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A Problem on Eigenvalues of Differential Operator of the Third Order with Non-Local Boundary Value Conditions

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Abstract. We study distribution of eigenvalues of differential operators of the third order on a segment with nonlocal integral boundary value conditions, connected with the zeros of entire functions having integral representations.

Keywords: operator, eigenvalues, boundary value conditions, entire functions, quasi-polynomial, zeros.

AMS Mathematics Subject Classification (2010): 47A05, 47E05

1. Introduction

Necessity of study asymptotic behavior and distribution of zeros of entire functions arises in the study of spectral problems of differential operators. A wide class of entire functions was studied by Levin [10], different classes of entire functions were considered in [11], [2]. In particular, functions of class **K** have been introduced by Lidskii and Sadovnichii [11], where for the wide class of boundary value problems, generated by ordinary differential expressions on a finite interval with complex entry of the spectral parameter, it is reduced to the study of regularized sums of roots of entire functions with certain asymptotic structure. It turned out that the functions of the functions of the class **K** correspond to boundary conditions have a nonlocal character, then the corresponding entire functions may not belong to the class **K**.Study of distribution and asymptotic behavior of the zeros of entire functions, which do not belong to the class of **K**, was considered in the oldest papers [4,17,26]. In [10, Chapter 6] the similar features have been studied by Levin. On studying these issues in recent years monographs of Kanguzhin and Sadybekov [7], Mitrokhin [14] appeared. As the more recent works we mention papers of Siedleckii [22,23,24], where for the operator of differentiation as the system of root vectors there arises a system of exponents, which was studied in detail. Moreover, in this area we should mention papers of Marchenko [15], Malamud [12], Anderson, Ma [1], Bondarenko [3].

To the study of zeros of entire functions, which have an integral representation, papers [4,17,26] and [23] have been dedicated.

The zeros of entire functions in the form of quasi-polynomials where investigated in [2,9,13]. Connection of zeros of quasi-polynomials with spectral problems was reflected in [16, 25].

In [18], [19, p. 177] zeros of entire functions

$$\Delta(\lambda)=\sum_{k=1}^{n}P_k\lambda^{m_k}e^{\beta_k\lambda},$$

where $\beta_k\,(\mathrm{k}=\overline{1,n})$ are commensurate complex numbers.

In the case when $m_k=0, k=\overline{1,n}$, zeros of entire functions have been considered in [7,9]. Moreover, this case, when $\beta_k\,(k=\overline{1,n})$ are not commensurate complex numbers, was studied in [2,5], where first thousand zeros of an entire function were calculated. In [20] the technique of construction of the characteristic determinant of the spectral problem with an integral perturbation, which is an entire analytic function of the variable λ, was suggested.

2. Statement of the problem and main results

We consider the differential operator, given by the following differential expression

$$L_0(u)=u'''(x)+P_1(x)u'(x)+P_0(x)u(x)=\lambda u(x),\quad 0<x<1, \tag{1}$$

with non-local integral boundary value conditions:

$$u(0)=\lambda\int_0^1(x)\sigma_1(x)dx, \tag{2}$$

$$u'(0)=\lambda\int_0^1(x)\sigma_2(x)dx, \tag{3}$$

$$u''(0)=\lambda\int_0^1(x)\sigma_3(x)dx, \tag{4}$$

where $\sigma_n\in L_2(0,1), n=\overline{1,3},\ \ P_1(x),\ P_0(x)\in C^1(0,1)$ among them multipoint problems, everywhere solvable in $L_2(0,1)$ are contained.

In [6] the spectral problem (1)-(4) was reduced to the study of distribution of zeros of entire functions, which has an integral representation of the following form:

$$\Delta_0(\lambda)=1+A\lambda+\lambda^2+\int_0^1\int_0^1\sum_{m=0}^{2}e^{\sqrt[3]{\lambda}(\nu_m x+\nu_{m+1}y)}P(x,y)dxdy, \tag{5}$$

where $A-const,\ \ \nu_m=e^{\frac{2\pi i}{3}m},\ \ i=\sqrt{-1},\ \ P(x,y)$ is an absolute integrable function, which is a characteristic determinant of the problem (1) – (4), where they proved the case of improvement of asymptotic behavior of zeros of an entire function $\Delta_0(\lambda)$, when in a square [0,1] × [0,1] the integrand function $P(x,y)$ has the property of smoothness up to the third order by the variables x,y.

Assume that $\sqrt[3]{\lambda}$ changes along the line $\sqrt[3]{\lambda}=\left|\sqrt[3]{\lambda}\right|e^{i\psi}$, where $\psi-const.$ Choose the point (x_0,y_0) and index m_0 from the conditions:

$$\max_{(x,y)\in[0,1]\times[0,1]}\ \max_{0\le m\le 2} Re\sqrt[3]{\lambda}\left(v_m x+v_{m+1}y\right)=Re\sqrt[3]{\lambda}\left(v_{m_0}x_0+v_{m_0+1}y_0\right). \tag{6}$$

We consider the case, when the maximum point (x_0,y_0) in unique. Suppose, that in a neighbourhood of the square [0,1]×[0,1] the integrand function $P(x,y)$ from (5) with bounded variation, and $P(x,y)\neq 0$.

Lower estimation of the function $\Delta_0(\lambda)$ is carried out as follows:

$$\begin{gathered}\left|\Delta_0(\lambda)\right|\ge|\lambda|^2\int_{x_0-\delta}^{x_0}dx\int_{y_0-\delta}^{y_0}e^{Re\sqrt[3]{\lambda}(v_{m_0}x+v_{m_0+1}y)}\left|P(x_0,y_0)\right|dy-\\ -|\lambda|^2\int_{x_0-\delta}^{x_0}dx\int_{y_0-\delta}^{y_0}e^{Re\sqrt[3]{\lambda}(v_{m_0}x+v_{m_0+1}y)}\left|P(x,y)-P(x_0,y_0)\right|dy-\\ -|\lambda|^2\iint_M e^{Re\sqrt[3]{\lambda}(v_{m_0}x+v_{m_0+1}y)}\left|P(x,y)\right|dy-\\ -|\lambda|^2\int_0^1dx\int_0^1e^{Re\sqrt[3]{\lambda}(v_m x+v_{m+1}y)}\left|P(x,y)\right|dxdy-1-|A||\lambda|,\end{gathered} \tag{7}$$

where M is the set $[0,1]\times[0,1]-[x_0-\delta,x_0]\times[y_0-\delta,y_0]$.

According to the Rouche theorem [21], the first term can be estimated from below, at the same time, the remaining terms will be estimated from above.
Calculation of the first term:

$$\begin{gathered}I_1=|\lambda|^2\frac{\left|P(x_0,y_0)\right|}{Re\sqrt[3]{\lambda}v_{m_0},\ Re\sqrt[3]{\lambda}v_{m_0+1}}\left(e^{Re\sqrt[3]{\lambda}v_{m_0}x_0}-e^{Re\sqrt[3]{\lambda}v_{m_0}(x_0-\delta)}\right)\times\\ \times\left(e^{Re\sqrt[3]{\lambda}v_{m_0+1}y_0}-e^{Re\sqrt[3]{\lambda}v_{m_0+1}(y_0-\delta)}\right)\ge|\lambda|^2\frac{\left|P(x_0,y_0)\right|}{\left|Re\sqrt[3]{\lambda}\right|^2}e^{Re\sqrt[3]{\lambda}(v_{m_0}x_0+v_{m_0+1}y_0)}\times\\ \times\left\{\left(1-e^{-Re\sqrt[3]{\lambda}v_{m_0}\delta}\right)\left(1-e^{-Re\sqrt[3]{\lambda}v_{m_0+1}\delta}\right)\right\}.\end{gathered}$$

Choosing $\left|\sqrt[3]{\lambda}\right|$ large enough, we can satisfy the following inequality:

$$\left(1-e^{-Re\sqrt[3]{\lambda}v_{m_0}\delta}\right)\left(1-e^{-Re\sqrt[3]{\lambda}v_{m_0+1}\delta}\right)\ge\frac{1}{2},$$

since (x_0,y_0) is unique. Therefore, we get

$$I_1 \geq |\lambda|^2 \frac{|P(x_0, y_0)|}{\left|Re\sqrt[3]{\lambda}\right|^2} e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x_0 + \nu_{m_0+1} y_0)}. \tag{8}$$

Now we estimate the remaining terms of (7) above. Note that at least one of the three numbers $Re\sqrt[3]{\lambda}(\nu_m x + \nu_{m+1} y),\ m = 0,1,2,$ is positive, since the angles between adjacent complex numbers $\sqrt[3]{\lambda}(\nu_m x + \nu_{m+1} y),\ m = 0,1,2$ constitute at 120^0. Consequently, we have

$$1 + |A||\lambda| \leq e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x_o + \nu_{m_0+1} y_0)}, \tag{9}$$

where $\left|\sqrt[3]{\lambda}\right|$ is large enough. On the other hand, taking the following inequality into account:

$$e^{Re\sqrt[3]{\lambda}(\nu_m x + \nu_{m+1} y)} < e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x_o + \nu_{m_0+1} y_0)},$$

we get the estimate of the integrals with exponents:

$$\int_0^1 dx \int_0^1 \sum_{\substack{m=0\\ m\neq m_0}}^{2} e^{Re\sqrt[3]{\lambda}(\nu_m x + \nu_{m+1} y)} |P(x,y)| dxdy \leq \frac{|P(x_0, y_0)|}{C\cdot\left|Re\sqrt[3]{\lambda}\right|^2} 2e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x_0 + \nu_{m_0+1} y_0)},\ C \gg 1, \tag{10}$$

when $\left|\sqrt[3]{\lambda}\right| \gg 1.$ Here we assume, that $|P(x_0, y_0)| > 0.$

Consider the evaluation of the integral separately:

$$\int_{x_0-\delta}^{x_0} dx \int_{y_0-\delta}^{y_0} e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x + \nu_{m_0+1} y)} |P(x,y) - P(x_0, y_0)| dy \leq$$

$$\leq \sup_{(x-x_0)^2+(y-y_0)^2 \leq 2\delta^2} |P(x,y) - P(x_0, y_0)||P(x,y) - P(x_0, y_0)| \frac{4}{\left|Re\sqrt[3]{\lambda}\right|^2} e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x + \nu_{m_0+1} y)}.$$

We introduce modulus of continuity [8]:

$$\omega(\sqrt{2}\delta) = \sup_{(x-x_0)^2+(y-y_0)^2 \leq 2\delta^2} |P(x,y) - P(x_0, y_0)|.$$

Then

$$\int_{x_0-\delta}^{x_0} dx \int_{y_0-\delta}^{y_0} e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x + \nu_{m_0+1} y)} |P(x,y) - P(x_0, y_0)| dy \leq$$

$$\leq \frac{4\omega\sqrt{2}\delta}{\left|Re\sqrt[3]{\lambda}\right|^2} e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x + \nu_{m_0+1} y)} \leq \frac{|P(x_0, y_0)|}{C\cdot\left|Re\sqrt[3]{\lambda}\right|^2} e^{Re\sqrt[3]{\lambda}(\nu_{m_0} x_0 + \nu_{m_0+1} y_0)},\ C \gg 1, \tag{11}$$

with small enough δ. Hence, due to (7), (8), (9), (10), (11), we have

$$\left|\Delta_0(\lambda)\right| \geq \frac{\left|P(x_0, y_0)\right|}{\tilde{C} \cdot \left|Re\sqrt[3]{\lambda}\right|^2} e^{Re\sqrt[3]{\lambda}(\nu_{m_0}x_0+\nu_{m_0+1}y_0)}, \ \tilde{C} \gg 1, \ \tilde{C} \geq C \qquad (12)$$

witha suitable choice of the module of $\sqrt[3]{\lambda}$. Inequality (12) implies the conclusion: along those rays on the complex plane λ for which the equation (6) holds at only one point, the function $\Delta_0(\lambda)$ may have only a finite number of eigenvalues.

We formulate the given result as a theorem

Theorem 1. Let $P(x, y)$ be an absolutely integrable in the square [0,1]×[0,1] function. If the function $P(x, y)$ with the bounded variation and $P(x, y) \neq 0$ in a neighbourhood of [0,1]×[0,1], then eigenvalues of the operator $L_0(u)$ can be only in the sectors of arbitrarily small angle ε:

$$\left|Arg\lambda \pm \frac{\pi}{2}\right| < \varepsilon,$$

outside of the sectors it can only have a finite number of eigenvalues.

Remark 1. Here we took into account the theorem [8] that any function of bounded variation has a finite derivative almost everywhere.

Rays, at which the equation (6) is achieved in a set of points, are called critical. According to the monograph [9], the critical rays are exactly six in the complexplane $\sqrt[3]{\lambda}$, i.e.

$$Arg\sqrt[3]{\lambda} = \frac{\pi}{6} + \frac{\pi k}{3}, k = 0,1,2,3,4,5.$$

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