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Fuzzy Congruences on MS-algebras

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Abstract. In this paper, we study the fuzzy congruence relation of MS-algebra L and the fuzzy congruence relation generated by a given fuzzy relation on L. We also investigate some properties of the fuzzy congruence relation generated by a given fuzzy relation on L.

Keywords: MS-algebras, fuzzy congruences, fuzzy ideals.

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1.Introduction

In the papers (in particular [7,5,8,1]) have investigated the properties of fuzzy equivalence (congruence) relations of algebras. In particular Yuan and Wangming [9] investigated the relationship between fuzzy ideals and fuzzy congruences on a distributive lattice L. In this paper, we discuss the fuzzy congruence relations in MS-algebras and we study the properties a fuzzy congruence relation generated by $\mu \times \mu$ on L.

2. Preliminaries

In this section, we recall some definitions and basic results on MS-algebras.

Definition 2.1. [3] An MS-algebra is an algebra $(L,\vee,\wedge,^\circ,0,1)$ of type $(2,2,1,0,0)$, such that $(L,\vee,\wedge,0,1)$ is a bounded distributive lattice and $a \to a^\circ$ is a unary operation satisfies: $a \leq a^{\circ\circ}$, $(a \wedge b)^\circ = a^\circ \vee b^\circ$ and $1^\circ = 0$.

Lemma 2.2. [3] For any two elements a, b of an MS-algebra L, we have the following:

1. $0^\circ = 1$,
2. $a \leq b \Rightarrow b^\circ \leq a^\circ$,
3. $a^{\circ\circ\circ} = a^\circ$,
4. $(a \vee b)^\circ = a^\circ \wedge b^\circ$,
5. $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,
6. $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Definition 2.3. [4] Let L be a lattice and let $H \subseteq L \times L$. We denote by $\Theta(H)$ the smallest congruence relation such that $a \equiv b$ for all $(a, b) \in H$, and call it the congruence relation generated by H. If $H = I \times I$, where I is an ideal, we write $\Theta[I]$ for all $\Theta(H)$.

Definition 2.4. [2] An equivalence relation θ is a congruence relation in MS-algebra L, if it is a lattice congruence and $(a, b) \in \theta$ implies $(a^\circ, b^\circ) \in \theta$ for all $a, b \in L$.

If we delete the operation $^\circ$, we shall speak the lattice congruence. To distinguish these two types, we shall use the subscript $'lat'$ to denote lattice congruence. Let I be an ideal of the MS-algebra L. Define $I_\circ^{\geq} = \{x \in L : i^\circ \leq x, \text{ for some } i \in I\}$. $I_{\circ\circ} = \{x \in L : x \leq i^{\circ\circ}, \text{ for some } i \in I\}$. Then $I_\circ^{\geq}$ is filter of L and $I_{\circ\circ}$ is a ideal of L.

Theorem 2.5. [3] Let I be an ideal of the MS-algebra L. Then $\Theta[I] = \Theta_{lat}[I_\circ^{\geq}] \vee \Theta_{lat}[I_{\circ\circ}]$. We recall that for any nonempty set L, the characteristic function of L defined as

$$\chi_L(x) = \begin{cases} 1 & if x \in L, \\ 0 & if x \notin L. \end{cases}$$

Let μ be a fuzzy subset of L. For any $\alpha \in [0,1]$, we shall denote the level subset $\mu^{-1}([\alpha, 1])$ by simply μ_α, i.e. $\mu_\alpha = \{x \in L : \alpha \leq \mu(x)\}$.

Theorem 2.6. [8] Let μ be a fuzzy subset of L. Then μ is a fuzzy ideal of L if and only if any one of the following conditions is satisfied:

1. $\mu(0) = 1$ and $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for all$x, y \in L$,
2. $\mu(0) = 1$ and $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ and $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$ for all $x, y \in L$.

A fuzzy relations on a set X are maps $\theta : X \times X \to [0,1]$. For any $x, y \in X$ and fuzzy relations $(\theta \cap \phi)(x, y) = min\{\theta(x, y), \phi(x, y)\}$,
$(\theta \cup \phi)(x, y) = max\{\theta(x, y), \phi(x, y)\}$, $\theta \subseteq \phi$ means $\theta(x, y) \leq \phi(x, y)$.

Definition 2.7. [5] Suppose that θ and ϕ are two fuzzy relations on a set X. Then $(\theta \circ \phi)(x, y) = sup_{z \in X}((\theta(x, z) \wedge (\phi)(z, y))$.

Definition 2.8. [5] A fuzzy relation ϕ on X is said two be a fuzzy equivalence relation on X if

1. $\phi(x, x) = 1$ for all $x \in X$ (reflexive),
2. $\phi(x, y) = \phi(y, x)$ for all $x, y \in L$ (symmetric),
3. $\phi(x, z) \geq \phi(x, y) \wedge \phi(y, z)$ for all x, $y, z \in L$ (transitive).

Throughout the next sections, L stands for MS-algebra.

3. Fuzzy congruences on ms-algebras

In this section, we give various characterization for fuzzy congruences on MS-algebra L.

Definition 3.1. A fuzzy equivalence relationϕon an MS-algebra L is called fuzzy congruence relation on L if the following are satisfied:

1. $\phi(x \wedge z, y \wedge w) \wedge \phi(x \vee z, y \vee w) \geq \phi(x, y) \wedge \phi(z, w)$ for all $x, y, z, w \in L$,

2. $\phi(x^\circ, y^\circ) \geq \phi(x, y)$ for all $x, y \in L$.

Example 3.2. Consider the MS-algebra L described as in following figure

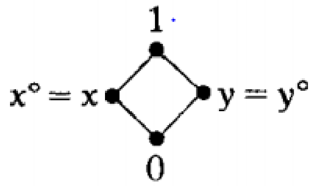

Figure 1:

Define a fuzzy relationϕ on L as follows: $\phi(0,0) = \phi(1,1) = \phi(x, x) = \phi(y, y) = 1$, $\phi(x, y) = \phi(y, x) = \phi(0, y) = \phi(y, 0) = \phi(x, 0) = \phi(0, x) = 0.6$. Then it can be easily verify that ϕ is a fuzzy congruence relation on L.

Theorem 3.3. A fuzzy equivalence relation$\phi$is fuzzy congruence on$L$if and only if $\phi(x, y) \leq \phi(x \wedge z, y \wedge z) \wedge \phi(x \vee z, y \vee z) \wedge \phi(x^\circ, y^\circ)$ for all $x, y, z \in L$.
Proof: The forward proof is straightforward.Conversely, suppose that ϕ is a fuzzy equivalence relation on L that satisfies $\phi(x, y) \leq \phi(x \wedge z, y \wedge z) \wedge \phi(x \vee z, y \vee z) \wedge \phi(x^\circ, y^\circ)$ for all $x, y, z \in L$. This implies $\phi(x, y) \leq \phi(x \wedge z, y \wedge z)$, $\phi(x, y) \leq \phi(x \vee z, y \vee z)$ and $\phi(x, y) \leq \phi(x^\circ, y^\circ)$ for all $x, y, z \in L$.

$\phi(x, y) \wedge \phi(z, w) \leq \phi(x \wedge z, y \wedge z) \wedge \phi(y \wedge z, y \wedge w) \leq \phi(x \wedge z, y \wedge w)$ for all $x, y, z, w \in L$. Similarly $\phi(x, y) \wedge \phi(z, w) \leq \phi(x \vee z, y \vee w)$ for all $x, y, z, w \in L$. Thus ϕ is a fuzzy congruence relation on L.

Theorem 3.4. A relation ϕ on L is a fuzzy congruence on$L$if and only if every level subset ϕ_α of ϕ at $\alpha \in [0,1]$ is congruence relation on L.

Theorem 3.5. A congruence relation ϕ is a congruence relation on L if and only if its characteristic function χ_ϕ is a fuzzy congruence on L.

Theorem 3.6. If $\{\phi_i : i \in \Delta\}$ is a family of fuzzy congruence of L, then $\cap_{i \in \Delta} \phi_i$ is a fuzzy congruence on L.

We denoted that the set of all fuzzy congruences of L by $\mathcal{FC}(L)$ and the set of all congruences of L by $\mathcal{C}(L)$. $\omega = \{(x, y) \in L \times L : x = y\}$ is the smallest and $\iota = L \times L$ is the largest element of $\mathcal{C}(L)$.

$$\chi_\omega(x, y) = \begin{pmatrix} 1 & \text{if } (x, y) \in \omega \\ 0 & \text{if otherwise} \end{pmatrix}$$

for all $x, y \in L$ is the smallest and $\chi_\iota(x, y) = 1$ for all $x, y \in L$ is the largest elements of $\mathcal{FC}(L)$.

Definition 3.7. Let ϕ and φ be any two fuzzy congruence relation of L. Then define $\phi \vee \varphi = \cap \{\Theta \in \mathcal{FC}(L) : \phi \subseteq \Theta$ and $\varphi \subseteq \Theta\}$, *i.e* $\phi \vee \varphi$ is the fuzzy congruence generated by $\phi \cup \varphi$.

Theorem 3.8. $(\mathcal{FC}(L), \subseteq)$ is complete lattice.

Proof: We note that both fuzzy congruences relations χ_ω and χ_ι are the least and the greatest elements of $\mathcal{FC}(L)$, respectively. Clearly $(\mathcal{FC}(L), \subseteq)$ is poset and $\cap_{i\in\Delta} \phi_i$ is lower bound of any family $\{\phi_i : i \in \Delta\}$ of fuzzy congruences of L. Let Θ be any a lower bound of $\{\phi_i : i \in \Delta\}$. Then $\Theta \subseteq \phi_i$ for all $i \in \Delta$ and so $\Theta \subseteq \cap_{i\in\Delta} \phi_i$. This implies $\cap_{i\in\Delta} \phi_i$ is a greatest lower bound of $\{\phi_i : i \in \Delta\}$. Hence $(\mathcal{FC}(L), \subseteq)$ is a complete lattice.

Theorem 3.9. Let ϕ and φ be any fuzzy congruence relations on an MS-algebra L. Then $\phi \vee \varphi = \cup_{n=1}^{\infty} \Theta_n$, where $\Theta_1 = \phi \circ \varphi \circ \phi,\ \Theta_2 = \phi \circ \varphi \circ \phi \circ \varphi \circ \phi,\ \Theta_3 = \phi \circ \varphi \circ \phi \circ \varphi \circ \phi \circ \varphi \circ \phi$.

Proof: Let $\kappa = \cup_{n=0}^{\infty} \Theta_n$. We prove that κ is the smallest fuzzy congruence relation in MS-algebra L containing ϕ and φ. It can be easily verify that $\phi \subseteq \Theta_1 \subseteq \Theta_2 \subseteq, \ldots$ and $\varphi \subseteq \Theta_1 \subseteq \Theta_2 \subseteq, \ldots$ and so $\Theta_n \subseteq \phi \vee \varphi$. Now we see that κ is a fuzzy congruence in MS-algebra L.

1. $1 = \phi(x,x) \leq \Theta_1(x,x) \leq \cup_{n=0}^{\infty} \Theta_n = \kappa(x,x)$. Hence $\kappa(x,x) = 1$.
2. Symmetric is straightforward.
3. $\kappa(x,y) \wedge \kappa(y,z) = \cup_{n=1}^{\infty} \Theta_n(x,y) \wedge \cup_{n=1}^{\infty} \Theta_n(y,z) = \sup_n \Theta_n(x,y) \wedge$

$\sup_n \Theta_n(y,z) \leq \cup_{n=1}^{\infty} \Theta_n(x,z)$. Since $\Theta_n(x,y) \wedge \Theta_m(y,z) \leq \Theta_{n+m}(x,z)$ for any real number n and m.

4. $\kappa(x,y) = \cup_{n=1}^{\infty} \Theta_n(x,y)$

$$= \sup_{z_1,z_2,\ldots,z_{2n}} (\phi(x,z_1) \wedge \varphi(z_1,z_2) \wedge \phi(z_2,z_3) \wedge \ldots \wedge \phi(z_{2n},y))$$
$$\leq \sup_{z_1\wedge c, z_2\wedge c,\ldots,z_{2n}\wedge c} (\phi(x\wedge c, z_1\wedge c) \wedge \varphi(z_1\wedge c, z_2\wedge c) \wedge \phi(z_2\wedge c, z_3\wedge c) \ldots \wedge \phi(z_{2n} \wedge c, y\wedge c))$$
$$= \cup_{n=1}^{\infty} \Theta_n(x\wedge c, y\wedge c) = \kappa(x\wedge c, y\wedge c)$$

Similarly $\kappa(x,y) \leq \kappa(x\vee c, y\vee c)$

5. $\kappa(x,y) = \cup_{n=1}^{\infty} \Theta_n(x,y)$

$$= \sup_{z_1,z_2,\ldots,z_{2n}} (\phi(x,z_1) \wedge \varphi(z_1,z_2) \wedge \phi(z_2,z_3) \wedge \ldots \wedge \Theta(z_{2n},y))$$
$$\leq \sup_{z_1^\circ,z_2^\circ,\ldots,z_{2n}^\circ} (\phi(x^\circ,z_1^\circ) \wedge \varphi(z_1^\circ,z_2^\circ) \wedge \phi(z_2^\circ,z_3^\circ) \ldots \wedge \Theta(z_{2n}^\circ,y^\circ))$$
$$= \cup_{n=1}^{\infty} \Theta_n(x^\circ,y^\circ) = \kappa(x^\circ,y^\circ)$$

This implies κ is fuzzy congruence of an MS-algebra L.

Finally let τ be any fuzzy congruence relation such that $\phi \subseteq \tau$ and $\varphi \subseteq \tau$. We prove that $\kappa \subseteq \tau$.

$$\kappa(x,y) = \cup_{n=1}^{\infty} \Theta_n(x,y)$$
$$= \sup_{z_1,z_2,\ldots,z_{2n}} (\phi(x,z_1) \wedge \varphi(z_1,z_2) \wedge \phi(z_2,z_3) \ldots \wedge \Theta(z_{2n},y))$$
$$\leq \sup_{z_1,z_2,\ldots,z_{2n}} (\tau(x,z_1) \wedge \tau(z_1,z_2) \wedge \ldots \wedge \tau(z_{2n},y))$$
$$= \tau_n(x,y)$$

Thus κ is the smallest fuzzy congruence such that $\phi \subseteq \tau$ and $\varphi \subseteq \tau$.
Hence $\phi \vee \varphi = \cup_{n=0}^{\infty} \Theta_n$.

Definition 3.10. Let L be an MS-algebra. The smallest fuzzy congruence generated by the fuzzy relation ϕ on L is defined by $\overline{\Theta}(\phi) = \cap\{\vartheta \in FC(L): \phi \subseteq \vartheta\}$.
If $\phi = \mu \times \mu$ is the product of fuzzy ideal μ by itself, where$(\mu \times \mu)(x,y) = \mu(x) \wedge \mu(y)$ for all$(x,y) \in L \times L$. We write $\overline{\Theta}[\mu]$ instead of $\overline{\Theta}(\phi)$.

Theorem 3.11. Let ϕ is a fuzzy relation of L. Then $\overline{\Theta}(\phi)(x,y) = sup\{\alpha: (x,y) \in \Theta(\phi_\alpha)\}$ for any $(x,y) \in L \times L$ and $\alpha \in [0,1]$.
Proof: Let $\varphi(x,y) = \sup\{\alpha: (x,y) \in \Theta(\phi_\alpha)\}$ for any $(x,y) \in L \times L$ and $\alpha \in [0,1]$. We see that $\varphi = \overline{\Theta}(\phi)$. First we prove that φ is a fuzzy congruence of L.

1. $\varphi(x,y) = \sup\{\alpha: (x,x) \in \Theta(\phi_\alpha)\} = 1$.
2. $\varphi(x,y) = \sup\{\alpha: (x,y) \in \Theta(\phi_\alpha)\} = \sup\{\alpha: (y,x) \in \Theta(\phi_\alpha)\} = \varphi(y,x)$
3. $\varphi(x,y) \wedge \varphi(y,z) = \sup\{\alpha: (x,y) \in \Theta(\phi_\alpha)\} \wedge \sup\{\lambda: (y,z) \in \Theta(\phi_\lambda)\} = \sup\{\alpha \wedge \lambda: (x,y) \in \Theta(\phi_\alpha),\ (y,z) \in \Theta(\phi_\lambda)\} \leq \sup\{\alpha \wedge \lambda: (x,y) \in \Theta(\phi_{\alpha\wedge\lambda}),\ (y,z) \in \Theta(\phi_{\alpha\wedge\lambda})\} \leq \sup\{\alpha \wedge \lambda: (x,z) \in \Theta(\phi_{\alpha\wedge\lambda})\} = \varphi(x,z)$.
4. $\varphi(x,y) \wedge \varphi(w,z) = \sup\{\alpha: (x,y) \in \Theta(\phi_\alpha)\} \wedge \sup\{\lambda: (w,z) \in \Theta(\phi_\lambda)\} = \sup\{\alpha \wedge \lambda: (x,y) \in \Theta(\phi_\alpha),\ (w,z) \in \Theta(\phi_\lambda)\} \leq \sup\{\alpha \wedge \lambda: (x,y) \in \Theta(\phi_{\alpha\wedge\lambda}),\ (w,z) \in \Theta(\phi_{\alpha\wedge\lambda})\} \leq \sup\{\alpha \wedge \lambda: (x \wedge w, y \wedge z) \in \Theta(\phi_{\alpha\wedge\lambda})\} = \varphi(x \wedge w, y \wedge z)$.

   Similarly $\varphi(x,y) \wedge \varphi(y,z) \leq \varphi(x \vee w, y \vee z)$.
5. $\varphi(x,y) = \sup\{\alpha: (x,y) \in \Theta(\phi_\alpha)\} \leq \sup\{\alpha: (x^\circ, y^\circ) \in \Theta(\phi_\alpha)\} = \varphi(x^\circ, y^\circ)$. Thus φ is a fuzzy congruence of MS-algebra L. Next, we prove that $\phi \subseteq \varphi$.

Now $\phi(x,y) = \{\alpha: (x,y) \in \phi_\alpha\} \leq \{\alpha: (x,y) \in \Theta(\phi_\alpha)\} = \varphi(x,y)$. Hence $\phi \subseteq \varphi$.

Finally, let τ be any fuzzy congruence of an MS-algebra L such that $\phi \subseteq \tau$. We see that $\varphi \subseteq \tau$.

If $\phi \subseteq \tau$, then $\phi_\alpha \subseteq \tau_\alpha$ and so $\Theta(\phi_\alpha) \subseteq \Theta(\tau_\alpha) = \tau_\alpha$. Now $\varphi(x,y) = \{\alpha: (x,y) \in \Theta(\phi_\alpha)\} \leq \{\alpha: (x,y) \in \tau_\alpha\} = \tau(x,y)$. Thus $\varphi = \overline{\Theta}(\phi.)$

4. Fuzzy ideals and congruences of MS-algebras

In this section, we study the relation of fuzzy ideals and fuzzy congruences of MS-algebras.

Let μ be a fuzzy ideal of an MS-algebra L. Define $\mu_\circ^{\gtrless}(x) = \sup\{\mu(i): i^\circ \leq x, i \in L\}$ for all $x \in L$ and $\mu_{\circ\circ}(x) = \sup\{\mu(i): x \leq i^{\circ\circ}\}$. Easily verify that $\mu_\circ^{\gtrless}$ is a fuzzy filter of L and $\mu_{\circ\circ}$ is a fuzzy ideal of L.

The following is due to Yuan Bo and Wu Wangming [9].

Definition 4.1. Let μ be a fuzzy ideal of a distributive lattice L. A fuzzy relation $\Theta_{lat}[\mu]$ on L defined by setting $\Theta_{lat}[\mu](x,y) = \{\mu(i): x \vee i = y \vee i,\ i \in L\}$ for all $x, y \in L$ is called the fuzzy relation induced by μ.

It can be easily verify that it is the smallest lattice fuzzy congruence containing the product fuzzy ideals $\mu \times \mu$ of $L \times L$. Using this definition the smallest lattice fuzzy congruence contains $\mu_{\circ\circ} \times \mu_{\circ\circ}$ defined as $\Theta[\mu_{\circ\circ}]$ on L by setting $\overline{\Theta}_{lat}[\mu_{\circ\circ}](x,y) = \{\mu_{\circ\circ}(i): x \vee i = y \vee i,\ i \in L\}$ for all $x, y \in L$.

Lemma 4.2. Let μ be a fuzzy filter of a distributive lattice L. Let us define fuzzy relation $\overline{\Theta}_{lat}[\mu]$ on L by, $\overline{\Theta}_{lat}[\mu](x,y) = \{\mu(i): x \wedge i = y \wedge i,\ i \in L\}$ for all $x, y \in L$. Then $\overline{\Theta}_{lat}[\mu]$ is the smallest fuzzy congruence on$L$containing the product of fuzzy filter $\mu \times \mu$ $of\, L \times L$.

By Lemma 4.2, let μ be a fuzzy ideal of an MS-algebra L. The smallest fuzzy relation containing the product of fuzzy filter $\mu_\circ^{\geq} \times \mu_\circ^{\geq}$ is $\overline{\Theta}[\mu_\circ^{\geq}](x,y) = \{\mu_\circ^{\geq}(i): x \wedge i = y \wedge i,\ i \in L\}$ for all $x, y \in L$. Then $\Theta_{\mu_\circ^{\geq}}$ is the smallest lattice fuzzy congruence containing the product of fuzzy filter $\mu_\circ^{\geq} \times \mu_\circ^{\geq}$.

Lemma 4.3. Let μ be a fuzzy ideal of an MS-algebra L. Then the following conditions are hold.

1. $\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x,y) \leq \overline{\Theta}_{lat}[\mu_{\circ\circ}](x^\circ, y^\circ)$ for all $x, y \in L$
2. $\overline{\Theta}_{lat}[\mu_{\circ\circ}](x,y) \leq \overline{\Theta}_{lat}[\mu_\circ^{\geq}](x^\circ, y^\circ)$ for all $x, y \in L$

Proof: (1) $\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x,y) = \sup\{(\mu_\circ^{\geq})(i): x \wedge i = y \wedge i,\ i, j \in L\} = \sup\{(\mu)(j),\ j^\circ \leq i: x \wedge i = y \wedge i,\ i \in L\} \leq \sup\{(\mu)(j),\ i^\circ \leq j^{\circ\circ}: x^\circ \vee i^\circ = y^\circ \vee i^\circ,\ i, j \in L\} = \overline{\Theta}_{lat}[\mu_{\circ\circ}](x^\circ, y^\circ)$. Similarly (2) holds.

Theorem 4.4. For every fuzzy ideal$\mu$of an MS-algebra L. Then $\overline{\Theta}[\mu] = \overline{\Theta}_{lat}[\mu_\circ^{\geq}] \vee \overline{\Theta}_{lat}[\mu_{\circ\circ}]$.

Proof: Let $\varphi = \overline{\Theta}_{lat}[\mu_\circ^{\geq}] \vee \overline{\Theta}_{lat}[\mu_{\circ\circ}]$. We prove that φ is the smallest fuzzy congruence containing $\mu \times \mu$. Easily verify that $\mu \times \mu \subseteq \varphi$. Now we prove that φ is a fuzzy congruence of L. Since $\overline{\Theta}_{lat}[\mu_\circ^{\geq}]$ and $\overline{\Theta}_{lat}[\mu_{\circ\circ}]$ are lattice fuzzy congruences, φ is a lattice fuzzy congruence of L.

By Theorem 3.9 and Lemma 4.3,

$\varphi(x,y) = \sup_{z_1, z_2, \ldots, z_{2n}}(\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x, z_1) \wedge \overline{\Theta}_{lat}[\mu_{\circ\circ}](z_1, z_2) \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](z_2, z_3) \wedge \ldots \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](z_{2n}, y)) \leq \sup_{z_1^\circ, z_2^\circ, \ldots, z_{2n}^\circ}(\overline{\Theta}_{lat}[\mu_{\circ\circ}](x^\circ, z_1^\circ) \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](z_1^\circ, z_2^\circ) \wedge \overline{\Theta}_{lat}[\mu_{\circ\circ}](z_2^\circ, z_3^\circ) \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](z_3^\circ, z_4^\circ) \wedge \ldots \wedge \overline{\Theta}_{lat}[\mu_{\circ\circ}](z_{2n}^\circ, y^\circ)) = \varphi(x^\circ, y^\circ)$.

Thus φ is fuzzy congruence of MS-algebra L. Finally, we see that φ is the smallest fuzzy congruence containing $\mu \times \mu$. Since φ a fuzzy congruence containing $\mu \times \mu$, $\overline{\Theta}[\mu] \subseteq \varphi$.

By Theorem 2.5 and Theorem 3.12, $\overline{\Theta}_{lat}[\mu_{\circ\circ}](x,y) = \sup\{\alpha: (x,y) \in \Theta_{lat}[(\mu_{\circ\circ})_\alpha]\} \leq \sup\{\alpha: (x,y) \in \Theta_{lat}[\mu_\alpha] = \overline{\Theta}[\mu](x,y)$.

Similarly $\overline{\Theta}_{lat}[\mu_\circ^{\geq}] \subseteq \overline{\Theta}[\mu]$. Thus $\overline{\Theta}_{lat}[\mu_\circ^{\geq}] \vee \overline{\Theta}_{lat}[\mu_{\circ\circ}] \subseteq \overline{\Theta}[\mu]$. Hence $\varphi \subseteq \overline{\Theta}[\mu]$. Therefore $\varphi = \overline{\Theta}[\mu]$.

The following Theorem is other useful description of $\overline{\Theta}[\mu]$.

Theorem 4.5. For every fuzzy ideal μ of an MS-algebra L, $\overline{\Theta}[\mu](x,y) = sup\{\mu_{\circ\circ}(i) \wedge \mu_\circ^{\geq}(j): (x \vee i) \wedge j = (y \vee i) \wedge j,\quad i, j \in L\}$ for any $x, y \in L$.

Proof: Put $\varphi(x,y) = \sup\{\mu_{\circ\circ}(i) \wedge \mu_\circ^{\geq}(j): (x \vee i) \wedge j = (y \vee i) \wedge j,\quad i, j \in L\}$ for any $x, y \in L$. We show that φ is fuzzy congruence of lattice L. For any $x, y, z, i, j, a, b, c, d \in L$

1. $\varphi(x,x) = \sup\{\mu_{\circ\circ}(i) \wedge \mu_{\circ}^{\geq}(j) : (x \vee i) \wedge j = (x \vee i) \wedge j,\quad i,j \in L\} \geq \mu_{\circ\circ}(0) \wedge \mu_{\circ}^{\geq}(1) = 1$. Since in particular $(x \vee 0) \wedge 1 = (x \vee 0) \wedge 1$. Hence $\varphi(x,x) = 1$ for any $x \in L$.

2. $\varphi(x,y) = \sup\{\mu_{\circ\circ}(i) \wedge \mu_{\circ}^{\geq}(j) : (x \vee i) \wedge j = (y \vee i) \wedge j,\quad i,j \in L\} = \sup\{\mu_{\circ\circ}(i) \wedge \mu_{\circ}^{\geq}(j) : (y \vee i) \wedge j = (x \vee i) \wedge j,\quad i,j \in L\} = \varphi(y,x)$

3. If $(x \vee a) \wedge b = (y \vee a) \wedge b,\ (y \vee c) \wedge d = (z \vee c) \wedge d$, then

$((x \vee (a \vee c)) \wedge (b \wedge d) = \{(x \vee a) \vee c) \wedge b\} \wedge d) = (\{(x \vee a) \wedge b\} \vee \{c \wedge b\}) \wedge d = (\{(y \vee a) \wedge b\} \vee \{c \wedge b\}) \wedge d = (\{y \vee (a \vee c)\} \wedge (b \wedge d) = (\{(y \vee c) \wedge d\} \vee \{a \wedge d\}) \wedge b = (\{(z \vee c) \wedge d\} \vee \{a \wedge d\}) \wedge b = (\{z \vee (a \vee c)\} \wedge (b \wedge d) = \{(z \vee c) \wedge d\} \vee \{a \wedge d\}) \wedge d = (z \vee (a \vee c)) \wedge (b \wedge d)$.

$\varphi(x,y) \wedge \varphi(y,z) = \sup\{\mu_{\circ\circ}(a) \wedge \mu_{\circ}^{\geq}(b) : (x \vee a) \wedge b = (y \vee a) \wedge b,\quad a,b \in L\} \wedge \sup\{\mu_{\circ\circ}(c) \wedge \mu_{\circ}^{\geq}(d) : (y \vee c) \wedge d = (z \vee c) \wedge d,\quad c,d \in L\} = \sup\{\mu_{\circ\circ}(a) \wedge \mu_{\circ\circ}(c) \wedge \mu_{\circ}^{\geq}(b) \wedge \mu_{\circ}^{\geq}(d) : (x \vee a) \wedge b = (y \vee a) \wedge b, (y \vee c) \wedge d = (z \vee c) \wedge d a,b,c,d \in L\} \leq \sup\{\mu_{\circ\circ}(a \vee c) \wedge \mu_{\circ}^{\geq}(b \wedge d) : (x \vee (a \vee c)) \wedge (b \wedge d) = ((y \vee (a \vee c)) \wedge (b \wedge d),\ a,b,c,d \in L\} = \varphi(x,z)$.

Hence $\varphi(x,y) \wedge \varphi(y,z) \leq \varphi(x,z)$.

4. If $(x \vee a) \wedge b = (y \vee a) \wedge b,\ (z \vee c) \wedge d = (w \vee c) \wedge d$, then after routine work we have got, $((x \wedge z) \vee (a \vee c)) \wedge (b \wedge d) = ((y \wedge w) \vee (a \vee c)) \wedge (b \wedge d)$.

$\varphi(x,y) \wedge \varphi(z,w) = \sup\{\mu_{\circ\circ}(a) \wedge \mu_{\circ}^{\geq}(b) : (x \vee a) \wedge b = (y \vee a) \wedge b,\ a,b \in L\} \wedge \sup\{\mu_{\circ\circ}(c) \wedge \mu_{\circ}^{\geq}(d) : (z \vee c) \wedge d = (w \vee c) \wedge d,\quad c,d \in L\} = \sup\{\mu_{\circ\circ}(a) \wedge \mu_{\circ\circ}(c) \wedge \mu_{\circ}^{\geq}(b) \wedge \mu_{\circ}^{\geq}(d) : (x \vee a) \wedge b = (y \vee a) \wedge b, (z \vee c) \wedge d = (w \vee c) \wedge d a,b,c,d \in L\} \leq \sup\{\mu_{\circ\circ}(a \vee c) \wedge \mu_{\circ}^{\geq}(b \wedge d) : ((x \wedge z) \vee (a \vee c)) \wedge (b \wedge d) = ((y \wedge w) \vee (a \vee c)) \wedge (b \wedge d),\ a,b,c,d \in L\} = \varphi(x \wedge z, y \wedge w)$. Hence $\varphi(x,y) \wedge \varphi(z,w) = \varphi(x \wedge z, y \wedge w)$.

Similarly $\varphi(x,y) \wedge \varphi(z,w) = \varphi(x \vee z, y \vee w)$. This implies φ is a lattice fuzzy congruence of L.

5. Now for every fuzzy ideal μ of an MS-algebra L, $\varphi(x,y) = \sup\{\mu_{\circ\circ}(i) \wedge \mu_{\circ}^{\geq}(j) : (x \vee i) \wedge j = (y \vee i) \wedge j,\quad i,j \in L\} \leq \sup\{\mu_{\circ}^{\geq}(j) : x \wedge j = x \wedge (x \vee i) \wedge j = x \wedge (y \vee i) \wedge j,\quad i,j \in L\} = \overline{\Theta}_{lat}(\mu_{\circ}^{\geq})(x, x \wedge (y \vee i))$.

6. Again, $\varphi(x,y) = \sup\{\mu_{\circ\circ}(i) \wedge \mu_{\circ}^{\geq}(j) : (x \vee i) \wedge j = (y \vee i) \wedge j,\quad i,j \in L\} \leq \sup\{\mu_{\circ\circ}(i) : (x \wedge (y \vee i)) \vee i = (x \vee i) \wedge ((y \vee i) \vee i) = (x \wedge y) \vee i,\quad i,j \in L\} = \overline{\Theta}_{lat}(\mu_{\circ\circ})(x \wedge (y \vee i), x \wedge y)$.

From (5) and (6) $\varphi(x,y) \leq \overline{\Theta}_{lat}(\mu_{\circ}^{\geq})(x, x \wedge (y \vee i)) \wedge \overline{\Theta}_{lat}(\mu_{\circ\circ})(x \wedge (y \vee i), x \wedge y)) \leq (\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ}))(x, x \wedge y)$.

Similarly $\varphi(x,y) \leq (\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ}))(x \wedge y, y)$. Again $\varphi(x,y) \leq (\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ}))(x, x \wedge y) \wedge (\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ}))(x \wedge y, y) \leq \sup_{x \wedge y}\{(\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ}))(x, x \wedge y) \wedge (\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ}))(x \wedge y, y)\} \leq ((\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ})) \circ (\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ}))(x,y) \leq (\overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \vee \overline{\Theta}_{lat}(\mu_{\circ\circ}))(x,y) = \overline{\Theta}[\mu](x,y)$.

Conversely, $\overline{\Theta}(\mu)(x,y) = (\overline{\Theta}_{lat}(\mu_{\circ\circ}) \vee \overline{\Theta}_{lat}(\mu_{\circ}^{\geq}))(x,y) = ((\overline{\Theta}_{lat}(\mu_{\circ\circ}) \circ \overline{\Theta}_{lat}(\mu_{\circ}^{\geq}) \circ (\overline{\Theta}_{lat}(\mu_{\circ\circ}) \circ \ldots \circ \overline{\Theta}_{lat}(\mu_{\circ\circ})(x,y) = \sup_{z_1,z_2,z_3,\ldots,z_{2n}}\{(\overline{\Theta}_{lat}(\mu_{\circ\circ})(x,z_1) \wedge \overline{\Theta}_{lat}(\mu_{\circ}^{\geq})(z_1,z_2) \wedge (\overline{\Theta}_{lat}(\mu_{\circ\circ}))(z_2,z_3) \wedge \ldots\} = \{\mu_{\circ\circ}(i_1) \wedge \mu_{\circ\circ}(i_2) \wedge \ldots \mu_{\circ\circ}(i_n) \wedge \mu_{\circ}^{\geq}(j_1) \wedge \mu_{\circ}^{\geq}(j_2) \wedge \ldots \mu_{\circ}^{\geq}(j_n) : x \vee i_1 = z_1 \vee i_1, z_1 \wedge j_1 = z_2 \wedge j_1, z_2 \vee i_2 = z_3 \vee i_2, z_3 \wedge$

$j_2 = z_4 \wedge j_2, \ldots,$ for each $k, i_k, j_k \in L\} \leq \{\mu_{\circ\circ}(\vee i_k) \wedge \mu_\circ^{\gtrless}(\wedge j_k): (x \vee \vee i_k) \wedge \wedge j_k) = (y \vee \vee i_k) \wedge \wedge j_k): \vee i_k, \wedge j_k \in L\} = \varphi(x, y)$.

Because for each $i_k,\ j_k \in L$, we have $\vee i_k,\ \wedge j_k \in L$.

$$
\begin{aligned}
(x \vee \vee i_n) \wedge \wedge j_n &= (z_1 \vee \vee i_n) \wedge \wedge j_n \\
&= (z_1 \wedge \wedge j_n) \vee (\vee i_n \wedge \wedge j_n) \\
&= (z_2 \wedge \wedge j_n) \vee (\vee i_n \wedge \wedge j_n) \\
&= (z_2 \vee \vee i_n) \wedge \wedge j_n \\
&= \ldots \\
&= (y \vee \vee i_n) \wedge \wedge j_n
\end{aligned}
$$

Hence $\overline{\Theta}(\mu)(x, y) = \varphi(x, y)$. Therefore $\overline{\Theta}(\mu)(x, y) = \sup\{\mu_{\circ\circ}(i) \wedge \mu_\circ^{\gtrless}(j): (x \vee i) \wedge j = (y \vee i) \wedge j,\quad i, j \in L\}$ for any $x, y \in L$.

Definition 4.6. A kernel fuzzy ideal μ of an MS-algebra L is a fuzzy ideal μ of L for which there exists a fuzzy congruence ϕ of L such that $\mu = ker\phi$ *i.e.* μ is the kernel fuzzy ideal of ϕ, where $ker\phi(x) = \phi(x, 0)$ for all $x \in L$.

Definition 4.7. A cokernel fuzzy filter δ of an MS-algebra L is a fuzzy filter δ of L for which there exists a fuzzy congruence φ of L such that $\delta = coker\varphi$ *i.e.* δ is the cokernel fuzzy filter of φ, where $coker\varphi(x) = \varphi(x, 1)$ for all $x \in L$.

Lemma 4.8. Let μ be a kernel fuzzy ideal of an MS-algebra L. Then $\mu = ker\overline{\Theta}_{lat}[\mu]$.

Theorem 4.9. A fuzzy ideal μ of an MS-algebra$L$is a kernel fuzzy ideal if and only if

1. $\mu_{\circ\circ} = \mu$ (i.e $\mu(i^{\circ\circ}) \geq \mu(i)$), for all $i \in L$,
2. $\mu(x \wedge j) \wedge \mu_\circ^{\gtrless}(j) \leq \mu(x)$ for all $x, j \in L$.

Proof: If μ is a kernel a fuzzy ideal of fuzzy congruence ψ on L, then for $\forall x \in L$, $\mu(x) = \psi(x, 0) \leq \psi(x^{\circ\circ}, 0) = \mu(x^{\circ\circ})$.*

Now $\mu_{\circ\circ}(x) = \sup\{\mu(i): i \leq x^{\circ\circ}\} \leq \mu(x^{\circ\circ}) \leq \mu(x)$. (Since μ is an ideal and and by (*)). Clearly $\mu \subseteq \mu_{\circ\circ}$. This implies $\mu = \mu_{\circ\circ}$.

For any $x, j \in L$, $\mu_\circ^{\gtrless}(j) = \sup\{\mu(i): i^\circ \leq j\} = \sup\{\psi(i, 0): i^\circ \leq j\} \leq \psi(i^\circ, 1) = \psi(i^\circ, 1) \leq \psi(j, 1) \leq \psi(j \wedge x, x)$.

Now $\mu(x \wedge j) \wedge \mu_\circ^{\gtrless}(j) \leq \psi(j \wedge x, x) \wedge \psi(x \wedge j, 0) \leq \psi(x, 0) = \mu(x)$.

Conversely, suppose that (1) and (2) hold. By Theorem 4.5 for any $x \in L$,

$$
\begin{aligned}
\overline{\Theta}[\mu](x, 0) &= \sup\{\mu_{\circ\circ}(i) \wedge \mu_\circ^{\gtrless}(j): (x \vee i) \wedge j = i \wedge j,\quad i, j \in L\} \\
&= \sup\{\mu(i) \wedge \mu_\circ^{\gtrless}(j) ☹ x \vee i) \wedge j = i \wedge j,\quad i, j \in L\}\ (by\ (1)\ \mu_{\circ\circ} = \mu) \\
&\leq \mu((x \vee i) \wedge \mathrm{j}) \wedge \mu_\circ^{\gtrless}(j)\ \ for\ \ (x \vee i) \wedge j = i \wedge j,\quad i, j \in L \\
&\leq \mu(x \vee i)\ \ by (2) \\
&\leq \mu(x)
\end{aligned}
$$

This implies $ker\overline{\Theta}[\mu] \subseteq \mu$. By Lemma 4.8, $\mu = ker\overline{\Theta}_{lat}[\mu]$ and $\overline{\Theta}_{lat}[\mu] \subseteq \overline{\Theta}[\mu]$, then $\mu = ker\overline{\Theta}_{lat}[\mu] \subseteq ker\overline{\Theta}[\mu]$. Hence $ker\overline{\Theta}[\mu] = \mu$. Thus μ is kernel fuzzy ideal of L.

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REFERENCES

1. M.Attallah, Principal fuzzy congruence relations of distributive lattices, *J. Egypt. Math. Soc.*, 9(1) (2000) 165-171.
2. T.S.Blyth and J.C.Varlet, *Ockham Algebras*, Oxford University Press, *(1994)*.
3. T.S.Blyth and J.C.Varlet, On a common abstraction of de Morgan and Stone algebras, *Proc. Roy. Soc. Edinburgh Sect.*, 94A (1983) 301-308.
4. G.Gratzer, *General Lattice Theory*, Academic Press, NewYork, (1978).
5. V.Murali, Fuzzy equivalence relations, *Fuzzy Sets and Systems*, 30 (1989) 155-163.
6. A.Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.,* 35 (1971) 512-517.
7. M.A.Samhan, Fuzzy congrunces on Semigroups, *Information Sciences,* 74 (1993) 165-175.
8. U.M.Swamy and D.V.Raju, Fuzzy ideals and congruences of lattices, *Fuzzy Sets and Systems,* 95 (1998) 249-253.
9. B.Yuan and W.Wu, Fuzzy ideals on a distributive lattice, *Fuzzy Sets and Systems*, 35 (1990) 231-240.
10. L.A.Zadeh, Similarity relations and fuzzy ordering, *Information Sciences,* 3 (1971) 177-200.