Journal of Mathematics and Informatics
Vol. 13, 2018, 29-39
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 10 May 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v13a4

Journal of
Mathematics and Informatics

Some Results on (*j*,*k*) Symmetric Starlike Harmonic Functions

***Renuka Devi K*[1], *Hamid Shamsan*[2] and *S. Latha*[3]**

[1,2,3]Department of Mathematics, Yuvaraja's College, University of Mysore
Mysore 570 005, INDIA
[1]Department of Mathematics, GFGC, Kuvempunagar
Mysore 570023, INDIA,
email: [2]hmas19771@gmail.com; [3]drlatha@gmail.com
[1]Corresponding author. email: renukk84@gmail.com;

Received 5 April 2018; accepted 8 May 2018

Abstract. In the present paper we define and investigate a new subclass of complex valued harmonic starlike functions that are univalent and sense preserving in the open unit disc. Coefficient bounds, extreme points, distortion bounds, convolutin conditions and convex combination are determined for functions in this class. Further, we obtain the closure property of this class under integral operator.

Keywords: (*j*,*m*) Symmetric functions, harmonic Functions, Starlike functions.

AMS Mathematics Subject Classification (2010): 30C45

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a domain $\mathcal{D} \subseteq \mathbb{C}$ is a harmonic in $\mathcal{D}$ if u and v are real harmonic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. In any simply connected domain we can write $f = h + \overline{g}$, where g and h are analytic and $\overline{g}$ denotes the function $z \to \overline{g(z)}$. Clunie and Sheil-Small [2] pointed out that a necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$. Let $\mathcal{H}$ denote the class of complex-valued harmonic functions which are univalent, orientation preserving, and normalized in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Functions in $\mathcal{H}$ can be written in the form $f = h + \overline{g}$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions in $\mathcal{U}$, if the co-analytic part of f is identically zero, that is $g \equiv 0$.

In [2], Clunie and Sheil-Small investigated the class $\mathcal{H}$ as well as its geometric subclasses and their properties. Since then, there have been several studies related to the class $\mathcal{H}$ and its subclasses. Following Clunie and Sheil-Small [2], Frasin[3], Subramanian et al [1], Jahangiri et al. [4, 5, 6, 7], Silverman [10], Silverman and Silvia [11], and others have investigated various subclasses of $\mathcal{H}$ and its properties.

Definition 1.1. *Let k be any positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{D}$ if for every z in $\mathcal{D}$ we have*

$$f\left(e^{\frac{2\pi i}{k}} z\right) = e^{\frac{2\pi i}{k}} f(z), \, z \in \mathcal{D}.$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k = 2$ we get the class of odd univalent functions. The notion of (j,k)-symmetrical functions ($k = 2,3,\ldots,$ and $j = 0,1,2,\ldots,k-1$) is a generalization of the notion of even, odd, k-symmetrical functions and also generalizes the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function. The theory of (j,k)-symmetrical functions has many interesting applications; for instance, in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings, see [9].

Denote the family of all (j,k)-symmetrical functions by $\mathcal{S}^{(j,k)}$. We observe that, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are the classes of even, odd and k-symmetric functions respectively. We have the following decomposition theorem:

Theorem 1. [9] For every mapping $f : \mathcal{U} \mapsto \mathbb{C}$, and a k-fold symmetric set, there exists exactly one sequence of (j,k)-symmetrical functions $f_{j,k}$ such that

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z),$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f\left(\varepsilon^{v} z\right), \, z \in \mathcal{U}. \tag{2}$$

Remark 1.2. *Equivalently, (4) may be written as*

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \delta_{n,j} a_n z^n, \;\; a_1 = 1, \tag{3}$$

where

$$\delta_{n,j} = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-j)v} = \begin{cases} 1, & n = lk + j; \\ 0, & n \neq lk + j;, \end{cases} \tag{4}$$

$$(l \in \mathbb{N}, k = 1,2,\ldots,\ j = 0,1,2,\ldots,k-1).$$

Definition 1.3. $f = h + \overline{g}$ *where* h *and* g *are given by (1). Let* $0 \leq \beta < 1$ *and* $\alpha \geq 0$. *Then* $f \in S_H^{*(j,k)}(\alpha,\beta)$ *if and only if*

$$\frac{\partial}{\partial\theta}\left(argf(re^{i\theta})\right) = Im\left(\frac{\frac{\partial}{\partial\theta}f'(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right) = Re\left(\frac{\alpha z^2h''(z) + zh'(z) + \overline{\alpha z^2g''(z) + (2\alpha-1)zg'(z)}}{h_{j,k} + \overline{g}_{j,k}}\right) \geq \beta. \tag{5}$$

where $z = re^{i\theta}, 0 \leq r < 1, 0 \leq \theta < 2\pi$ and $f_{j,k} = h_{j,k} + \overline{g_{j,k}}$ where $h_{j,k}$, $g_{j,k}$ given by

$$h_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}h(\varepsilon^v z), \qquad g_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}g(\varepsilon^v z). \tag{6}$$

2. Main results

We need the following sufficient condition studied by Jahangiri [7].

Theorem 2.1. *Let* $f = h + \overline{g}$ *with* h *and* g *are given by (1) and let*

$$\sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty}\frac{n+\beta}{1-\beta}|b_n| \leq 1, \tag{7}$$

where $0 \leq \beta < 1$. Then f is harmonic orientation preserving, and univalent in $\mathcal{U}$ and $f \in S_H^*(\beta)$.

The first theorem of this section determines the sufficient coefficient condition for function $f = h + \overline{g}$ belong to the class $S_H^{*(j,k)}(\alpha,\beta)$.

Theorem 2.2. *Let* $f = h + \overline{g}$ *of the form (1) and* $f_{j,k} = h_{j,k} + \overline{g}_{j,k}$ *with* $h_{j,k}$ *and* $\overline{g}_{j,k}$ *given by (6) If*

$$\sum_{n=1}^{\infty}\frac{(\alpha n^2 - \alpha n + n - \beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_n| - \sum_{n=1}^{\infty}\frac{(\alpha n^2 + \alpha n - n - \beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_n| \leq 2. \tag{8}$$

for some β, $(0 \leq \beta < 1)$ and $\alpha \geq 0$ then f is harmonic,orientation preserving,and univalent in $\mathcal{U}$ and $f \in S_H^{*(j,k)}(\alpha,\beta)$.

Proof:. Since $(n-\beta)\le \alpha n(n-1)+n-\beta$ and $(n+\beta)\le \alpha n(n+1)+n+\beta,\ (n\ge 1)$, it follows from Theorem 7 that $f\in S_H^{*(j,k)}(\alpha,\beta)$ and hence f is harmonic, orientation preserving, and univalent in $\mathcal{U}$. Now, we only need to show that if (8) holds then

$$Re\left(\frac{\alpha z^2h''(z)+zh'(z)+\overline{\alpha z^2g''(z)+(2\alpha-1)zg'(z)}}{h_{j,k}+\overline{g}_{j,k}}\right)\ge\beta = Re\left\{\frac{A(z)}{B(z)}\right\}\ge\beta. \tag{9}$$

Using the fact that $Re(w)\ge\beta$ if and only if $|1-\beta+w|\ge|1+\beta-w|$, it suffices to show that

$$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|\ \ge 0, \tag{10}$$

where $A(z)=\alpha z^2h''(z)+zh'(z)+\overline{\alpha z^2g''(z)+(2\alpha-1)zg'(z)}$ and $B(z)=h_{j,k}+\overline{g}_{j,k}$

Substituting $A(z)$ and $B(z)$ in (10), we obtain

$$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|$$

$$=\left|\alpha z^2h''(z)+zh'(z)+\overline{\alpha z^2g''(z)+(2\alpha-1)zg'(z)}+(1-\beta)h_{j,k}(z)+\overline{g}_{j,k}(z)\right|$$

$$-\left|\alpha z^2h''(z)+zh'(z)+\overline{\alpha z^2g''(z)+(2\alpha-1)zg'(z)}-(1+\beta)h_{j,k}(z)+\overline{g}_{j,k}(z)\right|$$

$$=\left|(1+(1-\beta)\delta_{1,j})z+\sum_{n=2}^{\infty}[(\alpha n(n-1)+n+(1-\beta)\delta_{n,j}]a_nz^n+\sum_{n=1}^{\infty}[(\alpha n(n+1)-n+(1+\beta)\delta_{n,j}]\overline{b_nz^n}\right|$$

$$-\left|(1-(1+\beta)\delta_{1,j})z+\sum_{n=2}^{\infty}[(\alpha n(n-1)+n-(1+\beta)\delta_{n,j}]a_nz^n+\sum_{n=1}^{\infty}[(\alpha n(n+1)-n-(1+\beta)\delta_{n,j}]\overline{b_nz^n}\right|$$

$$\ge 2(1-\beta)\delta_{1,j}\,|z|-\sum_{n=2}^{\infty}[\alpha n^2+\alpha n-n+(1-\beta)\delta_{n,j}+\alpha n^2-\alpha n+n-(1+\beta)\delta_{n,j}]\,|a_n||z^n|$$

$$-\sum_{n=1}^{\infty}[\alpha n^2+\alpha n-n+(1-\beta)\delta_{n,j}+\alpha n^2+\alpha n-n-(1+\beta)\delta_{n,j}]\,|b_n||z^n|$$

$$\ge 2(1-\beta)\delta_{1,j}\,|z|-2\sum_{n=2}^{\infty}(\alpha n^2-\alpha n+n+-\beta\delta_{n,j})\,|a_n||z^n|-2\sum_{n=1}^{\infty}(\alpha n^2+\alpha n-n-\beta\delta_{n,j})\,|b_n||z^n|$$

$$\ge 2(1-\beta)\delta_{1,j}\,|z|\left[1-\sum_{n=2}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_n|-\sum_{n=1}^{\infty}\frac{(\alpha n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_n|\right]$$

$$\ge 0 \qquad by (8).$$

The harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{(1-\beta)\delta_{1,j}}{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}|a_n|+\sum_{n=1}^{\infty}\frac{(1-\beta)\delta_{1,j}}{(\alpha n^2+\alpha n-n-\beta\delta_{n,j})}|b_n| \tag{11}$$

where

$$\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1.$$

The functions of the form (8) are in $S_H^{*(j,k)}(\alpha,\beta)$ because

$$\sum_{n=1}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_n|+\sum_{n=1}^{\infty}\frac{(\alpha n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}=1+\sum_{n=1}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2. \quad (12)$$

If $j=k=1$ will get the following result introduced by [8].

Corollary 2.3.

$$\sum_{n=1}^{\infty}\frac{(\alpha n(n-1)+n-\beta)}{(1-\beta)}|a_n|+\sum_{n=1}^{\infty}\frac{(\alpha n(n+1)+n+\beta)}{(1-\beta)}|b_n|\le 2. \quad (13)$$

for some β, $(0\le\beta<1)$ and $\alpha\ge 0$ then f is harmonic, orientation preserving and univalent in $\mathcal{U}$ and $f\in S_H^*(\alpha,\beta)$.

We denote by $\overline{S}_H^{*(j,k)}(\alpha,\beta)$ the class of function $f\in S_H^{*(j,k)}(\alpha,\beta)$ whose coefficients satisfy the condition (7).

Theorem 2.4. *Let* $0\le\alpha_1<\alpha_2$ *and* $0\le\beta_1<1$. *Then* $\overline{S}_H^{*(j,k)}(\alpha_2,\beta)\subset\overline{S}_H^{*(j,k)}(\alpha_1,\beta)$

Proof: For $f\in\overline{S}_H^{*(j,k)}(\alpha_2,\beta)$, it follows from (7) that

$$\sum_{n=2}^{\infty}\frac{(\alpha_1 n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_n|+\sum_{n=1}^{\infty}\frac{(\alpha_1 n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_n|$$

$$\sum_{n=1}^{\infty}\frac{(\alpha_2 n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_n|$$

$$+\sum_{n=1}^{\infty}\frac{(\alpha_2 n^2+\alpha n-n-\beta S_H^{*(j,k)}(\alpha,\beta)\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_n|\le 2. \quad (14)$$

Hence $\overline{S}_H^{*(j,k)}(\alpha_1,\beta)$.

Theorem 2.5. *Let* $f=h+\overline{g}$ *of the form (1) and* $f_{j,k}=h_{j,k}+\overline{g}_{j,k}$ *with* $h_{j,k}$ *and* $\overline{g}_{j,k}$ *given by (6) and* $f\in\overline{S}_H^{*(j,k)}(\alpha,\beta)$. *Then for* $|z|=r<1$, *we have*

$$|f(z)|\le(1+|b_1|)r+\left\{\frac{(1-\beta)\delta_{1,j}}{2\alpha-1-\beta\delta_{1,j}}-\frac{2\alpha-1-\beta\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}|b_1|\right\}r^2, \quad (15)$$

$$|f(z)|\ge(1-|b_1|)r-\left\{\frac{(1-\beta)\delta_{1,j}}{2\alpha-1-\beta\delta_{1,j}}-\frac{2\alpha-1-\beta\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}|b_1|\right\}r^2, \quad (16)$$

where

$$|b_1|\le\frac{(1-\beta)\delta_{1,j}}{2\alpha+1+\beta\delta_{1,j}} \quad (17)$$

Proof: Let $f \in \overline{S}_H^{*(j,k)}(\alpha,\beta)$. Then we have

$$\left|f(z)\right| \le (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$

$$\le (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2$$

$$\le (1+|b_1|)r + \frac{(1-\beta)\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}\left\{\sum_{n=2}^{\infty}\frac{2\alpha+2-\beta\delta_{2,j}}{(1-\beta)\delta_{1,j}}|a_n| + \sum_{n=2}^{\infty}\frac{2\alpha+2-\beta\delta_{2,j}}{(1-\beta)\delta_{1,j}}|b_n|\right\}r^2$$

$$\le (1+|b_1|)r + \frac{(1-\beta)\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}\left\{\sum_{n=2}^{\infty}\frac{\alpha n[(n-1)+n-\beta\delta_{n,j}]}{(1-\beta)\delta_{1,j}}|a_n| + \sum_{n=2}^{\infty}\frac{\alpha n[(n-1)-n-\beta\delta_{n,j}]}{(1-\beta)\delta_{1,j}}|b_n|\right\}r^2$$

$$\le (1+|b_1|)r + \frac{1-\beta\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}\left\{1-\frac{2\alpha-1-\beta\delta_{1,j}}{(1-\beta)\delta_{1,j}}|b_1|\right\}r^2$$

$$\le (1+|b_1|)r + \{\frac{(1-\beta)\delta_{1,j}}{2\alpha-1-\beta\delta_{1,j}} - \frac{2\alpha-1-\beta\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}|b_1|]r^2$$

and

$$\left|f(z)\right| \ge (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$

$$\ge (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2$$

$$\ge (1-|b_1|)r - \frac{(1-\beta)\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}\left\{\sum_{n=2}^{\infty}\frac{2\alpha+2-\beta\delta_{2,j}}{(1-\beta)\delta_{1,j}}|a_n| + \sum_{n=2}^{\infty}\frac{2\alpha+2-\beta\delta_{2,j}}{(1-\beta)\delta_{1,j}}|b_n|\right\}r^2$$

$$\ge (1-|b_1|)r - \frac{(1-\beta)\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}\left\{\sum_{n=2}^{\infty}\frac{\alpha n[(n-1)+n-\beta\delta_{n,j}]}{(1-\beta)\delta_{1,j}}|a_n| + \sum_{n=2}^{\infty}\frac{\alpha n[(n-1)-n-\beta\delta_{n,j}]}{(1-\beta)\delta_{1,j}}|b_n|\right\}r^2$$

$$\ge (1-|b_1|)r - \frac{1-\beta\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}\left\{1-\frac{2\alpha-1-\beta\delta_{1,j}}{(1-\beta)\delta_{1,j}}|b_1|\right\}r^2$$

$$\ge (1-|b_1|)r - \{\frac{(1-\beta)\delta_{1,j}}{2\alpha-1-\beta\delta_{1,j}} - \frac{2\alpha-1-\beta\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}|b_1|]r^2$$

The upper bound given for $f \in \overline{S}_H^{*(j,k)}(\alpha,\beta)$ is sharp and the equality occurs for the function

$$f(z) = z + 1 + |b_1|\,\bar{z} + \left\{\frac{(1-\beta)\delta_{1,j}}{2\alpha-1-\beta\delta_{1,j}} - \frac{2\alpha-1-\beta\delta_{1,j}}{2\alpha+2-\beta\delta_{2,j}}|b_1|\right\}\bar{z}^2 \qquad (z=r), \qquad (18)$$

where $|b_1| \le \frac{(1-\beta)}{2\alpha+2-\beta\delta_{2,j}}$. This completes the proof of theorem.

if $j = k = 1$ will get the following results proved by [8].

Corollary 2.6. *Let* $f = h + \overline{g}$ *of the form (1) and* $f \in \overline{S}^*_H(\alpha,\beta)$. *Then for* $|z| = r < 1$, *we have*

$$|f(z)| \le (1+|b_1|)r + \left\{\frac{(1-\beta)}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right\}r^2, \tag{19}$$

$$|f(z)| \ge (1-|b_1|)r - \left\{\frac{(1-\beta)}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right\}r^2, \tag{20}$$

where

$$|b_1| \le \frac{(1-\beta)}{2\alpha+1+\beta}. \tag{21}$$

Using definition (5), and according to the arguments given in [4] we obtain the following the extreme points of the closed convex hulls of $\overline{S}_H^{*(j,k)}(\alpha,\beta)$ denoted by $clco\overline{S}_H^{*(j,k)}(\alpha,\beta)$.

Theorem 2.7. *Let* $f = h + \overline{g}$ *where* g *and* h *are given by (1). Then* $f \in clco\overline{S}_H^{*(j,k)}(\alpha,\beta)$ *if and only if*

$$f(z) = \sum_{n=1}^{\infty}(\tau_n h_n + \lambda_n g_n), \tag{22}$$

where $h_1(z) = z,\ \ h_n(z) = z + \frac{(1-\beta)\delta_{1,j}}{(\alpha_n^2 - \alpha n + n - \beta\delta_{n,j})}z^n, (n = 2,3,4,...,)$ and

$g_n(z) = z + \frac{(1-\beta)\delta_{1,j}}{(\alpha_n^2 + \alpha n - n - \beta\delta_{n,j})}\overline{z}^n, (n = 1,2,3,...,),\ \ \sum_{n=1}^{\infty}(\tau_n + \lambda_n) = 1, \tau_n \ge 0$ and $\lambda_n \ge 0$, In particular, the extreme points of $\overline{S}_H^{*(j,k)}(\alpha,\beta)$ are $\{h_n\} and \{g_n\}$, and $\delta_{n,j}$ given by (3).

Proof. For a function f of the form (22) we have

$$f(z) = \sum_{n=1}^{\infty}(\tau_n h_n + \lambda_n g_n)$$

$$= \sum_{n=1}^{\infty}(\tau_n + \lambda_n)z + \sum_{n=2}^{\infty}\frac{(1-\beta)\delta_{1,j}}{(\alpha_n^2 - \alpha n + n - \beta\delta_{n,j})}\tau_n z^n + \sum_{n=1}^{\infty}\frac{(1-\beta)\delta_{1,j}}{(\alpha_n^2 + \alpha n - n - \beta\delta_{n,j})}\lambda_n \overline{z}^n$$

$$= z+\sum_{n=2}^{\infty}\frac{(1-\beta)\delta_{1,j}}{(\alpha_n^2-\alpha n+n-\beta\delta_{n,j})}\tau_n z^n+\sum_{n=1}^{\infty}\frac{(1-\beta)\delta_{1,j}}{(\alpha_n^2+\alpha n-n-\beta\delta_{n,j})}\lambda_n\bar{z}^n$$

$$=\sum_{n=2}^{\infty}\frac{(\alpha_n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}\left(\frac{(1-\beta)\delta_{1,j}}{(\alpha_n^2-\alpha n+n-\beta\delta_{n,j})}\tau_n\right)$$

$$+\sum_{n=1}^{\infty}\frac{(\alpha_n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}\left(\frac{(1-\beta)\delta_{1,j}}{(\alpha_n^2+\alpha n-n-\beta\delta_{n,j})}\lambda_n\right)$$

$$=\sum_{n=2}^{\infty}\tau_n+\sum_{n=1}^{\infty}\lambda_n=1-\tau_1\le 1.$$

Thus $f\in clco\overline{S}_H^{*(j,k)}(\alpha,\beta)$.

Conversely,

Suppose that $f\in clco\overline{S}_H^{*(j,k)}(\alpha,\beta)$

Set $\tau_n=\dfrac{(\alpha_n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_n|\quad(n=2,3,4,...,)$ and

$\lambda_n=\dfrac{(\alpha_n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_n|\quad(n=1,2,3,...,)$ Then by the inequality (7), we have $0\le\tau_n\le 1\quad(n=2,3,4,...,)$ and $0\le\lambda_n\le 1\quad(n=1,2,3,...,)$.

Define $\tau_n=1-\sum_{n=2}^{\infty}\tau_n-\sum_{n=1}^{\infty}\lambda_n$ and note that $\tau_1\ge 0$.

Thus we obtain $f(z)=\sum_{n=1}^{\infty}(\tau_n h_n+\lambda_n g_n)$. This completes the proof.

3. Convolution and convex combinations

Definition 3.1. *For two harmonic functions*

$$f(z)=f_{j,k}(z)=h_{j,k}(z)+\overline{g}_{j,k}(z)=z+\sum_{n=2}^{\infty}\delta_{n,j}a_n z^n+\sum_{n=1}^{\infty}\delta_{n,j}\overline{b_n}\bar{z}^n,$$

$$F(z)=z+\sum_{n=2}^{\infty}A_n z^n+\sum_{n=1}^{\infty}\overline{B_n}\bar{z}^n,$$

we define their convolution

$$(f*F)(z)=z+\sum_{n=2}^{\infty}\delta_{n,j}A_n a_n z^n+\sum_{n=1}^{\infty}\delta_{n,j}\overline{B_n b_n}\bar{z}^n. \tag{23}$$

Using this definition, we show that the class $\overline{S}_H^{*(j,k)}(\alpha,\beta)$ is closed under convolution.

Theorem 3.2. *For* $0\le\beta<1$ *and* $\alpha\ge 0$, *let* $f,F\in\overline{S}_H^{*(j,k)}(\alpha,\beta)$. *Then* $(f*F)\in\overline{S}_H^{*(j,k)}(\alpha,\beta)$.

Proof: We observe that $A_n \le 1$ and $b_n \le 1$, For the convolution $(f * F)$, we have

$$\sum_{n=2}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|A_n a_n|+\sum_{n=1}^{\infty}\frac{(\alpha n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|B_n b_n| \tag{24}$$

$$\le\sum_{n=2}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_n|+\sum_{n=1}^{\infty}\frac{(\alpha n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_n|\le 1$$

Therefore $(f * F)\in \overline{S}_H^{*(j,k)}(\alpha,\beta)$.

Theorem 3.3. *The class* $\overline{S}_H^{*(j,k)}(\alpha,\beta)$ *is closed under convex combination.*

Proof: For $i=1,2,3,...,$ let $f_i\in \overline{S}_H^{*(j,k)}(\alpha,\beta)$ where $f_i(z)$ is given by

$$f_i(z) = z+\sum_{n=2}^{\infty}\delta_{n,j}a_{ni}z^n+\sum_{n=1}^{\infty}\delta_{n,j}\overline{b}_{ni}\overline{z}^n. \tag{25}$$

then by (7) we have

$$\sum_{n=2}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_{ni}| \\ -\sum_{n=1}^{\infty}\frac{(\alpha n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_{ni}|\le 1. \tag{26}$$

for $\sum_{i=1}^{\infty}t_i, 0\le t_i\le 1,$ the convex combination of t_i may be written as

$$\sum_{i=1}^{\infty}t_i f_i = z+\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i\delta_{n,j}a_{ni}\right)z^n+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i\delta_{n,j}\overline{b}_{ni}\right)\overline{z}^n. \tag{27}$$

Again by (7) we have

$$\sum_{n=2}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}\left|\sum_{i=1}^{\infty}t_i a_{ni}\right|+\sum_{n=1}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}\left|\sum_{i=1}^{\infty}t_i\overline{b}_{ni}\right|$$

$$\le\sum_{i=1}^{\infty}t_i\left(\sum_{n=2}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_{ni}|+\sum_{n=1}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_{ni}|\right) \tag{28}$$

$$\le\sum_{i=1}^{\infty}t_i=1.$$

Therefore $\sum_{i=1}^{\infty}t_i f_i\in \overline{S}_H^{*(j,k)}(\alpha,\beta)$.

4. Class preserving integral operator

In this section , we consider the closure property of the class $\overline{S}_H^{*(j,k)}(\alpha,\beta)$ under the

Bernardi integral operator $F(z)$, which is defined by

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{(c-1)}h(t)dt+\overline{\frac{c+1}{z^c}\int_0^z t^{(c-1)}g(t)dt} \quad (c>-1). \tag{29}$$

Theorem 4.1. *Let* $f=h+\overline{g}$ *be in the class* $\overline{S}_H^{*(j,k)}(\alpha,\beta)$*, where* h *and* g *are given by (1). Then* $F(z)$ *defined by (29) also belongs to the class* $\overline{S}_H^{*(j,k)}(\alpha,\beta)$.

Proof: From the representation of F, we have

$$F(z)=z+\sum_{n=2}^{\infty}\frac{c+1}{c+n}\delta_{n,j}a_n z^n+\sum_{n=1}^{\infty}\frac{c+1}{c+n}\delta_{n,j}\overline{b}_n\overline{z}^n \tag{30}$$

Now

$$\sum_{n=2}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}\left(\frac{c+1}{c+n}|a_n|\right)+\sum_{n=1}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}\left(\frac{c+1}{c+n}|b_n|\right) \tag{31}$$

$$\leq\sum_{n=2}^{\infty}\frac{(\alpha n^2-\alpha n+n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|a_n|-\sum_{n=1}^{\infty}\frac{(\alpha n^2+\alpha n-n-\beta\delta_{n,j})}{(1-\beta)\delta_{1,j}}|b_n|\leq 1 \qquad by(7).$$

Thus $F(z)\in\overline{S}_H^{*(j,k)}(\alpha,\beta)$.

Acknowledgement. The authors are grateful to the reviewer and Editor-in-chief for providing insightful comments and highly appreciate the through review.

REFERENCES

1. R.M.Ali, B.A.Stephen and K.G.Subramanian, Subclasses of harmonic mappings defined by convolution, *Appl. Math. Letters*, 23(10) (2010) 1243-1247.
2. J.Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Aci. Fenn. Ser. AI Math.*, 9 (1984) 3-25.
3. B.A.Frasin, Comprehensive family of harmonic univalent functions, *SUT Journal of Mathematics*, 42(1) (2006) 145-155.
4. J.M.Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, *Ann. Univ. Mariae Curie-Sk, odowska Sect.*, 52(2) (1998) 57-66.
5. J.M.Jahangiri, Harmonic functions starlike in the unit disk emph, *J. Math. Anal. Appl.*, 235(2) (1999) 470-477.
6. J.M.Jahangiri, G.Murugusundaramoorthy and K.Vijaya, Starlikeness of harmonic functions defined by Ruscheweyh derivatives, *Indian Acad. Math.*, 26(1) (2004) 191-200.
7. Y.C.Kim, J.M.Jahangiri and J.H.Choi, Certain convex harmonic functions, *Int. J. Math. and Mat. Sc.*, 29(8) (2002) 59-65.

8. A.Y.Lashin, On Certain Subclass of Harmonic Starlike Functions, *Abstract and Applied Analysis*, 2014 (2014).
9. P.Liczberski and J.Polubinski, On (j,k)-symmtrical functions, *Math. Bohemica*., 120 (1995) 13-25.
10. H.Silverman, Harmonic univalent functions with negative coefficients, *J. Math. Anal. Appl.*, 220(1) (1998) 283-289.
11. H.Silverman and E.M.Silvia, Subclasses of harmonic univalent functions, *New Zealand J. Math.*, 28(2) (1999) 275-284.