

$$\left.\begin{aligned} w_1 &= 2w_0 \\ v_1 &= 2v_0 + h \\ u_1 &= 2u_0 + h \end{aligned}\right\} \qquad (25)$$

be the second solution of (3)

In this case, the corresponding integer solutions to (1) is given by

$$x_n = u_n + v_n$$

$$= u_0\left[\frac{(2)^n}{4}(3) + \frac{(-2)^n}{-4}(-1) + \frac{(2)^n}{4}(-1) + \frac{(-2)^n}{-4}(-1)\right] + v_0\left[\frac{(2)^n}{4}(-3) + \frac{(-2)^n}{-4}(-3) + \frac{(2)^n}{4}(1) + \frac{(-2)^n}{-4}(-3)\right]$$

$$y_n = u_n - v_n$$

$$= u_0\left[\frac{(2)^n}{4}(3) + \frac{(-2)^n}{-4}(-1) + \frac{(2)^n}{4}(-1) + \frac{(-2)^n}{-4}(-1)\right] - v_0\left[\frac{(2)^n}{4}(-3) + \frac{(-2)^n}{-4}(-3) + \frac{(2)^n}{4}(1) + \frac{(-2)^n}{-4}(-3)\right] z_n$$

$$= 2\left[\frac{(2)^n}{4}(3) + \frac{(-2)^n}{-4}(-1)\right]u_0 + \left[\frac{(2)^n}{4}(-3) + \frac{(-2)^n}{-4}(-3)\right]v_0$$

$$w_n = 2^n w_0$$

5. Special relations

Employing the solutions (x, y) of (1) each of following expressions among the special polygonal, centred polygonal, pronic numbers and pyramidal numbers is a congruent to zero under modulo 7.

$$(i)\left(\frac{3P_x^3}{t_{3,x+1}}\right)^3 + \left(\frac{18P_x^3 - 2}{Ct_{6,x-2} - 1}\right)^3$$

$$(ii)\left(\frac{6P_x^5}{Ct_{6,x^{-1}}}\right)^3 + \left(\frac{3P_y^3}{t_{3,x+1}}\right)^3$$

$$(iii)\left(\frac{P^5 x}{t_{3,x}}\right)^3 + \left(\frac{6P_y^3}{Ct_{6,y^{-1}}}\right)^3$$

$$(iv)\left(\frac{6P_x^3}{P_r x}\right)^3 + \left(\frac{P_y^3}{t_{3,y}}\right)^3$$

$$(v)\left(\frac{2P^5 x}{t_{4,y}}\right)^3 + \left(\frac{P^5 y}{t_{3,y}}\right)^3$$

6. Conclusion

To conclude one may search for other patterns of solutions and their corresponding properties to the considered on the homogeneous cubic equation with four unknowns.

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