Journal of Mathematics and Informatics
Vol. 15, 2019, 19-31
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 27 January 2019
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.127v15a3

Journal of Mathematics and Informatics

Existence of Positive Solutions for a Coupled System of Kirchhoff Type Equations with Sobolev Critical Exponent

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Received 20 December 2018; accepted 9 January 2019

Abstract. In this paper, we consider a coupled system of Kirchhoff type equations with Sobolev critical exponent. Based upon the Nehari manifold, fibering maps and variational methods, we prove an existence result of positive solutions.

Keywords: Kirchhoff type system; Sobolev critical exponent; positive solutions

AMS Mathematics Subject Classification (2010): 35B33, 35J48, 35J50

1. Introduction

This paper is concerned with the existence of positive solutions to the following coupled elliptic system of Kirchhoff type equations with Sobolev critical exponent

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda |u|^{r-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & \text{in } \Omega, \\ -\left(a+b\int_\Omega |\nabla v|^2 dx\right)\Delta v = \mu |v|^{r-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \Omega, \\ u=v=0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $a, b, \lambda, \mu > 0$, $1<r<2$, $\alpha, \beta > 1$, $\alpha+\beta = 6$ is the Sobolev critical exponent.

System (1) is a nonlocal problem because of the terms $b\int_\Omega |\nabla u|^2 dx$, and the operator $b\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u$ appears in the Kirchhoff equation

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

This problem is analogous to the stationary case of the following equation

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^{2}\,dx\right)\Delta u=f\left(x,u\right),$$

which was proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The nonlocal effect also appears in biological systems where u describes a process. Equation (2) has attracted a considerable attention only after Lions [2] presented an abstract framework to this problem. In recent years, some interesting results can be found in [3-6] and the references therein.

Without the nonlocal terms, (1) is related to the following semilinear elliptic system

$$\begin{cases}-\Delta u=\lambda|u|^{r-2}u+\dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & \text{in } \Omega,\\ -\Delta v=\mu|v|^{r-2}v+\dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \Omega,\\ u=v=0, & \text{on } \partial\Omega,\end{cases} \tag{3}$$

where $\alpha+\beta=2^{*}=\frac{2N}{N-2}(N\geq 3)$. In fact, system (3) is a special case of (1) when $a=1$, $b=0$. Fan [7] researched system (3) by using the Nehari manifold and the Lusternik-Schnirelman category, and proved that the system admits at least $\operatorname{cat}(\Omega)+1$ positive solutions. Recently, the existence of multiple positive solutions for the p-q-Laplacian system with critical nonlinearities has been received much attention. Especially, Li and Yang [8] obtained the existence of $\operatorname{cat}(\Omega)+1$ positive solutions, Yin [9] obtained the existence of $\operatorname{cat}(\Omega)$ positive solutions.

Motivated by [7, 9], in this work we are concerned with the existence of positive solutions for system (1). System (1) arouses some interesting results due to the lack of compactness of the embedding $W_{0}^{1,2}(\Omega)\underset{\rightarrow}{\subseteq}L^{2^{*}}(\Omega)$. Hence, the corresponding energy functional to system (1) do not satisfy the Palais-Smale condition, which is the difficulty of this paper for system (1). Therefore, we can’t use the standard variational methods. Moreover, as far as we known, there are few works on the existence of positive solutions for system (1). we shall extend the study of [8] to overcome the difficulty and obtain the existence of positive solutions for (1). The following theorem is our main result.

Theorem 1.1. There exists $\Lambda_{*}>0$ such that if $\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\in\left(0,\Lambda_{*}\right)$, system (1) admits at least one positive solutions.

Remark 1.1. The existence result of system (1) in the case $a=1$, $b=0$ has been obtained in [7]. The result has been extended to the p-q Laplacian equations in [8, 9].

2. Preliminaries results

For a bounded domain $\Omega\subset\mathbb{R}^{3}$ with smooth boundary, we denote by $\|\cdot\|$, $|\cdot|_{p}$ the norm of $H_{0}^{1}(\Omega)$ and $L^{p}(\Omega)$ respectively as follows

$$\|u\| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}},\ \ |u|_p = \left(\int_\Omega |u|^p\, dx\right)^{\frac{1}{p}},\ \ (p>1).$$

Obviously, $H_0^1(\Omega)$ is a normal Sobolev space and $H := \left(H_0^1(\Omega)\right)^2$ is a Hilbert space. Let H' be dual of H. The norm on H is given by

$$\|(u,v)\|_H = \left(\|u\|^2 + \|v\|^2\right)^{\frac{1}{2}}.$$

We define

$$S := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_\Omega |u|^{2^*}\, dx\right)^{\frac{2}{2^*}}},\quad S_{\alpha,\beta} := \inf_{(u,v)\in H\setminus\{(0,0)\}} \frac{\|(u,v)\|_H^2}{\left(\int_\Omega |u|^\alpha |v|^\beta\, dx\right)^{\frac{2}{\alpha+\beta}}}. \tag{4}$$

It is known that S is independent of Ω and never achieved except when $\Omega = \mathbb{R}^3$ (see [10]). Obviously, by (4), we have $\int_\Omega |u|^\alpha |v|^\beta\, dx \le S_{\alpha,\beta}^{-3}\|(u,v)\|_H^6$.

We associate with system (1) a functional given by

$$I(u,v) = \frac{1}{2}\|(u,v)\|_H^2 + \frac{b}{4}\left(\|u\|^4 + \|v\|^4\right) - \frac{1}{r}\int_\Omega\left[\lambda\left(u^+\right)^r + \mu\left(v^+\right)^r\right]dx - \frac{2}{\alpha+\beta}\int_\Omega\left(u^+\right)^\alpha\left(v^+\right)^\beta dx, \tag{5}$$

where $u^+ = \max\{u,0\},\ v^+ = \max\{v,0\}$. It is trivial to verify that the energy functional (5) is of C^1. Then the nontrivial critical points of the functional (5) are the nonzero weak solutions of system (1). Accurately, (u,v) is a weak solution of system (1) if $(u,v)\in H$. Thus for all $(\varphi_1,\varphi_2)\in H$ it holds

$$\begin{aligned} & a\int_\Omega\left(\nabla u\nabla\varphi_1 + \nabla v\nabla\varphi_2\right)dx + b\|u\|^2\int_\Omega \nabla u\nabla\varphi_1 dx + b\|v\|^2\int_\Omega\nabla v\nabla\varphi_2 dx \\ & - \int_\Omega\left[\lambda\left(u^+\right)^{r-1}\varphi_1 + \mu\left(v^+\right)^{r-1}\varphi_2\right]dx - \frac{2\alpha}{\alpha+\beta}\int_\Omega\left(u^+\right)^{\alpha-1}\left(v^+\right)^\beta\varphi_1 dx \\ & - \frac{2\beta}{\alpha+\beta}\int_\Omega\left(u^+\right)^\alpha\left(v^+\right)^{\beta-1}\varphi_2 dx = 0. \end{aligned} \tag{6}$$

In order to obtain existence results, we introduce the following Nehari manifold

$$\mathcal{N}_{\lambda,\mu} := \left\{(u,v)\in H\setminus\{(0,0)\} : \left\langle I'(u,v),(u,v)\right\rangle = 0\right\}.$$

Then by (6), we find $(u,v)\in\mathcal{N}_{\lambda,\mu}$ if and only if

$$a\|(u,v)\|_H^2 + b\left(\|u\|^4 + \|v\|^4\right) - \int_\Omega\left[\lambda\left(u^+\right)^r + \mu\left(v^+\right)^r\right]dx - 2\int_\Omega\left(u^+\right)^\alpha\left(v^+\right)^\beta dx = 0.$$

Obviously, $\mathcal{N}_{\lambda,\mu}$ contains all positive weak solutions of system (1). If $(u,v)\in\mathcal{N}_{\lambda,\mu}$, we deduce that

$$I(u,v)=\frac{a}{2}\|(u,v)\|_H^2+\frac{b}{4}\left(\|u\|^4+\|v\|^4\right)-\frac{1}{r}\int_\Omega\left[\lambda(u^+)^r+\mu(v^+)^r\right]dx-\frac{2}{2^*}\int_\Omega(u^+)^\alpha(v^+)^\beta dx$$
$$=a\left(\frac{1}{2}-\frac{1}{r}\right)\|(u,v)\|_H^2+b\left(\frac{1}{4}-\frac{1}{r}\right)\left(\|u\|^4+\|v\|^4\right)+2\left(\frac{1}{r}-\frac{1}{6}\right)\int_\Omega(u^+)^\alpha(v^+)^\beta dx$$
$$=\frac{a}{3}\|(u,v)\|_H^2+\frac{b}{12}\left(\|u\|^4+\|v\|^4\right)+\left(\frac{1}{6}-\frac{1}{r}\right)\int_\Omega\left[\lambda(u^+)^r+\mu(v^+)^r\right]dx.$$

Then, we conclude from $1<r<2$ and the Hölder inequality that $I(u,v)$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.

For each $t>0$, we define the fibering maps

$$\phi_{u,v}(t)=\frac{at^2}{2}\|(u,v)\|_H^2+\frac{bt^4}{4}\left(\|u\|^4+\|v\|^4\right)-\frac{t^r}{r}\int_\Omega\left[\lambda(u^+)^r+\mu(v^+)^r\right]dx-\frac{t^6}{3}\int_\Omega(u^+)^\alpha(v^+)^\beta dx. \tag{7}$$

Then for each $(u,v)\in\mathcal{N}_{\lambda,\mu}$, we have

$$\phi''_{u,v}(1)=a(2-r)\|(u,v)\|_H^2+\mathrm{b}(4-r)\left(\|u\|^4+\|v\|^4\right)-2(6-r)\int_\Omega(u^+)^\alpha(v^+)^\beta dx$$
$$=-4a\|(u,v)\|_H^2-2\mathrm{b}\left(\|u\|_2^4+\|v\|_2^4\right)+(6-r)\int_\Omega\left[\lambda(u^+)^r+\mu(v^+)^r\right]dx.$$

We split the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to the local minima, the local maxima and the points of inflection

$$\mathcal{N}_{\lambda,\mu}^+:=\left\{(u,v)\in\mathcal{N}_{\lambda,\mu}:\phi''_{u,v}(1)>0\right\},$$
$$\mathcal{N}_{\lambda,\mu}^0:=\left\{(u,v)\in\mathcal{N}_{\lambda,\mu}:\phi''_{u,v}(1)=0\right\},$$
$$\mathcal{N}_{\lambda,\mu}^-:=\left\{(u,v)\in\mathcal{N}_{\lambda,\mu}:\phi''_{u,v}(1)<0\right\},$$

which is similar to the method used in [8].

Throughout this paper, we denote the weak convergence by $\rightharpoonup$, and the strong convergence by $\to$. We denote positive constants by C_0, Λ_*, $\Lambda_i^*(i=1,2)$. Then we have the following lemmas.

Lemma 2.1. (See [11, Lemma 2.4]) Assume that (u_0,v_0) is a local minimizer for $I(u,v)$ on $\mathcal{N}_{\lambda,\mu}$ and $(u_0,v_0)\notin\mathcal{N}_{\lambda,\mu}^0$. Then $I'(u_0,v_0)=0$ in H'.

Lemma 2.2 For $(u,v)\in\mathcal{N}_{\lambda,\mu}$, there exists a positive constant C_0 (depending on $p,\ r,$ $N,S,|\Omega|$) such that $I(u,v)\geq -C_0\left(\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\right)$.

Proof: The proof is similar to that of [12, Lemma 2.2].

Lemma 2.3 There exists $\Lambda_1^*>0$ such that $\mathcal{N}_{\lambda,\mu}^0=\varnothing$ if $\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\in\left(0,\Lambda_1^*\right)$.

Proof: Now we define

$$\Lambda_1^* := \left[\frac{a(2-r)}{2(6-r)} S_{\alpha,\beta}^3\right]^{\frac{1}{2}} \left[\frac{6-r}{2a}|\Omega|^{\frac{6-r}{6}} S^{-\frac{r}{2}}\right]^{-\frac{2}{2-r}}.$$

Arguing by the way of contradiction we assume that there exists $\lambda^{\frac{2}{2-r}} + \mu^{\frac{2}{2-r}} \in \left(0, \Lambda_1^*\right)$, such that $\mathcal{N}_{\lambda,\mu}^0 \neq \varnothing$. Then for any $(u,v) \in \mathcal{N}_{\lambda,\mu}^0$, we obtain

$$a(2-r)\|(u,v)\|_H^2 \leq a(2-r)\|(u,v)\|_H^2 + \mathrm{b}(4-r)\left(\|u\|^4 + \|v\|^4\right) = 2(6-r)\int_\Omega \left(u^+\right)^\alpha \left(v^+\right)^\beta dx,$$

and

$$4a\|(u,v)\|_H^2 \leq 4a\|(u,v)\|_H^2 + 2\mathrm{b}\left(\|u\|^4 + \|v\|^4\right) = (6-r)\int_\Omega \left[\lambda\left(u^+\right)^r + \mu\left(v^+\right)^r\right] dx.$$

Based upon the Hölder inequality and the Sobolev embedding theorem, we obtain

$$2(6-r)\int_\Omega \left(u^+\right)^\alpha \left(v^+\right)^\beta dx \leq 2(6-r) S_{\alpha,\beta}^{-3} \|(u,v)\|_H^6,$$

and

$$\begin{aligned}(6-r)\int_\Omega \left[\lambda\left(u^+\right)^r + \mu\left(v^+\right)^r\right] dx &\leq (6-r)|\Omega|^{\frac{6-r}{6}} S^{-\frac{r}{2}} (\lambda + \mu)\|(u,v)\|_H^r \\ &\leq 2(6-r)|\Omega|^{\frac{6-r}{6}} S^{-\frac{r}{2}} \left(\lambda^{\frac{2}{2-r}} + \mu^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \|(u,v)\|_H^r.\end{aligned}$$

Then, we deduce

$$\|(u,v)\|_H \geq \left[\frac{a(2-r)}{2(6-r)} S_{\alpha,\beta}^3\right]^{\frac{1}{4}}, \quad \|(u,v)\|_H \leq \left[\frac{6-r}{2a}|\Omega|^{\frac{6-r}{6}} S^{-\frac{r}{2}}\right]^{\frac{1}{2-r}} \left(\lambda^{\frac{2}{2-r}} + \mu^{\frac{2}{2-r}}\right)^{\frac{1}{2}}.$$

This implies

$$\lambda^{\frac{2}{2-r}} + \mu^{\frac{2}{2-r}} \geq \left[\frac{a(2-r)}{2(6-r)} S_{\alpha,\beta}^3\right]^{\frac{1}{2}} \left[\frac{6-r}{2a}|\Omega|^{\frac{6-r}{6}} S^{-\frac{r}{2}}\right]^{-\frac{2}{2-r}} = \Lambda_1^*,$$

which contradicts with the assumption. Therefore, we conclude that there exists a constant $\Lambda_1^* > 0$ such that $\mathcal{N}_{\lambda,\mu}^0 = \varnothing$ for $\lambda^{\frac{2}{2-r}} + \mu^{\frac{2}{2-r}} \in \left(0, \Lambda_1^*\right)$. □

By Lemma 2.3, for $\lambda^{\frac{2}{2-r}} + \mu^{\frac{2}{2-r}} \in (0, \Lambda_1^*)$, $\mathcal{N}_{\lambda,\mu} = \mathcal{N}_{\lambda,\mu}^+ \cup \mathcal{N}_{\lambda,\mu}^-$, and we define

$$\gamma_{\lambda,\mu} := \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} I(u,v), \quad \gamma_{\lambda,\mu}^+ := \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^+} I(u,v), \quad \gamma_{\lambda,\mu}^- := \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^-} I(u,v).$$

Then we have the following results.

Lemma 2.4. There exists $\Lambda_2^* > 0$ such that if $\lambda^{\frac{2}{2-r}} + \mu^{\frac{2}{2-r}} \in \left(0, \Lambda_2^*\right)$ we obtain

(i) $\gamma_{\lambda,\mu}^+ < 0$;

(ii) $\gamma_{\lambda,\mu}^- \geq d_0$ for some $d_0 > 0$.

Proof: (i) For any $(u,v) \in \mathcal{N}_{\lambda,\mu}^+ \subset \mathcal{N}_{\lambda,\mu}$, we have

$$a(2-r)\|(u,v)\|_H^2 + \mathrm{b}(4-r)\left(\|u\|^4+\|v\|^4\right) > 2(6-r)\int_\Omega (u^+)^\alpha (v^+)^\beta dx,$$

and hence

$$\begin{aligned} I(u,v) &= a\left(\frac{1}{2}-\frac{1}{r}\right)\|(u,v)\|_H^2 + b\left(\frac{1}{4}-\frac{1}{r}\right)\left(\|u\|^4+\|v\|^4\right) + 2\left(\frac{1}{r}-\frac{1}{6}\right)\int_\Omega (u^+)^\alpha (v^+)^\beta dx \\ &< -\frac{a(2-r)}{3r}\|(u,v)\|_H^2 - \frac{b(4-r)}{12r}\left(\|u\|^4+\|v\|^4\right) < 0. \end{aligned}$$

This implies $\gamma_{\lambda,\mu} \le \gamma_{\lambda,\mu}^+ < 0.$

(ii) For any $(u,v)\in \mathcal{N}_{\lambda,\mu}^- \subset \mathcal{N}_{\lambda,\mu}$, we deduce

$$\begin{aligned} &a(2-r)\|(u,v)\|_H^2 + \mathrm{b}(4-r)\left(\|u\|^4+\|v\|^4\right) \\ &< 2(6-r)\int_\Omega (u^+)^\alpha (v^+)^\beta dx \le 2(6-r) S_{\alpha,\beta}^{-3}\|(u,v)\|_H^6, \end{aligned}$$

which implies that

$$\|(u,v)\|_H \ge \left[\frac{a(2-r)}{2(6-r)} S_{\alpha,\beta}^3\right]^{\frac{1}{4}}.$$

Then, we obtain from (5), the Hölder inequality and the Sobolev embedding theorem that

$$\begin{aligned} I(u,v) &= a\left(\frac{1}{2}-\frac{1}{2^*}\right)\|(u,v)\|_H^2 + b\left(\frac{1}{4}-\frac{1}{2^*}\right)\left(\|u\|^4+\|v\|^4\right) + \left(\frac{1}{2^*}-\frac{1}{r}\right)\int_\Omega \left[\lambda (u^+)^r + \mu (v^+)^r\right]dx \\ &\ge \frac{a}{3}\|(u,v)\|_H^2 - 2\left(\frac{1}{r}-\frac{1}{6}\right)|\Omega|^{\frac{6-r}{6}} S^{-\frac{r}{2}}\left(\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\right)^{\frac{2}{2-r}}\|(u,v)\|_H^r \\ &= \|(u,v)\|_H^r\left[\frac{a}{3}\|(u,v)\|_H^{2-r} - 2\left(\frac{1}{r}-\frac{1}{6}\right)|\Omega|^{\frac{6-r}{6}} S^{-\frac{r}{2}}\left(\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\right)^{\frac{2}{2-r}}\right] \\ &\ge \left[\frac{a(2-r)}{2(6-r)} S_{\alpha,\beta}^3\right]^{\frac{r}{4}}\left\{\frac{1}{3}\left[\frac{a(2-r)}{2(6-r)} S_{\alpha,\beta}^3\right]^{\frac{2-r}{4}} - 2\left(\frac{1}{r}-\frac{1}{6}\right)|\Omega|^{\frac{6-r}{6}} S^{-\frac{r}{2}}\left(\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\right)^{\frac{2}{2-r}}\right\}. \end{aligned}$$

Then, we conclude that there exists $\Lambda_2^* > 0$ small enough and $d_0 > 0$ such that $\gamma_{\lambda,\mu}^- \ge d_0 > 0$ for all $(u,v)\in \mathcal{N}_{\lambda,\mu}^-$ if $\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}} \in (0,\Lambda_2^*)$. □

According to the fibering maps and Nehari manifold, we consider the following function $J_{u,v}(t):\mathbb{R}^+ \to \mathbb{R}$, which is defined by

$$J_{u,v}(t) = at^{-4}\|(u,v)\|_H^2 + bt^{-2}\left(\|u\|^4+\|v\|^4\right) - t^{r-6}\int_\Omega \left[\lambda (u^+)^r + \mu (v^+)^r\right]dx. \tag{8}$$

Then by (7), we derive

$$\phi_{u,v}'(t) = t^5\left[J_{u,v}(t) - 2\int_\Omega (u^+)^\alpha (v^+)^\beta dx\right].$$

Taking into account (8), we have $\lim_{t\to 0^+} J_{u,v}(t) = -\infty$, $\lim_{t\to+\infty} J_{u,v}(t) = 0$ and

$$J'_{u,v}(t)=t^{r-7}\left\{-4at^{2-r}\left\|(u,v)\right\|_H^2-2bt^{4-r}\left(\|u\|^4+\|v\|^4\right)-(r-6)\int_\Omega\left[\lambda\left(u^+\right)^r+\mu\left(v^+\right)^r\right]dx\right\}.$$

Now we define

$$\psi_{u,v}(t)=-4at^{2-r}\left\|(u,v)\right\|_H^2-2bt^{4-r}\left(\|u\|^4+\|v\|^4\right)-(r-6)\int_\Omega\left[\lambda\left(u^+\right)^r+\mu\left(v^+\right)^r\right]dx.$$

By direct computation, we obtain

$$\lim_{t\to0^+}\psi_{u,v}(t)=(6-r)\int_\Omega\left[\lambda\left(u^+\right)^r+\mu\left(v^+\right)^r\right]dx,\quad \lim_{t\to+\infty}\psi_{u,v}(t)=-\infty,$$

and

$$\psi'_{u,v}(t)=-4(2-r)t^{1-r}\left\|(u,v)\right\|_H^2-2(4-r)t^{3-r}\left(\|u\|^4+\|v\|^4\right)<0.$$

Thus, for $(u,v)\in H$ with $\int_\Omega\left[\lambda\left(u^+\right)^r+\mu\left(v^+\right)^r\right]dx>0$, $J_{u,v}(t)$ achieves its maximum at the point $t_{\max}=t_{\max}(u,v)$, where $t_{\max}$ is the unique solution of the equation

$$4at^{2-r}\left\|(u,v)\right\|_H^2+2bt^{4-r}\left(\|u\|^4+\|v\|^4\right)=(6-r)\int_\Omega\left[\lambda\left(u^+\right)^r+\mu\left(v^+\right)^r\right]dx.$$

Moreover, we deduce

$$J_{u,v}(t_{\max})=\frac{a(2-r)}{6-r}t_{\max}^{-4}\left\|(u,v)\right\|_H^2+\frac{\mathrm{b}(4-r)}{6-r}t_{\max}^{-2}\left(\|u\|^4+\|v\|^4\right)>0$$

and $J'_{u,v}(t)>0$, as $t\in(0,t_{\max})$; $J'_{u,v}(t)<0$, as $t\in(t_{\max},+\infty)$. This lemma is completed. □

Lemma 2.5. For each $(u,v)\in H\setminus\{(0,0)\}$, if $\int_\Omega\left(u^+\right)^\alpha\left(v^+\right)^\beta dx>0$, then there exists a unique t^+, a unique t^- and $0<t^+<t_{\max}<t^-$ such that $\left(t^+u,t^+v\right)\in\mathcal{N}^+_{\lambda,\mu}$, $\left(t^-u,t^-v\right)\in\mathcal{N}^-_{\lambda,\mu}$, and

$$I\left(t^+u,t^+v\right)=\inf_{0\le t\le t_{\max}}I(tu,tv),\qquad I\left(t^-u,t^-v\right)=\sup_{t\ge0}I(tu,tv).\tag{9}$$

Proof: Due to $\int_\Omega\left(u^+\right)^\alpha\left(v^+\right)^\beta dx>0$ and $J_{u,v}(t_{\max})>0$, we deduce that the equation $J_{u,v}(t)=2\int_\Omega\left(u^+\right)^\alpha\left(v^+\right)^\beta dx$ has two solutions $t^+<t_{\max}<t^-$ such that $J'_{u,v}\left(t^+\right)>0$ and $J'_{u,v}\left(t^-\right)<0$. Taking into account $\phi''_{u,v}\left(t^+\right)=\left(t^+\right)^5J'_{u,v}\left(t^+\right)>0$, we obtain $\left(t^+u,t^+v\right)\in\mathcal{N}^+_{\lambda,\mu}$, $\left(t^-u,t^-v\right)\in\mathcal{N}^-_{\lambda,\mu}$. Therefore, we conclude that $\phi_{u,v}(t)$ is decreasing on $\left(0,t^+\right)$, increasing on $\left(t^+,t^-\right)$ and decreasing on $\left(t^-,+\infty\right)$. Then (9) holds. □

In the following, we give the corresponding definition of the Palais-Smale sequence in H for $I(u,v)$.

Definition 2.6. (i) For $c\in R$, a sequence $\{(u_n,v_n)\}\subset H$ is a $(PS)_c$ sequence for $I(u,v)$ if $I(u_n,v_n)\to c$ and $I'(u_n,v_n)\to 0$ strongly in H' as $n\to\infty$;

(ii) $c\in R$ is a (PS) value in H for $I(u,v)$ if there exist a $(PS)_c$ sequence in H for $I(u,v)$;

(iii) $I(u,v)$ satisfies the $(PS)_c$ condition in H if any $(PS)_c$ sequence in H for $I(u,v)$ contains a convergent subsequence.

Lemma 2.7. For any $\lambda,\ \mu>0$ and $1<r<2$, $I(u,v)$ satisfies the $(PS)_c$ condition for

$$c\in\left(-\infty,\ \tfrac{2}{3}\left(\tfrac{aS_{\alpha,\beta}}{2}\right)^{\frac{3}{2}}-C_0\left(\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\right)\right). \tag{10}$$

Proof: Assume that $\{(u_n,v_n)\}\subset H$ is a $(PS)_c$ sequence for $I(u,v)$ satisfying (10). Then we have

$$I(u_n,v_n)=c+o(1),\qquad I'(u_n,v_n)=o(1). \tag{11}$$

In the following, we will prove that $\{(u_n,v_n)\}$ is bounded in H. Assume that

$$\|(u_n,v_n)\|_H\to\infty,\quad (\hat u_n,\hat v_n)=\left(\tfrac{u_n}{\|(u_n,v_n)\|_H},\tfrac{u_n}{\|(u_n,v_n)\|_H}\right).$$

Obviously, $\|(\hat u_n,\hat v_n)\|_H=1$ is bounded in H. Up to a subsequence, we suppose that

$$(\hat u_n,\hat v_n)\rightharpoonup(\hat u,\hat v)\ \text{ in } H,$$

which implies that

$$\hat u_n\rightharpoonup\hat u,\quad \hat v_n\rightharpoonup\hat v\ \text{ in } H_0^1(\Omega),$$

$$\hat u_n\to\hat u,\quad \hat v_n\to\hat v\ \text{ in } L^s(\Omega),\ \text{where } 1\le s<6,$$

$$\int_\Omega\left[\lambda(\hat u_n^+)^r+\mu(\hat v_n^+)^r\right]dx\to\int_\Omega\left[\lambda(\hat u^+)^r+\mu(\hat v^+)^r\right]dx.$$

By means of (10), we deduce

$$c+o(1)=\frac{a}{2}\|(u_n,v_n)\|_H^2\|(\hat u_n,\hat v_n)\|_H^2+\frac{b}{4}\|(u_n,v_n)\|_H^4\left(\|\hat u\|^4+\|\hat v\|^4\right)-\frac{1}{3}\|(u_n,v_n)\|_H^6\int_\Omega(\hat u_n^+)^\alpha(\hat v_n^+)^\beta dx$$
$$-\frac{1}{r}\|(u_n,v_n)\|_H^r\int_\Omega\left[\lambda(\hat u_n^+)^r+\mu(\hat v_n^+)^r\right]dx,\quad \text{as } n\to\infty,$$

and

$$o(1)=a\|(u_n,v_n)\|_H^2\|(\hat u_n,\hat v_n)\|_H^2+b\|(u_n,v_n)\|_H^4\left(\|\hat u\|^4+\|\hat v\|^4\right)-2\|(u_n,v_n)\|_H^6\int_\Omega(\hat u_n^+)^\alpha(\hat v_n^+)^\beta dx$$
$$-\|(u_n,v_n)\|_H^r\int_\Omega\left[\lambda(\hat u_n^+)^r+\mu(\hat v_n^+)^r\right]dx,\quad \text{as } n\to\infty.$$

Therefore, by $\|(u_n,v_n)\|_H\to\infty$, we obtain

$$o(1)=\frac{a}{2}\left\|\left(\hat{u}_n,\hat{v}_n\right)\right\|_H^2+\frac{b}{4}\left\|\left(u_n,v_n\right)\right\|_H^2\left(\left\|\hat{u}_n\right\|^4+\left\|\hat{v}_n\right\|^4\right)-\frac{1}{3}\left\|\left(u_n,v_n\right)\right\|_H^4\int_\Omega\left(\hat{u}_n^+\right)^\alpha\left(\hat{v}_n^+\right)^\beta dx$$
$$-\frac{1}{r}\left\|\left(u_n,v_n\right)\right\|_H^{r-2}\int_\Omega\left[\lambda\left(\hat{u}_n^+\right)^r+\mu\left(\hat{v}_n^+\right)^r\right]dx,\quad \text{as } n\to\infty,$$

and

$$o(1)=a\left\|\left(\hat{u}_n,\hat{v}_n\right)\right\|_H^2+b\left\|\left(u_n,v_n\right)\right\|_H^2\left(\left\|\hat{u}_n\right\|^4+\left\|\hat{v}_n\right\|^4\right)-2\left\|\left(u_n,v_n\right)\right\|_H^4\int_\Omega\left(\hat{u}_n^+\right)^\alpha\left(\hat{v}_n^+\right)^\beta dx$$
$$-\left\|\left(u_n,v_n\right)\right\|_H^{r-2}\int_\Omega\left[\lambda\left(\hat{u}_n^+\right)^r+\mu\left(\hat{v}_n^+\right)^r\right]dx,\quad \text{as } n\to\infty.$$

Then, as $n\to\infty$, we get

$$\left\|\left(\hat{u}_n,\hat{v}_n\right)\right\|_H^2=\frac{b(r-4)}{2a(2-r)}\left\|\left(u_n,v_n\right)\right\|_H^2\left(\left\|\hat{u}_n\right\|^4+\left\|\hat{v}_n\right\|^4\right)+\frac{2(6-r)}{3a(2-r)}\left\|\left(u_n,v_n\right)\right\|_H^4\int_\Omega\left(\hat{u}_n^+\right)^\alpha\left(\hat{v}_n^+\right)^\beta dx+o(1).$$

Therefore, we deduce that $\left\|\left(\hat{u}_n,\hat{v}_n\right)\right\|_H^2\to\infty$, as $n\to\infty$, which contradicts with $\left\|\left(\hat{u}_n,\hat{v}_n\right)\right\|_H^2=1$, thus the sequence $\left\{\left(\hat{u}_n,\hat{v}_n\right)\right\}$ is bounded in H.

Going if necessary to a subsequence, we suppose that

$$\left(u_n,v_n\right)\rightharpoonup(u,v) \text{ in } H,$$
$$u_n\to u,\quad v_n\to v \text{ in } L^s(\Omega), \text{ where } 1\le s<6,$$
$$\int_\Omega\left[\lambda\left(u_n^+\right)^r+\mu\left(v_n^+\right)^r\right]dx\to\int_\Omega\left[\lambda\left(u^+\right)^r+\mu\left(v^+\right)^r\right]dx.$$

According to the Brezis-Lieb lemma and [11, Lemma 2.1], we obtain

$$\left\|\left(u_n-u,v_n-v\right)\right\|_H^2=\left\|\left(u_n,v_n\right)\right\|_H^2-\left\|(u,v)\right\|_H^2+o(1),$$

and

$$\int_\Omega\left(u_n-u\right)_+^\alpha\left(v_n-v\right)_+^\beta dx=\int_\Omega\left(u_n^+\right)^\alpha\left(v_n^+\right)^\beta dx-\int_\Omega\left(u^+\right)^\alpha\left(v^+\right)^\beta dx+o(1).$$

Then for any $\left(\varphi_1,\varphi_2\right)\in H$, there holds

$$\lim_{n\to\infty}\left\langle I'\left(u_n,v_n\right),\left(\varphi_1,\varphi_2\right)\right\rangle=\left\langle I'(u,v),\left(\varphi_1,\varphi_2\right)\right\rangle=0,$$

which means (u,v) is a critical point of $I(u,v)$. Now, we only need to show $\left(u_n,v_n\right)\to(u,v)$ in H. By (11), we deduce

$$c+o(1)=I(u,v)+\frac{a}{2}\left\|\left(u_n-u,v_n-v\right)\right\|_H^2+\frac{b}{4}\left(\left\|u_n-u\right\|^4+\left\|v_n-v\right\|^4\right)-\frac{1}{3}\int_\Omega\left(u_n-u\right)_+^\alpha\left(v_n-v\right)_+^\beta dx, \tag{12}$$

and

$$o(1)=\left\langle I'(u,v),\left(u_n-u,v_n-v\right)\right\rangle$$
$$=a\left\|\left(u_n-u,v_n-v\right)\right\|_H^2+b\left(\left\|u_n-u\right\|^4+\left\|v_n-v\right\|^4\right)-2\int_\Omega\left(u_n-u\right)_+^\alpha\left(v_n-v\right)_+^\beta dx.$$

Without loss of generality, we set

$$\left\|\left(u_n-u,v_n-v\right)\right\|_H^2=m+o(1),\qquad \left\|u_n-u\right\|^4+\left\|v_n-v\right\|^4=l+o(1).$$

Then

$$2\int_\Omega\left(u_n-u\right)_+^\alpha\left(v_n-v\right)_+^\beta dx=am+bl+o(1).$$

If $a=0$, then the conclusion of this lemma holds. On the contrary, now we suppose $a>0$. Then by the Sobolev inequality, we derive

$$m\geq S_{\alpha,\beta}\left(\frac{am+bl}{2}\right)^{\frac{1}{3}}\geq S_{\alpha,\beta}\left(\frac{am}{2}\right)^{\frac{1}{3}}. \tag{13}$$

In view of (12), (13) and $\left(u,v\right)\in\mathcal{N}_{\lambda,\mu}\setminus\{(0,0)\}$, we deduce

$$c=I\left(u,v\right)+\frac{am}{2}+\frac{bl}{4}-\frac{am+bl}{6}\geq I\left(u,v\right)+\frac{am}{3}\geq\frac{2}{3}\left(\frac{aS_{\alpha,\beta}}{2}\right)^{\frac{3}{2}}-C_0\left(\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\right),$$

which contradicts (10). Therefore, we conclude $m=0$, and hence $(u_n,v_n)\to(u,v)$ strongly in H. The lemma is proved. □

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 3.1. If $\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\in\left(0,\Lambda_*\right)$, there exists a $(PS)_{\gamma_{\lambda,\mu}}$ sequence $\left\{\left(u_n,v_n\right)\right\}\subset\mathcal{N}_{\lambda,\mu}^+$ in H for $I\left(u,v\right)$.

Proof: The proof is almost the same as that of Wu [13, Proposition 9].

To obtain the existence of a local minimum for $I\left(u,v\right)$ on $\mathcal{N}_{\lambda,\mu}^+$, we need the following lemma:

Lemma 3.2. If $\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\in\left(0,\Lambda_*\right)$, there exists $\Lambda_*>0$ such that $I\left(u,v\right)$ has a minimizer $\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)\in\mathcal{N}_{\lambda,\mu}^+$ and it satisfies

(i) $I\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)=\gamma_{\lambda,\mu}=\gamma_{\lambda,\mu}^+<0$;

(ii) $\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)$ is a positive solution of system (1).

Proof: From Lemma 3.1, there exists a minimizing sequence $\left\{\left(u_n,v_n\right)\right\}$ for $I\left(u,v\right)$ on $\mathcal{N}_{\lambda,\mu}$ such that

$$I(u_n,v_n)=\gamma_{\lambda,\mu}+o(1)\quad\text{and}\quad I'(u_n,v_n)=o(1)\ \text{ in }\ H'. \tag{14}$$

Since $I\left(u,v\right)$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$, $\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)$ is bounded in H. Then there exists a subsequence $\left\{\left(u_n,v_n\right)\right\}$ and $\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)\in H$ such that

$$\begin{cases} u_n \rightharpoonup u_{\lambda,\mu},\ v_n \rightharpoonup v_{\lambda,\mu} & \text{in } H_0^1(\Omega), \\ u_n \to u_{\lambda,\mu},\ v_n \to v_{\lambda,\mu} & \text{a.e. in } \Omega, \\ u_n \to u_{\lambda,\mu},\ v_n \to v_{\lambda,\mu} & \text{in } L^s(\Omega),\ 1 \le s < 6, \end{cases}$$

which implies

$$\int_\Omega \left[\lambda\left(u_n^+\right)^r + \mu\left(v_n^+\right)^r \right] dx \to \int_\Omega \left[\lambda\left(u_{\lambda,\mu}^+\right)^r + \mu\left(v_{\lambda,\mu}^+\right)^r \right] dx \quad \text{as } n \to \infty. \tag{15}$$

Thus, by using a standard argument, we have $I'\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right) = 0$ and $\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right)$ is a nontrivial solution of system (1). From $\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right) \in \mathcal{N}_{\lambda,\mu}$, we deduce that

$$\int_\Omega \left[\lambda\left(u_n^+\right)^r + \mu\left(v_n^+\right)^r \right] dx = \frac{2ar}{6-r}\left\|(u_n, v_n)\right\|_H^2 + \frac{br}{2(6-r)}\left(\|u_n\|^4 + \|v_n\|^4\right) - \frac{6r}{6-r} I\left(u_n, v_n\right). \tag{16}$$

Taking $n \to \infty$ in (16), we deduce by (14), (15) and $\gamma_{\lambda,\mu} \le \gamma_{\lambda,\mu}^+ < 0$ that

$$\int_\Omega \left[\lambda\left(u_{\lambda,\mu}^+\right)^r + \mu\left(v_{\lambda,\mu}^+\right)^r \right] dx \ge -\frac{6r}{6-r}\gamma_{\lambda,\mu} > 0.$$

Thus, $\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right) \not\equiv (0,0)$ and $\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right)$ is a nontrivial solution of system (1). Now we prove that $\left(u_n, v_n\right) \to \left(u_{\lambda,\mu}, v_{\lambda,\mu}\right)$ in H and $I\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right) = \gamma_{\lambda,\mu} = \gamma_{\lambda,\mu}^+$. Applying the Fatou lemma, if $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we conclude from (5) that

$$\begin{aligned} \gamma_{\lambda,\mu} &\le I\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right) \\ &= \frac{a}{3}\left\|\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right)\right\|_H^2 + \frac{b}{12}\left(\|u_{\lambda,\mu}\|^4 + \|v_{\lambda,\mu}\|^4\right) - \left(\frac{1}{r} - \frac{1}{6}\right)\int_\Omega \left[\lambda\left(u_{\lambda,\mu}^+\right)^r + \mu\left(v_{\lambda,\mu}^+\right)^r \right] dx \\ &\le \liminf_{n\to\infty} \left\{ \frac{a}{3}\left\|(u_n, v_n)\right\|_H^2 + \frac{b}{12}\left(\|u_n\|^4 + \|v_n\|^4\right) - \left(\frac{1}{r} - \frac{1}{6}\right)\int_\Omega \left[\lambda\left(u_n^+\right)^r + \mu\left(v_n^+\right)^r \right] dx \right\} \\ &\le \liminf_{n\to\infty} I\left(u_n, v_n\right) = \gamma_{\lambda,\mu}. \end{aligned}$$

This implies that $I(u_{\lambda,\mu}, v_{\lambda,\mu}) = \gamma_{\lambda,\mu}$ and

$$\lim_{n\to\infty}\left\|(u_n, v_n)\right\|_H^2 = \left\|\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right)\right\|_H^2, \quad \lim_{n\to\infty}\left(\|u_n\|^4 + \|v_n\|^4\right) = \|u_{\lambda,\mu}\|^4 + \|v_{\lambda,\mu}\|^4.$$

Let $\left(\tilde{u}_n, \tilde{v}_n\right) = \left(u_n, v_n\right) - \left(u_{\lambda,\mu}, v_{\lambda,\mu}\right)$, then the Brezis-Lieb lemma implies

$$\left\|(\tilde{u}_n, \tilde{v}_n)\right\|_H^2 = \left\|(u_n, v_n)\right\|_H^2 - \left\|(u_{\lambda,\mu}, v_{\lambda,\mu})\right\|_H^2 + o(1), \quad \text{as } n \to \infty.$$

Thus $\left(u_n, v_n\right) \to \left(u_{\lambda,\mu}, v_{\lambda,\mu}\right)$ in H. In the following, we will prove $\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right) \in \mathcal{N}_{\lambda,\mu}^+$. On the contrary, if $\left(u_{\lambda,\mu}, v_{\lambda,\mu}\right) \in \mathcal{N}_{\lambda,\mu}^-$, then by Lemma 2.5, there exist unique t^+, t^-

such that $\left(t^{+}u_{\lambda,\mu},t^{+}v_{\lambda,\mu}\right)\in \mathcal{N}_{\lambda,\mu}^{+}$ and $\left(t^{-}u_{\lambda,\mu},t^{-}v_{\lambda,\mu}\right)\in \mathcal{N}_{\lambda,\mu}^{-}$. In particular, we have $t^{+}<t^{-}=1$. By means of

$$\frac{d}{dt}I\left(t^{+}u_{\lambda,\mu},t^{+}v_{\lambda,\mu}\right)=0,\quad \frac{d^{2}}{dt^{2}}I\left(t^{+}u_{\lambda,\mu},t^{+}v_{\lambda,\mu}\right)>0,$$

we deduce that there exists $t^{+}<\overline{t}<t^{-}$ such that $I\left(t^{+}u_{\lambda,\mu},t^{+}v_{\lambda,\mu}\right)<I\left(\overline{t}u_{\lambda,\mu},\overline{t}v_{\lambda,\mu}\right)$. From Lemma 2.5, we have

$$\gamma_{\lambda,\mu}^{+}\le I\left(t^{+}u_{\lambda,\mu},t^{+}v_{\lambda,\mu}\right)<I\left(\overline{t}u_{\lambda,\mu},\overline{t}v_{\lambda,\mu}\right)\le I\left(t^{-}u_{\lambda,\mu},t^{-}v_{\lambda,\mu}\right)=I\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)=\gamma_{\lambda,\mu},$$

which is a contradiction, and thus $\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)\in \mathcal{N}_{\lambda,\mu}^{+}$. In view of $I\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)=I\left(|u_{\lambda,\mu}|,|v_{\lambda,\mu}|\right)$ and $\left(|u_{\lambda,\mu}|,|v_{\lambda,\mu}|\right)\in \mathcal{N}_{\lambda,\mu}^{+}$, we conclude from Lemma 2.1 that $\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)$ is a nontrivial nonnegative solution of system (1). Moreover, if $\lambda,\ \mu>0$, then $\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)$ is a positive solution of system (1) by the strongly maximum principle. Therefore, we obtain $\gamma_{\lambda,\mu}^{+}=\gamma_{\lambda,\mu}=I\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)$. □

Proof of Theorem 1.1. Taking into account Lemma 3.2, we obtain that for all λ, $\mu>0$ and $\lambda^{\frac{2}{2-r}}+\mu^{\frac{2}{2-r}}\in\left(0,\Lambda_{*}\right)$, system (1) has a positive solution $\left(u_{\lambda,\mu},v_{\lambda,\mu}\right)\in \mathcal{N}_{\lambda,\mu}^{+}$. Then the conclusion of Theorem 1.1 follows. □

Acknowledgements. This work is supported by the National Natural Science Foundation of China (Nos. 11471235; 11601052) and Chongqing Research Program of Basic Research and Frontier Technology (No. cstc2017jcyjBX0037).

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