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Integer Solution of the Homogeneous Bi-Quadratic Diophantine Equation with Five Unknowns $(x-y)(x^3-y^3)=(z^2-w^2)p^2$

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Abstract. The homogeneous equation with five unknown $(x-y)(x^3-y^3)=(z^2-w^2)p^2$ is analyzed for its nonzero distinct integer solutions. Employing the transformation and applying the method of factorization, different patterns of nonzero distinct integer solutions to the above bi-quadratic equation are obtained. A few interesting relations between the solutions and special number patterns namely polygonal and pyramidal numbers are presented.

Keywords: Homogeneous bi-quadratic, Bi-quadratic equation with five unknown, integer solutions.

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

Bi-quadratic Diophantine equations, homogeneous and non -homogeneous, have aroused the interest of numerous mathematicians since ambiguity as can be seen from [1-2] particularly In [3-5] bi-quadratic diophantine equations with three unknowns are considered In [6-9] bi-quadratic equation with four unknowns are considered In [10-12] bi-quadratic equation with five unknowns are considered. In this paper, another interesting bi-quadratic equation with five unknown given by

$$(x-y)(x^3-y^3)=(z^2-w^2)p^2$$

is considered and five different patterns of integral solutions are illustrated. A few interesting properties between the solutions and special number patterns are exhibited.

2. Notation

- $T_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$ - Polygonal number of rank n with side m.
- $S_n=6n(n-1)+1$ - Star number of rank n

- $PR_n = n(n+1)$ - Pronic number of rank n
- $CP_{n,3} = \frac{n^3 + n}{2}$ - Centered triangular pyramidal number of rank n
- $CP_{n,6} = n^3$ -Centered hexagonal pyramidal number of rank n
- $G_{no_2} = 2n-1$ - Gnomonic number
- $SO_n = n(2n^2 - 1)$ - Stella Octangular number of rank n
- $J_n = \frac{1}{3}(2^n - (-1)^n)$ - Jacobsthal number of rank n.

3. Method of analysis

The Diophantine equation representing the biquadratic equation with five unknowns under consideration is.

$$(x-y)(x^3-y^3) = (z^2-w^2)p^2 \quad (1)$$

The substitution of the linear transformations in (1) gives

$$x = u+v,\ y = u - v,\ z = uv+1,\ w = uv-1 \quad (2)$$

$$u^2 + 3v^2 = p^2 \quad (3)$$

We solve (3) through different methods and thus obtained different patterns of solutions to (1)

3.1. Pattern 1

Equation (3) can be written as

$$\frac{p+u}{3v} = \frac{v}{p-u} = \frac{A}{B} \quad \text{wher e } B \neq 0 \quad (4)$$

Equation (4) is equivalent to the system of double equations.

$$\left.\begin{aligned} uA + vB - pA = 0 \\ uB + vA - pB = 0 \end{aligned}\right\} \quad (5)$$

Solving (5) by applying the method of cross multiplication and using (2) the corresponding non-zero integer solution to (1) are obtained as

$$\left.\begin{aligned} x = x(A, B) &= 3A^2 - B^2 + 2AB \\ y = y(A, B) &= 3A^2 - B^2 - 2AB \\ z = z(A, B) &= 6A^2B - 2AB^2 + 1 \\ w = w(A, B) &= 6A^3B - 2AB^3 + 1 \\ p = p(A, B) &= A^2 + 3B^2 \end{aligned}\right\} \quad (6)$$

Properties:

1. $5x(A,1) + 5y(A,1) + 4p(A,1) = 6^2A^2$ is a perfect square
2. $z(A+1, A) - p(A+1, A) + 2t_{4,A} - 20t_{4,A} + 4 = 0$
3. $p(2^n, 1) = 3J_{2A+4}$

Remark:
In addition to (4), (3) may also be expressed in the form of ratios as presented below

$$\frac{3\mathrm{v}}{p-u}=\frac{\mathrm{p+u}}{\mathrm{v}}=\frac{\mathrm{A}}{\mathrm{B}} \quad \text{where } \mathrm{B}\neq 0$$

Following the procedure as presented above the corresponding non zero integer solutions to (1) are found to be as given below.

$$x = x(\mathrm{A, B}) = 3\mathrm{B}^2 - A^2 - 2AB$$
$$y = y(\mathrm{A, B}) = 3\mathrm{B}^2 - A^2 + 2AB$$
$$z = \mathrm{z}(\mathrm{A, B}) = \mathrm{A}^4 + 9B^4 - 10A^2B^2 + 1$$
$$w = \mathrm{w}(\mathrm{A, B}) = \mathrm{A}^4 + 9B^4 - 10A^2B^2 - 1$$
$$p = \mathrm{p}(\mathrm{A, B}) = \mathrm{A}^2 + 3B^2$$

Properties:

1. $p(A,A) - 4t_{4,A} = 0$
2. $x(\mathrm{A,A}) + y(\mathrm{A,A}) = 4A^2$ is a perfect square
3. $x(1,\mathrm{B}) + p(1,\mathrm{B}) - 2\mathrm{PR}_{\mathrm{A}} - 4 = 0$

3.2. Pattern 2

Let $p = \mathrm{p}(\mathrm{a,b}) = a^2 + 3b^2$ (7)

where a and b are non-zero district integers
Using (3) and (7) and applying the method of factorization, define

$$\left(u + i\sqrt{3}v\right)\left(u - i\sqrt{3}v\right) = \left(a + i\sqrt{3}b\right)^2\left(a - i\sqrt{3}b\right)^2$$

Equating real and imaginary parts we have

$$\left.\begin{aligned} u &= a^2 - 3b^2 \\ v &= 2ab \end{aligned}\right\} \quad (8)$$

using (8) in (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$\left.\begin{aligned} x &= x(\mathrm{a,b}) = a^2 - 3b^2 + 2ab \\ y &= y(\mathrm{a,b}) = a^2 - 3b^2 - 2ab \\ z &= z(\mathrm{a,b}) = 2a^3b - 6ab^3 + 1 \\ w &= w(\mathrm{a,b}) = 2a^3 - 6ab^3 - 1 \end{aligned}\right\} \quad (9)$$

Thus, (7) and (9) represent the non- zero distinct integral solutions to (1)

Properties:

1. $x(A+1,1)+y(A+1,1)+8t_{4,A}-8t_{3,A}-2=0$
2. $y(B,B+1)+p(B,B+1)+4t_{3,A}-2t_{4,A}=0$
3. $y(B,B+1)+p(B,B+1)+4t_{3,A}-2(t_{4,A})^2+6t_{4,A}=0$
4. $w(A,1)+9p(A,1)-2CP_{A,6}-2t_{20,A}-G_{AO}-27=0$

3.3. Pattern 3

(3) can be written as

$$u^2+3v^2=p^2*1 \tag{10}$$

Write 1 as,

$$1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \tag{11}$$

Using (7) and (11) in (10) and applying the method of factorization, define

$$(u+i\sqrt{3})=(a+i\sqrt{3}b)^2\frac{(1+i\sqrt{3})}{2}$$

Equating the real and imaginary part, we have

$$u=u(\mathrm{a},\mathrm{b})=\frac{1}{2}(a^2-3b^2-6ab)$$

$$v=v(\mathrm{a},\mathrm{b})=\frac{1}{2}(a^2-3b^2+2ab)$$

As our interest is to find integer solution so we replace a by 2A and b by 2B we get,

$$\left.\begin{array}{l} u=u(\mathrm{A},\mathrm{B})=2A^2-6B^2-12AB \\ v=v(\mathrm{A},\mathrm{B})=2A^2-6B^2+4AB \end{array}\right\} \tag{12}$$

Using (12) in (2) the corresponding non-zero distinct integral solutions of (1) are given by

$$x=x(A,B)=4A^2-12B^2-8AB$$

$$y=y(A,B)=-16AB$$

$$z=z(A,B)=4A^4+32A^3+24A^2B^2-96AB^3+36B^4+1$$

$$w=w(A,B)=4A^4+32A^3B+24A^2B^2-96AB^3+36B^4-1$$

$$p=p(A,B)=A^2+3B^2$$

Properties:

1. $p(A^2,A)-y(A^2,A)+16\,CP_{A,6}-(t_{4,A})^2-3t_{4,A}=0$
2. $w(1,B)+z(1,B)-8(t_{4,A})^2-32SO_A+320t_{4,A}-208t_{3,A}-72=0$
3. $x(A,A+1)+y(A,A+1)+64t_{3,A}+16\,pr_A-16t_{4,A}+12=0$

3.4. Pattern 4

Write 1 as, $1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{7^2}$

Proceeding as in Pattern: 3 the non-zero distinct integral solutions to (1) are

$x = x(\text{A, B}) = 35A^2 - 105B^2 - 154AB$

$y = y(\text{A, B}) = -21A^2 + 63B^2 - 182AB$

$z = z(\text{A,B}) = 196\text{A}^4 - 3528A^2B^2 - 4606A^3B + 13818AB^3 + 1764B^4 + 1$

$w = w(\text{A,B}) = 196\text{A}^4 - 3528A^2B^2 - 4606A^3B + 13818AB^3 + 1764B^4 - 1$

$p = p(\text{A,B}) = 147B^2 + 49A^2$

Properties:

1. $p(2^n,1) = 147\ J_{2A} + 196$
2. $x(\text{A, A}) + y(\text{A, A}) + p(\text{A, A}) + 168t_{4,A} = 0$
3. $x(\text{A}^2, \text{A}) + p(\text{A}^2, \text{A}) - 84(t_{4,A}) - 42t_{4,A} - 154CP_{A,6}$

3.5. Pattern 5

Write equation (3) as

$$p^2 - 3v^2 = u^2 * 1 \quad (13)$$

Let $u = a^2 - 3b^2$ (14)

Write 1 as,

$$1 = (2+\sqrt{3})(2-\sqrt{3}) \quad (15)$$

Using (14) and (15) in (13) and applying the method of factorization, define

$(p+\sqrt{3}v) = (a+\sqrt{3}b)^2(2+\sqrt{3})$

Equating the positive and negative parts of the above equations, we have

$p = p(A,B) = 2\text{A}^2 + 6\text{B}^2 + 6AB$ (16)

$v = v(A,B) = 4AB + \text{A}^2 + 3\text{B}^2$ (17)

Substituting(13) and (17) in (2) we get

$$\left.\begin{aligned} x &= x(\text{A,B}) = 21A^2 + 4AB \\ y &= y(\text{A,B}) = -3B^2 + 4AB \\ z &= z(\text{A,B}) = A^4 + 4A^3B - 12A^3B - 9B^4 + 1 \\ w &= w(\text{A,B}) = A^4 + 4A^3B - 12A^3B - 9B^4 - 1 \end{aligned}\right\} \quad (18)$$

Thus, (16) and (18) represent the non-zero distinct integral solutions to (1)

Properties:

1. $x(\text{A}^2, \text{A}) + y(\text{A}^2, \text{A}) - 2(t_{4,A})^2 - SO_A - 3OH_A - 4CP_{A,6} + 3t_{4,A} = 0$
2. $z(\text{A},1) + w(\text{A},1) + 16CP_{A,6} - 2(t_{4,A})^2 + 18 = 0$

3. $p(A+1, A)+x(A+1, A)-2t_{4,A}-36t_{3,A}-4=0$

Note:
Write 1 as

$$1=(7+4\sqrt{3})(7-4\sqrt{3})$$

Proceeding as in the Pattern 5, the non-zero distinct integral solutions to (1) are

$$x=x(A,B)=2A^2+4AB$$
$$y=y(A,B)=-3B^2+4AB$$
$$z=z(A,B)=A^4+4A^3B-12A^3B+1$$
$$w=w(A,B)=A^4+4A^3B-12A^3B-1$$
$$p=p(A,B)=49A^2+147B^2$$

Properties:

1. $w(1, B)+p(1, B)-36(t_{4,A})^2-6CP_{A,7}-35CP_{A,6}-t_{44,A}-10=0 \pmod{3}$
2. $p(A, A)+y(A, A)-20t_{4,A}=0$
3. $x(A,1)+y(A,1)-4t_{3,A}+2\Pr_A+6=0$

4. Conclusion

In this paper, we illustrated different methods of obtaining integer solutions to the bi-quadratic equation with five unknowns $(x-y)(x^3-y^3)=(z^2-w^2)p^2$. As bi-quadratic equations are rich in variety, one may consider the other forms of equations and search for their corresponding integer solutions.

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