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Recursive Aggregation of OWA Operators for Fuzzy Number Based on Non-additive Measure with σ-λ Rules

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Abstract. In this paper, considering of the fuzziness and uncertainty of the objective things and the connection of the weight vector, the OWA operators for fuzzy number based on a non-additive measure with $\sigma-\lambda$ rules and its recursive aggregation theory are proposed and investigated by means of the $\sigma-\lambda$ rules of a non-additive measure. In addition, the calculation methods are present and an illustrative example is designed.

Keywords: Fuzzy measure, OWA operator, fuzzy number, recursive aggregation

AMS Mathematics Subject Classification (2010): 28E10, 04A72

1. Introduction

Since Yager [13] proposed Ordered Weighted Averaging operators in 1988, the OWA operators has been widely used in many fields [2,14]. In 2005, Yager proposed the recursive forms of OWA operators [11], and after that, Jin investigated the discrete and continuous recursive forms of OWA operator [7], However, the recursive forms of OWA operators based on the classical probability measure and Lebesgue integral integration operator. We must note that when using the existing aggregation results, an indisputable fact is that the attribute index is mostly related with each other, and not mutually independent. In other words, the weight vector is not able to be measured independently, and it may also not satisfy countable additivity of the classical probability measure. In 1974, Sugeon [9] proposed the concept of fuzzy measure, which can be used to study the correlation problem of attribute index [3, 5, 6, 12, 15]. On the other hand, due to the fuzziness and uncertainty of the objective things, the evaluation values involved in the decision problems are not always expressed as crisp numbers, and some of them are more suitable to be denoted by fuzzy number. So, the fuzzy sets theory introduced by Zadeh was a very good tool to deal with vagueness and uncertainty in real decision problems[15]. The fuzzy number, as a special fuzzy sets, has been applied to many aspects [1, 10]. Based on the above consideration, in this article, the OWA operators for fuzzy number based on a non-additive measure with $\sigma-\lambda$ rules and its recursive aggregation theory are proposed and investigated. In addition, the calculation methods are put forward and an illustrative example is given.

2. Preliminaries

Definition 1. [5, 6, 12, 15] *Let* X *be a nonempty set and* $\mathcal{A}$ *a* $\sigma-$*algebra on the* X .

A set function μ *is called a regular fuzzy measure if*

(1) $\mu(\varnothing)=0$;

(2) $\mu(X)=1$;

(3) for every A and $B\in\mathcal{A}$ such that $A\subseteq B$, $\mu(A)\le\mu(B)$.

A regular fuzzy measure μ is called Sugeno measure if μ satisfies $\sigma-\lambda$ rules, briefly denoted as g_λ. The fuzzy measure shown in this paper is Sugeno measure.

Definition 2. [5, 6, 12, 15] g_λ *is called a fuzzy measure based on* $\sigma-\lambda$ *rules if*

$$g_\lambda\left(\bigcup_{i=1}^{\infty}A_i\right)=\begin{cases}\dfrac{1}{\lambda}\left\{\prod_{i=1}^{\infty}[1+\lambda g_\lambda(A_i)]-1\right\}, & \lambda\ne 0,\\ \sum_{i=1}^{\infty}g_\lambda(A_i), & \lambda=0,\end{cases}$$

where $\lambda\in(-\frac{1}{\sup\mu},\infty)\bigcup\{0\}$, $\{A_i\}\subset\mathcal{A}$, $A_i\cap A_j=\varnothing$ for all $i,j=1,2,\cdots$ and $i\ne j$.

Particularly, if $\lambda=0$, then g_λ is a classic probability measure. If X is a finite set, for any subset A of X, then

$$g_\lambda(A)=\begin{cases}\dfrac{1}{\lambda}\left\{\prod_{x\in A}[1+\lambda g_\lambda(\{x\})]-1\right\}, & \lambda\ne 0,\\ \sum_{x\in A}g_\lambda(\{x\}), & \lambda=0.\end{cases}$$

If X is a finite set, then the parameter λ of a regular Sugeno measure based on $\sigma-\lambda$ rules is determined by the equation

$$1+\lambda=\prod_{i=1}^{n}(1+\lambda g_\lambda(x_i)).$$

Definition 3. [12] *Fuzzy set* $\tilde{A}\in\tilde{E}$ *is called a fuzzy number if* $\tilde{A}$ *is a normal, convex fuzzy set, upper semi-continuous and supp* $A_0=\overline{\{x\in\mathbb{R}:A(x)>0\}}$ *is compact. We use* $\tilde{E}$ *to denote the fuzzy number space .*

[4] Let $(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$, if $\tilde{a}_i\in\tilde{E}$, and its membership function is

$$u_{\tilde{a}_i}(x)=\begin{cases}\dfrac{x-a_i+\delta_{i,1}}{\delta_{i,1}} & a_i-\delta_{i,1}\le x\le a_i,\\ \dfrac{a_i+\delta_{i,2}-x}{\delta_{i,2}} & a_i<x\le a_i+\delta_{i,2},\\ 0 & \text{else},\end{cases}$$

then $\tilde{a}_i = (a_i - \delta_{i,1}, a_i, a_i + \delta_{i,2})$ is said to be triangle fuzzy number.

For any two triangle fuzzy numbers $\tilde{a}_i$ and $\tilde{a}_j$, $i \neq j$, then

$$\tilde{a}_i + \tilde{a}_j = (a_i - \delta_{i,1} + a_j - \delta_{j,1}, a_i + a_j, a_i + \delta_{i,1} + a_j + \delta_{j,1})$$
$$k \cdot \tilde{a}_i = (ka_i - k\delta_{i,1}, ka_i, ka_i + k\delta_{i,2}).$$

[8] Let $\tilde{u}, \tilde{v} \in \tilde{E}$, the partial order $\tilde{u} \preceq \tilde{v}$ means that it satisfy the conditions $[\tilde{u}]_r \leq [\tilde{v}]_r$, $r \in [0,1]$, i.e. $\tilde{u}_r^- \leq \tilde{v}_r^-$, $\tilde{u}_r^+ \leq \tilde{u}_r^+$, $r \in [0,1]$.

Denoting $A_i = \{x_1, x_2, \cdots, x_i\}$, $i = 1,2,\cdots,n$, let $g_\lambda(\{x_i\}) = g_i$, then $g_\lambda(A_1) = g_\lambda(\{x_1\}) = g_1$.

Definition 4. *Let* g_λ *be fuzzy measure satisfying* $\sigma - \lambda$ *rules. Denote* $A_i = \{x_1, x_2, \cdots, x_i\}$, $i = 1,2,\cdots,n$. *An OWA operator of dimension n for fuzzy number based on a non-additive measure with* $\sigma - \lambda$ *rules is a mapping* $\tilde{F}_n : \tilde{E}_1 \times \tilde{E}_2 \times \cdots \times \tilde{E}_n \to \tilde{E}$ *defined as*

$$\tilde{F}_n(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \sum_{i=1}^{n} (g_\lambda(A_i) - g_\lambda(A_{i-1})) \cdot \tilde{b}_i,$$

where $\tilde{b}_i$ is the i-th largest value out of $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ $(i.e., \tilde{b}_1 \geq \tilde{b}_2 \geq \cdots \geq \tilde{b}_n)$.

Corollary 1. *When* $\lambda = 0$ *and* $\tilde{b}_i$ *is a crisp number, then the definition 4 degenerates to the classic OWA operator.*

Definition 5. *The measure of orness (andness) associated with an OWA operator* $\tilde{F}_n$ *of dimension* n *for fuzzy number based on a non-additive measure with* $\sigma - \lambda$ *rules is defined as*

$$orness(g_\lambda^{(n)}) = \frac{1}{n-1} \sum_{i=1}^{n-1} g_\lambda^{(n)}(A_i),$$
$$andness(g_\lambda^{(n)}) = 1 - orness(g_\lambda^{(n)}) = 1 - \frac{1}{n-1} \sum_{i=1}^{n-1} g_\lambda^{(n)}(A_i).$$

3. Recursive forms of the OWA operator for fuzzy number based on a non-additive measure with $\sigma - \lambda$ rules

Theorem 1. *Let* g_λ *be fuzzy measure satisfying* $\delta - \lambda$ *rules. Denote* $A_i = \{x_1, x_2, \cdots, x_i\}$, $i = 1,2,\cdots,n$. $(g_\lambda^{(n)}(A_i) - g_\lambda^{(n)}(A_{i-1}))$ *is the i-th element for the weighting vector of dimension* n. $P_L^{(n)}$ *denotes correlation coefficient. Left Recursive form (LRF) of the OWA operator for fuzzy number based on a non-additive measure with* $\sigma - \lambda$ *rules can* *be written as*

$$\widetilde{F}_n = P_L^{(n)} \cdot (\sum_{i=1}^{n-1}(g_\lambda^{(n-1)}(A_i) - g_\lambda^{(n-1)}(A_{i-1})) \cdot \widetilde{b}_i) + (g_\lambda^{(n)}(A_n) - g_\lambda^{(n)}(A_{n-1})) \cdot \widetilde{b}_n$$

$$= P_L^{(n)} \cdot \widetilde{F}_{n-1} + (1 - g_\lambda^{(n)}(A_{n-1})) \cdot \widetilde{b}_n,$$

where

$$P_L^{(n)} = 1 - (1 - g_\lambda^{(n)}(A_{n-1})) = g_\lambda^{(n)}(A_{n-1}),$$

$$g_\lambda^{(n)}(A_i) - g_\lambda^{(n)}(A_{i-1}) = P_L^{(n)} \cdot (g_\lambda^{(n-1)}(A_i) - g_\lambda^{(n-1)}(A_{i-1})),$$

$$\sum_{i=1}^{n-1}(g_\lambda^{(n-1)}(A_i) - g_\lambda^{(n-1)}(A_{i-1})) = 1.$$

For a fixed level of orness α, we get

$$P_L^{(n)} = \frac{(n-1)\alpha}{1 + \sum_{i=1}^{n-2} g_\lambda^{(n-1)}(A_i)},$$

$$g_\lambda^{(n)}(A_1) = g_\lambda^{(n-1)}(A_1) \cdot P_L^{(n)},$$

$$\ldots$$

$$g_\lambda^{(n)}(A_{n-2}) = g_\lambda^{(n-1)}(A_{n-2}) \cdot P_L^{(n)},$$

$$g_\lambda^{(n)}(A_{n-1}) = P_L^{(n)}.$$

Proof: The simplest aggregation is for two elements, as

$$\widetilde{F}_2 = g_\lambda^{(2)}(A_1) \cdot \widetilde{b}_1 + (1 - g_\lambda^{(2)}(A_1)) \cdot \widetilde{b}_2, \quad orness(g_\lambda^{(2)}) = g_\lambda^{(2)}(A_1) = \alpha.$$

Let us now consider the aggregation $\widetilde{F}_3$. In this case

$$orness(g_\lambda^{(3)}) = \frac{1}{2}\sum_{i=1}^{2} g_\lambda^{(3)}(A_i) = \alpha.$$

This leads to the system of independent equations

$$\begin{cases} g_\lambda^{(3)}(A_1) + g_\lambda^{(3)}(A_2) = 2\alpha, \\ (g_\lambda^{(3)}(A_1) - 0) + (g_\lambda^{(3)}(A_2) - g_\lambda^{(3)}(A_1)) + (g_\lambda^{(3)}(A_3) - g_\lambda^{(3)}(A_2)) = 1, \\ g_\lambda^{(3)}(A_1) = g_\lambda^{(2)}(A_1) \cdot P_L^{(3)}, \\ (g_\lambda^{(3)}(A_2) - g_\lambda^{(3)}(A_1)) = (1 - g_\lambda^{(2)}(A_1)) \cdot P_L^{(3)}. \end{cases}$$

The solution is

$$P_L^{(3)} = \frac{2\alpha}{1 + g_\lambda^{(2)}(A_1)},$$

$$g_\lambda^{(3)}(A_1) = g_\lambda^{(2)}(A_1) \cdot P_L^{(3)},$$

$$g_\lambda^{(3)}(A_2) = P_L^{(3)}.$$

More generally, in the case of n arguments, we can get theorem 1.

Corollary 2. *It is interesting to notice that* $P_L^{(n)}$ *depends on* n *and* α*, as*

$$P_L^{(n)} = P_L(n,\alpha) = \frac{(n-1)\alpha}{(n-2)\alpha+1}.$$

Theorem 2. *Let* g_λ *be fuzzy measure satisfying* $\delta-\lambda$ *rules. Denote* $A_i^{(n-1)} = \{x_2,\cdots, x_i\}$, $i=2,\cdots,n$, $A_1^{(n-1)}=\varnothing$ *in* $\widetilde{F}_{n-1}$. $A_i^{(n)} = \{x_1, x_2,\cdots, , \ x_i\}$, $i=1,2,\cdots,n$, $A_0^{(n)}=\varnothing$. $(g_\lambda^{(n)}(A_i) - g_\lambda^{(n)}(A_{i-1}))$ *is the i-th element for the weighting vector of dimension* n . $P_L^{(n)}$ *denotes correlation coefficient. Right Recursive form (RRF) of the OWA operator for fuzzy number based on a non-additive measure with* $\sigma-\lambda$ *rules can be written as*

$$\begin{aligned}\widetilde{F}_n &= (g_\lambda^{(n)}(A_1^{(n)}) - g_\lambda^{(n)}(A_0^{(n)}))\cdot \widetilde{b}_1 + P_R^{(n)}\cdot(\sum_{i=2}^{n}(g_\lambda^{(n-1)}(A_i^{(n-1)}) - g_\lambda^{(n-1)}(A_{i-1}^{(n-1)}))\cdot\widetilde{b}_i)\\ &= g_\lambda^{(n)}(A_1^{(n)})\cdot\widetilde{b}_1 + P_R^{(n)}\cdot\widetilde{F}_{n-1},\end{aligned}$$

where

$$P_R^{(n)} = 1 - g_\lambda^{(n)}(A_1^{(n)}),$$

$$g_\lambda^{(n)}(A_i^{(n)}) - g_\lambda^{(n)}(A_{i-1}^{(n)}) = P_R^{(n)}\cdot(g_\lambda^{(n-1)}(A_i^{(n-1)}) - g_\lambda^{(n-1)}(A_{i-1}^{(n-1)})),$$

$$\sum_{i=2}^{n}(g_\lambda^{(n-1)}(A_i^{(n-1)}) - g_\lambda^{(n-1)}(A_{i-1}^{(n-1)})) = 1.$$

For a fixed level of andness $\overline{\alpha}$, we get

$$\begin{aligned}&P_R^{(n)} = \frac{(n-1)\overline{\alpha}}{-\sum_{i=2}^{n-1} g_\lambda^{(n-1)}(A_i^{(n-1)}) + (n-1)},\\ &g_\lambda^{(n)}(A_1^{(n)}) = 1 - P_R^{(n)},\\ &g_\lambda^{(n)}(A_2^{(n)}) = (g_\lambda^{(n-1)}(A_2^{(n-1)}) - 1)\cdot P_R^{(n)} + 1,\\ &\cdots\\ &g_\lambda^{(n)}(A_{n-1}^{(n)}) = (g_\lambda^{(n-1)}(A_{n-1}^{(n-1)}) - 1)\cdot P_R^{(n)} + 1.\end{aligned}$$

Proof: The simplest aggregation is between two elements

$$\widetilde{F}_2 = g_\lambda^{(2)}(A_1^{(2)})\cdot\widetilde{b}_1 + (1 - g_\lambda^{(2)}(A_1^{(2)}))\cdot\widetilde{b}_2,\quad andness(g_\lambda^{(2)}) = 1 - g_\lambda^{(2)}(A_1^{(2)}) = \overline{\alpha},$$

Let us now consider the aggregation $\widetilde{F}_3$. In this case

$$andness(g_\lambda^{(3)}) = 1 - \frac{1}{2}\sum_{i=1}^{2} g_\lambda^{(3)}(A_i^{(3)}) = \overline{\alpha}.$$

This leads to the system of independent equations

$$\begin{cases} -g_\lambda^{(3)}(A_1^{(3)}) - g_\lambda^{(3)}(A_2^{(3)}) + 2 = 2\overline{\alpha}, \\ (g_\lambda^{(3)}(A_1^{(3)}) - g_\lambda^{(3)}(A_0^{(3)})) + (g_\lambda^{(3)}(A_2^{(3)}) - g_\lambda^{(3)}(A_1^{(3)})) + (g_\lambda^{(3)}(A_3^{(3)}) - g_\lambda^{(3)}(A_2^{(3)})) = 1, \\ g_\lambda^{(3)}(A_1^{(3)}) = 1 - P_R^{(3)}, \\ (g_\lambda^{(3)}(A_2^{(3)}) - g_\lambda^{(3)}(A_1^{(3)})) = (g_\lambda^{(2)}(A_2^{(2)}) - g_\lambda^{(2)}(A_1^{(2)})) \cdot P_R^{(3)}, \end{cases}$$

the solution is

$$P_R^{(3)} = \frac{2\overline{\alpha}}{-g_\lambda^{(2)}(A_2^{(2)}) + 2},$$
$$g_\lambda^{(3)}(A_1^{(3)}) = 1 - P_R^{(3)},$$
$$g_\lambda^{(3)}(A_2^{(3)}) = (g_\lambda^{(2)}(A_2^{(2)}) - 1) \cdot P_R^{(3)} + 1.$$

More generally, in the case of n arguments, we can get theorem 2.

Corollary 3. *It is interesting to notice that $P_L^{(n)}$ depends on n and α, as*

$$P_R^{(n)} = P_L(n, \alpha) = \frac{(n-1)(1-\alpha)}{\alpha + (n-1)(1-\alpha)}.$$

Theorem 3. *Let g_λ be fuzzy measure satisfying $\delta - \lambda$ rules. Denote $A_i = \{x_1, x_2, \cdots, x_i\}$, $i = 1, 2, \cdots, n$. $(g_\lambda^{(n)}(A_i) - g_\lambda^{(n)}(A_{i-1}))$ is the i-th element for the weighting vector of dimension n. $P_L^{(n)}$ denotes correlation coefficient. A more general form of the OWA operator for fuzzy-number based on a non-additive measure with $\sigma - \lambda$ rules can be written as*

$$\begin{aligned} \widetilde{F}_n &= \Big(\sum_{i=1, i\neq k}^{n-1} (g_\lambda^{(n)}(A_i) - g_\lambda^{(n)}(A_{i-1})) \cdot \widetilde{b}_i\Big) + (g_\lambda^{(n)}(A_k) - g_\lambda^{(n)}(A_{k-1})) \cdot \widetilde{b}_k \\ &= P_k^{(n)} \cdot \Big(\sum_{i=1}^{n-1} (g_\lambda^{(n-1)}(A_i) - g_\lambda^{(n-1)}(A_{i-1})) \cdot \widetilde{b}_i\Big) + (g_\lambda^{(n)}(A_k) - g_\lambda^{(n)}(A_{k-1})) \cdot \widetilde{b}_k \\ &= P_k^{(n)} \cdot \widetilde{F}_{n-1} + (g_\lambda^{(n)}(A_k) - g_\lambda^{(n)}(A_{k-1})) \cdot \widetilde{b}_k, \end{aligned}$$

where

$$(g_\lambda^{(n)}(A_i) - g_\lambda^{(n)}(A_{i-1}))_{i\neq k} = \begin{cases} P_k^{(n)} \cdot (g_\lambda^{(n-1)}(A_i) - g_\lambda^{(n-1)}(A_{i-1})), & \text{i<k,} \\ P_k^{(n)} \cdot (g_\lambda^{(n-1)}(A_{i-1}) - g_\lambda^{(n-1)}(A_{i-2})), & \text{i>k.} \end{cases}$$

$$P_k^{(n)} = 1 - (g_\lambda^{(n)}(A_k) - g_\lambda^{(n)}(A_{k-1})) = \sum_{i=1}^{n-1} (g_\lambda^{(n)}(A_i) - g_\lambda^{(n)}(A_{i-1})).$$

For a fixed level of orness α, we get

$$P_k^{(n)} = \frac{(n-1)\alpha + (k-n)}{(n-2)\alpha + (k-n) + g_\lambda^{(n-1)}(A_{k-1})},$$
$$g_\lambda^{(n)}(A_1) = P_k^{(n)} \cdot g_\lambda^{(n-1)}(A_1),$$
$$\cdots$$

$$g_\lambda^{(n)}(A_{k-1}) = P_k^{(n)} \cdot g_\lambda^{(n-1)}(A_{k-1}),$$
$$g_\lambda^{(n)}(A_k) = P_k^{(n)} \cdot (g_\lambda^{(n-1)}(A_{k-1}) - 1) + 1,$$
$$\ldots$$
$$g_\lambda^{(n)}(A_{n-1}) = P_k^{(n)} \cdot (g_\lambda^{(n-1)}(A_{n-2}) - 1) + 1.$$

Proof: According to the given conditions, we have the linear equation system

$$\begin{cases} \sum_{i=1}^{n-1} g_\lambda^{(n)}(A_i) = (n-1)\alpha, \\ \sum_{i=1}^{n} (g_\lambda^{(n)}(A_i) - g_\lambda^{(n)}(A_{i-1})) = 1, \\ g_\lambda^{(n)}(A_1) - g_\lambda^{(n)}(A_0) = P_k^{(n)} \cdot (g_\lambda^{(n-1)}(A_1) - g_\lambda^{(n-1)}(A_0)), \\ \ldots \\ g_\lambda^{(n)}(A_{k-1}) - g_\lambda^{(n)}(A_{k-2}) = P_k^{(n)} \cdot (g_\lambda^{(n-1)}(A_{k-1}) - g_\lambda^{(n-1)}(A_{k-2})), \\ g_\lambda^{(n)}(A_k) - g_\lambda^{(n)}(A_{k-1}) = 1 - P_k^{(n)} \\ g_\lambda^{(n)}(A_{k+1}) - g_\lambda^{(n)}(A_k) = P_k^{(n)} \cdot (g_\lambda^{(n-1)}(A_k) - g_\lambda^{(n-1)}(A_{k-1})), \\ \ldots \\ g_\lambda^{(n)}(A_n) - g_\lambda^{(n)}(A_{n-1}) = P_k^{(n)} \cdot (g_\lambda^{(n-1)}(A_{n-1}) - g_\lambda^{(n-2)}(A_{n-1}). \end{cases}$$

The solution is theorem 3.

Corollary 4. *For a more general recursive form of the OWA operator for fuzzy number based on a non-additive measure with* $\sigma - \lambda$ *rules. When*

(1) $k = n$, *we get the LRF of the OWA operator for fuzzy number based on a non-additive measure with* $\sigma - \lambda$ *rules.*

(2) $k = 1$, *we get the RRF of the OWA operator for fuzzy number based on a non-additive measure with* $\sigma - \lambda$ *rules.*

4. Illustrative example

A management selects 4 experts to evaluate for a scheme. Information evaluation values as shown in table 1, where expert is x_i, evaluation value is denoted as $\tilde{a}_i = (a_i - \delta_{i,1}, a_i, a_i + \delta_{i,2})$ which is a triangle fuzzy number.

Table 1:

expert	evaluation	g_λ
x_1	(0.20,0.30,0.40)	0.1524
x_2	(0.80,0.90,1.00)	0.1586
x_3	(0.50,0.60,0.70)	0.2428
x_4	(0.10,0.20,0.30)	0.4292

Step 1: According to $1+\lambda = \prod_{i=1}^{n}(1+\lambda g_{\lambda}(\{x_i\}))$, we know $\lambda = 0.05$. Meanwhile

$$g_{\lambda}^{(4)}(A_1) = g_{\lambda}^{(4)}(\{x_1\}) = 0.1524,$$

$$g_{\lambda}^{(4)}(A_2) = \frac{1}{\lambda}[(1+\lambda g_{\lambda}^{(4)}(\{x_1\}))(1+\lambda g_{\lambda}^{(4)}(\{x_2\}))-1] = 0.3122,$$

$$g_{\lambda}^{(4)}(A_3) = \frac{1}{\lambda}[(1+\lambda g_{\lambda}^{(4)}(\{x_1\}))(1+\lambda g_{\lambda}^{(4)}(\{x_2\}))(1+\lambda g_{\lambda}^{(4)}(\{x_3\}))-1] = 0.5588.$$

Step 2: According to definition 4, we can know the comprehensive evaluation $\tilde{F}_4$ of 4 experts.

$$\begin{aligned}\tilde{F}_4 &= (g_{\lambda}^{(4)}(A_1))\cdot(0.80,0.90,1.00)+(g_{\lambda}^{(4)}(A_2)-g_{\lambda}^{(4)}(A_1))\cdot(0.50,0.60,0.70)\\ &\quad+(g_{\lambda}^{(4)}(A_3)-g_{\lambda}^{(4)}(A_2))\cdot(0.20,0.30,0.40)+(1-g_{\lambda}^{(4)}(A_3))\cdot(0.10,0.20,0.30)\\ &= (0.2953,0.3953,0.4953).\end{aligned}$$

Step 3: When a new expert x_5 gives a value for this scheme which is $(0.30,0.40,0.50)$. According to Definition 4 and Theorem 1, we know that

$$\alpha = orness(g_{\lambda}^{(4)}) = \frac{1}{3}\sum_{j=1}^{3} g_{\lambda}^{(4)}(A_j) = 0.3411,$$

$$P_L^{(5)} = \frac{(n-1)\alpha}{(n-2)\alpha+1} = \frac{4\alpha}{3\alpha+1} = 0.6743,$$

$$g_{\lambda}^{(5)}(A_4) = P_L^{(5)} = 0.6743,$$

$$\tilde{F}_5 = P_L^{(5)}\cdot\tilde{F}_4+(1-g_{\lambda}^{(5)}(A_4))\times(0.30,0.40,0.50)) = (0.2968,0.3968,0.4968).$$

Step 4: When a new expert x_6 gives a value for this scheme which is $(0.40,0.50,0.60)$, we can know the comprehensive evaluation $\tilde{F}_6$ of 4 experts.

$$P_L^{(6)} = \frac{5\alpha}{4\alpha+1} = 0.7213$$

$$g_{\lambda}^{(6)}(A_5) = P_L^{(6)} = 0.7213$$

$$\tilde{F}_6 = P_L^{(6)}\cdot\tilde{F}_5+(1-g_{\lambda}^{(6)}(A_5))\times(0.40,0.50,0.60) = (0.3256,0.4256,0.5256).$$

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