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Control and Synchronization of a Fractional-order Chaotic Financial System

Jiming Zheng[1] **and *Yang Qiu***[2]

School of Science, Chongqing University of Posts and Telecommunications
Chongqing-400065, Chongqing, China.
[1]Email: zhengjm@cqupt.edu.cn; [2]Email: amosqy@126.com

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Abstract. In this paper, we study the synchronization control problem of a class of fractional-order chaotic financial systems with uncertain parameters. Based on the theory of fractional order stability and adaptive control method, a simple fractional-order synchronous controller of two fractional-order chaotic financial systems is designed, and a simple theoretical analysis is given to prove that the error system is stable. Then two fractional-order hybrid synchronization methods are proposed. Finally numerical simulation demonstrates the validity of methods.

Keywords: Fractional order; Chaotic financial system; Chaotic control; Synchronization

AMS Mathematics Subject Classification (2010): 34H10

1. Introduction

Since Lorenz chaotic system was proposed in 1963, as an interesting nonlinear phenomenon, chaos has applications in many areas such as secure communication, economics, many other engineering systems, and so on. While in the financial field, some scholars have established a chaotic financial system with interest rates, investment needs and price indices as state variables [2]. The research object of the financial system is the nonlinear relationship in the system, which is embodied in chaotic synchronization and control. Chaotic synchronization of the financial system has become a hot issue in the research and application of chaos theory, and it is also a hot issue in the theory and method of economics. As in the literature [3], for the chaotic financial system with uncertain parameters, the generalized projection synchronization of the drive system is realized. Guo Rongwei et al. [4] studied the synchronization and anti-synchronization of the financial system.

In recent years, more and more scholars have begun to study the dynamic properties, control and synchronization of fractional-order chaotic systems, and have obtained some research results. In [5], two implementation schemes for the synchronization of fractional financial systems are given. The literature [6] studies the synchronization problem of a fractional-order system with uncertain factors, while the literature [7] uses the fractional order skillfully. The nature of the continuous time-varying Lyapunov function proves the synchronization of fractional-order financial systems with time-delay. In fact, whether it is a fractional order system or an integer order system, the research and application of stability theory is crucial, and it is an important basis for judging whether a system can operate normally. It is also an important basis to prove chaotic synchronization. At present, the main chaotic synchronization criterion is based on the synchronization criterion of Lyapunov exponent and the synchronization criterion based on Lyapunov function. The literature [5-7] is based on the synchronization criterion of the Lyapunov function. However, in the fractional order system, due to the more complicated system, there are of course some methods that are different from the integer order stability judgment. The literature [8,9] separately calculates the range of the value of the coefficient matrix of the analysis system, and then judges the stability criterion of the fractional-order linear system; Hu Jianbing [10-12] is a long-term commitment to fractional nonlinear stability. Research; literature [13] and literature [14] judge the stability of the system according to the Lyapunov equation; recently, Huang et al. [15] proposed a new method for judging the stability of fractional-order systems, that is, constructing a suitable function first. Then analyze the positive and negative of its eigenvalues to determine whether the system is stable. The hybrid synchronization problem of chaotic systems has only appeared in recent years. Hybrid synchronization, that is, synchronization and anti-synchronization coexist, is actually a generalization of synchronization and projection synchronization. In the literature [16] and [17], the definition of hybrid synchronization is given respectively. Further, the literature [18] designed a simple linear hybrid synchronous controller, and proposed a synchronization criterion, but the conditions of the criterion is related to the state variables of the drive system.

This paper will study the synchronization control problem of fractional-order chaotic financial systems. The rest of the paper is organized as follows. In section 2, the definitions of common fractional derivatives, the general form of linear fractional chaotic systems, the definitions of stability and synchronization of fractional-order chaotic systems are given respectively. Further, we study the chaotic control of a fractional-order chaotic financial system in section 3. The synchronization of the two fractional financial systems is discussed, and the controller of synchronizing two systems is designed in section 4. In section 5, two methods for realizing the hybrid synchronization of two different fractional financial systems are given respectively. Conclusions are finally drawn in Section 6.

2. Preliminaries

In this section, we introduce the common definition of fractional-order derivative, and then in fractional-order chaotic system, the definition of stability and synchronization are given. Further, a class of fractional-order chaotic financial system is given and its chaotic behavior and stability with fixed points are analyzed as follows.

Definition 2.1. [5,6] The Caputo fractional-order derivative is defined as follows:

$$ {}_{a}^{C}D_{t}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\xi)^{n-\alpha-1}u\left(\xi\right)d\xi , $$

where the order α of u(t) satisfies $n-1<\alpha<n$, n is the smallest integer greater than α. In order to facilitate the representation of fractional order operators, the following is replaced by D^{α}.

Therefore, according to the definition 1, we can get the general form of a fractional-order chaotic system as follows:

$$D^{\alpha}x(t) = Ax(t),\ 0<\alpha<1,\ x(0) = x_0, \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$ is the state variables of chaotic system(1), and shortly, $x = (x_1, x_2, \cdots, x_n)^T$. A is the coefficient matrix of system.

Definition 2.2. [19] (Fractional-order stability) For the above fractional chaotic system(1), the stability of system is defined as:

(1) System (1) is stable, if and only if $\forall x_0 \in R^n$, $\exists M>0$, $\forall t>0$, $\| x \| < M$ is established;

(2) System (1) is asymptotically stable, if and only if $\lim\limits_{t\to\infty} \| x(t) \| = 0$ is established.

Lemma 2.1. [20] If $x(t) \in R^n$ is a continuous and differentiable vector-value function, then for any time instant $t \ge t_0$, we have

$$\frac{1}{2} D^{\alpha} x^T(t)x(t) \le x^T(t) D^{\alpha} x(t),$$

where $0<\alpha<1$.

Let system (1) be the master system, then the response system is

$$D^{\alpha} y(t) = Ay(t) + u,\ 0<\alpha<1,\ y(0) = y_0, \tag{2}$$

where $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T \in R^n$ is the state variables of chaotic system(1), and shortly, $y = (y_1, y_2, \cdots, y_n)^T$. A is the coefficient matrix of system. u is the controller to desigen.

Definition 2.3. For any initial values, let the error is $e_i = y_i - x_i, i = 1,2,\cdots,n$. If $\lim\limits_{t\to\infty} \| e_i \| = \lim\limits_{t\to\infty} \| y_i - x_i \| = 0$, then system(1) and system (2) are called synchronization; and if $\lim\limits_{t\to\infty} \| E_i \| = \lim\limits_{t\to\infty} \| y_i + x_i \| = 0$, then system(1) and system (2) are called anti-synchronization.

Definition 2.4. [17] For any initial values, e_i is synchronous error, and E_i is anti-synchronous error. If the following conditions are satisfied

$$\lim_{t\to\infty} \| e_i \| = \lim_{t\to\infty} \| y_i - x_i \| = 0, i = 1,2,\cdots,m$$
$$\lim_{t\to\infty} \| E_i \| = \lim_{t\to\infty} \| y_i + x_i \| = 0, i = m+1,\cdots,n$$

Then we call system(1) and (2) is hybrid synchronization.

The paper study a class of fractional-order chaotic financial system[2] as follows:

$$\begin{aligned} D^{\alpha}x_1 &= (x_2 - a)x_1 + x_3 \\ D^{\alpha}x_2 &= 1 - bx_2 - x_1^2 \\ D^{\alpha}x_3 &= -x_1 - cx_3 \end{aligned} \tag{3}$$

where $(x_1, x_2, x_3) \in R^3$ is state variables , and stands for the interest rate, the investment demand, and the price index respectively; the constants a,b,c are positive system parameter and represent the saving amount, the cost per investment and the elasticity of demand of the commercial markets respectively.

Let the right side of the system (3) equation is zero, the equilibrium point of the system (3) is obtained as follow

$$P_0 = (0, \frac{1}{b}, 0), \quad P_{1,2} = (\pm\delta, \frac{ac+1}{c}, \mp\frac{1}{c}\delta)$$

where $\delta = \sqrt{\frac{c-b-abc}{c}}$ .This moment, $c-b-abc > 0$, otherwise, there is only one fixed point. Then Jacobi matrix of system(3) is

$$J(x) = \begin{pmatrix} x_2 - a & x_1 & 1 \\ -2x_1 & -b & 0 \\ -1 & 0 & -c \end{pmatrix}$$

Here only the more complex equilibrium points $P_{1,2}$ are discussed, and then the eigenvalues of the Jacobi matrix are $\lambda_1 = \text{-}0.7464, \lambda_{2,3} = 0.2732 \pm 1.2382\mathrm{i}$. It can be seen that at the equilibrium point, the system (3) is unstable; similarly, the stability of system (3) with fixed point P_2 is also unstable. As shown as Figure1,the system(3) is chaotic.

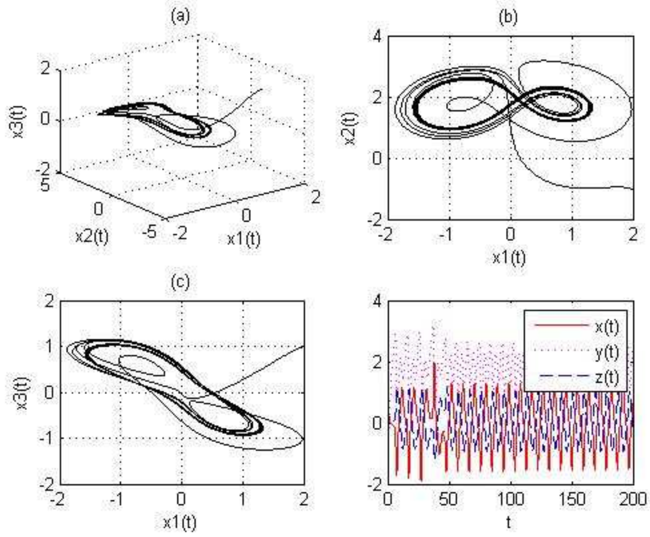

Figure 1: Chaos in system (3)

3. Control of fractional-order chaotic financial system

In this section, we discuss control of fractional-order chaotic financial system and give some results as follows.

Firstly, let system (3) change into the following:

$$D^{\alpha}x = Ax + f(x) + I_0$$
$$= \begin{pmatrix} -a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix} x + \begin{pmatrix} x_1x_2 \\ -x_1^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{4}$$

where A is the coefficient matrix of system (3), $f(x) = (f_1(x), f_2(x), f_3(x))^T$ is nonlinear function term, and then we can give the following result.

Theorem 1. For system (4), if $I_0 + u = 0$, u is a controller, and then system (4) will be stable.

Proof: We can let the Lyapunov function is $V = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$, and easy to know V is positive definite function. We can proof Fractional-order function $D^{\alpha}V$ is negative definite function as follows. Then, we have

$$\begin{aligned} D^{\alpha}V &= D^{\alpha}(\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)) \\ &\le x_1D^{\alpha}x_1 + x_2D^{\alpha}x_2 + x_3D^{\alpha}x_3 \\ &= x_1((x_2 - a)x_1 + x_3) + x_2(-bx_2 - x_1^2) + x_3(-x_1 - cx_3) \\ &= -ax_1^2 - bx_2^2 - cx_3^2 - x_1x_3 \\ &\le -(a-1)x_1^2 - bx_2^2 - (c-1)x_3^2 \end{aligned}$$

When $a - 1 \ge 0, b \ge 0, c - 1 \ge 0$, system(4) will be stable; further, if the above inequality is a strict inequality, then system(4) will be asymptotically stable. This moment, the three state variables tend to fixed points as shown as Figure 2.

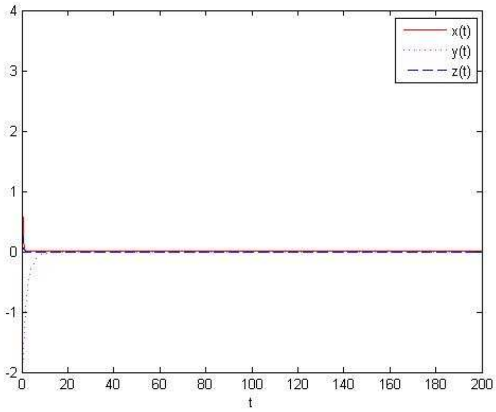

Figure 2: Control of a fractional-order chaotic financial system

4. Synchronization of fractional-order chaotic financial systems

In this section, in different initial values, synchronization of two fractional-order chaotic financial systems will be investigated as follows.

Firstly, let fractional-order chaotic financial systems (3) be the master system, and the salve system is as follows:

$$\begin{aligned} D^{\alpha} y_1 &= (y_2 - a) y_1 + y_3 + u_1 \\ D^{\alpha} y_2 &= 1 - b y_2 - y_1^2 + u_2 \\ D^{\alpha} y_3 &= -y_1 - c y_3 + u_3 \end{aligned} \tag{5}$$

Then let the error is $e_i = y_i - x_i, i = 1, 2, 3$, the error system of system (3) and system (5) is calculated as

$$\begin{aligned} D^{\alpha} e_1 &= g_1(e, x) = -a e_1 + e_3 + x_1 e_2 + x_2 e_1 + e_1 e_2 + u_1 \\ D^{\alpha} e_2 &= g_2(e, x) = -b e_2 - 2 x_1 e_1 - e_1^2 + u_2 \\ D^{\alpha} e_3 &= g_3(e, x) = -e_1 - c e_3 + u_3 \end{aligned} \tag{6}$$

We introduce the feedback gain k_1 satisfies $D^{\alpha} k_1 = -e_2^2$.

Theorem 2. The two systems (3) and systems (5) with different initial values will be synchronous when the controller $u = (k_1 e_1, 0, 0)^T$.

To prove theorem 2, we first introduce a conclusion as follows.

Lemma 2. [4] If system (6) is chaotic, then there exist $\lambda_i > 0$, which satisfies

$$e_i g_i(e, x) \le \lambda_i e_i, \ i = 1, 2, \ldots, n,$$

Proof: We can let the Lyapunov function is $V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2)$, and easy to know V is positive definite function. From Lemma1 and Lemma2, we can prove fractional-order function $D^{\alpha} V$ is negative definite function as follows. Then, when $M \ge \max\{\lambda_i\}$, we have

$$\begin{aligned} D^{\alpha} V &= D^{\alpha}\left(\frac{1}{2}(e_1^2 + e_2^2 + e_3^2)\right) \\ &\le e_1 D^{\alpha} e_1 + e_2 D^{\alpha} e_2 + e_3 D^{\alpha} e_3 + (k_1 + L) D^{\alpha} k_1 \\ &= e_1(g_1(e, x) + k_1) + e_2 g_2(e, x) + e_3 g_3(e, x) - (k_1 + L) e_1^2 \\ &\le M(e_1^2 + e_2^2 + e_3^2) - L e_1^2 \end{aligned}$$

When $L \ge M \dfrac{e_1^2 + e_2^2 + e_3^2}{e_1^2}$, system is stable; further, if the above inequality is a strict inequality, then system(4) will be asymptotically stable. This moment, the three state variables tend to fixed points as shown as Figure 3.

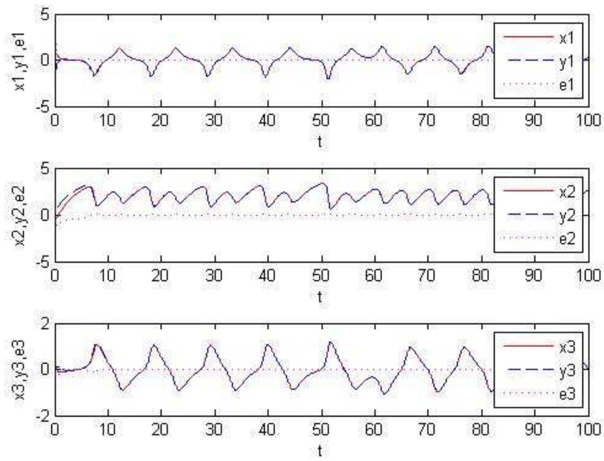

Figure 3: Synchronization of fractional-order chaotic financial systems

5. Coexistence of synchronization and anti-synchronization of fractional-order chaotic financial systems

In this section, we will discuss coexistence of synchronization and anti-synchronization of fractional-order chaotic financial systems, some results will be given as follows.

Firstly, let system (3) and (5) be drive system and response system, respectively. Secondly, let error variables $E_i = y_i + x_i, i = 1,3$, $e_2 = y_2 - x_2$, then we can get error system

$$\begin{aligned} D^{\alpha}E_1 &= -aE_1 - x_1e_2 + x_2E_1 + E_1e_2 + E_3 + u_1 \\ D^{\alpha}E_2 &= -be_2 - E_1^2 + 2x_1E_1 + u_2 \\ D^{\alpha}E_3 &= -E_1 - cE_3 + u_3 \end{aligned} \tag{8}$$

The hybrid synchronization criterion given in [18], which depends on the value range of the state variables of the drive system, while here the fractional synchronization controller in the hybrid synchronous error system (9) can be

$$\begin{cases} u_1 = k_1E_1 - x_2E_1 + x_1e_2 \\ u_2 = k_2e_2 \\ u_3 = k_3E_3 \end{cases} \tag{9}$$

Therefore,

$$\begin{cases} D^{\alpha}E_1 = -(a-k_1)E_1 + (x_1 - y_1)e_2 + E_3 \\ D^{\alpha}e_2 = -(b+k_2)e_2 - (x_1 - y_1)E_1 \\ D^{\alpha}E_3 = -E_1 - (c-k_3)E_3 \end{cases} . \tag{10}$$

$$D^{\alpha}E = BE,$$

where

$$E = (E_1, e_2, E_3)^T,$$

$$B=\begin{pmatrix} -(a-k_1) & x_1-y_1 & 1 \\ -x_1+y_1 & -(b+k_2) & 0 \\ -1 & 0 & -(c-k_3) \end{pmatrix}。$$

According to Lemma 3

Theorem 3. If the following conditions are satisfied
(1) $a-k_1>0$，(2) $b+k_2>0$，(3) $c-k_3>0$，
then the drive system (3) and the response system (5) can be realize hybrid synchronization.

To prove Theorem 3, we introduce Lemma 3 firstly as follows.

Lemma 3. [10] For Linear time-varying fractional order chaotic system,

$$D^{\alpha}x(t)=F(x)=A(t)x(t)\,,\,t\ge t_0 \tag{11}$$

Let $H(t)=A^T(t)+A(t)$, where $H(t)$ is positive symmetry matrix. For ease of expression, the following is a brief that is $H=A^T+A$, then the sufficient condition that the system (11) is asymptotically stable with equilibrium point $x_e=0$ is that the maximum eigenvalue of H satisfies $\lambda_{\max}<0$.Then we prove theorem 3 as follows.

Proof:

$$B^T+B=\begin{pmatrix} -2(a-k_1) & 0 & 0 \\ 0 & -2(b+k_2) & 0 \\ 0 & 0 & -2(c-k_3) \end{pmatrix}$$

Its determinant can be expressed as

$$|B^T+B|=\begin{vmatrix} -2(a-k_1) & 0 & 0 \\ 0 & -2(b+k_2) & 0 \\ 0 & 0 & -2(c-k_3) \end{vmatrix}$$
$$=-8(c-k_3)(a-k_1)(b+k_2)<0.$$

Hence, in order to achieve the hybrid synchronization, we let $a-k_1>0$, $b+k_2>0$, $c-k_3>0$, that is, B^T+B is negative definite function. Therefore, the drive system (3) and the response system (5) are to achieve the hybrid synchronization.

We take $a=1.0,\ b=0.2,\ c=1$, $k_1=0.5,\ k_2=1,\ k_3=0.8$, and $x_1(0)=2$, $x_2(0)=-1$, $x_3(0)=1$, $y_1(0)=-2.5$, $y_2(0)=2$, $y_3(0)=-1$.The error variables E_1,e_2,E_3 vary with time t are shown as Figure 4.

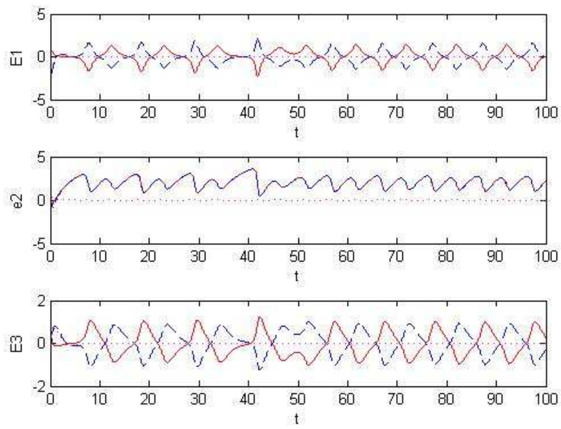

Figure 4: The error system realize hybrid synchronization with controller u

Corollary 5.1. If $k_3 = 0$, $c > 0$, $0 < k_1 < a$ and $b + k_2 > 0$, then the drive system (7) and the response system (8) to achieve the hybrid synchronization.

We take $a = 1.0,\ b = 0.2,\ c = 1$, $k_1 = 0.5,\ k_2 = 1,\ k_3 = 0.8$, which satisfies the conditions of corollary 5.1, and $x_1(0) = 2$, $x_2(0) = -1$, $x_3(0) = 1$, $y_1(0) = -2.5$, $y_2(0) = 2$, $y_3(0) = -1$.The error variables E_1, e_2, E_3 vary with time t are shown as Figure 5.

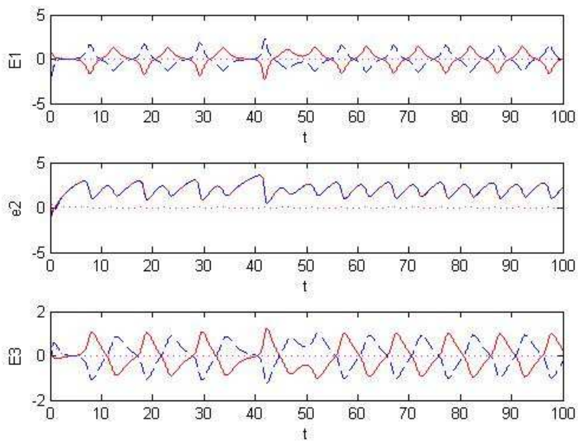

Figure 5: Hybrid synchronization with parameters $k_1 = 2,\ k_2 = 1.8,\ k_3 = 0$

Corollary 5.2. If $k_2 = k_3 = 0$, $c > 0$, $b > 0$ and $0 < k_1 < a$, then the drive system (7) and the response system (8) to achieve the hybrid synchronization.

We take $a=1.0,\ b=0.2,\ c=1, k_1=k_2=0,\ k_3=0.8$, which satisfies the conditions of corollary 5.1, and $x_1(0)=2$, $x_2(0)=-1$, $x_3(0)=1$, $y_1(0)=-2.5$, $y_2(0)=2$, $y_3(0)=-1$.The error variables E_1, e_2, E_3 vary with time t are shown as Figure 6.

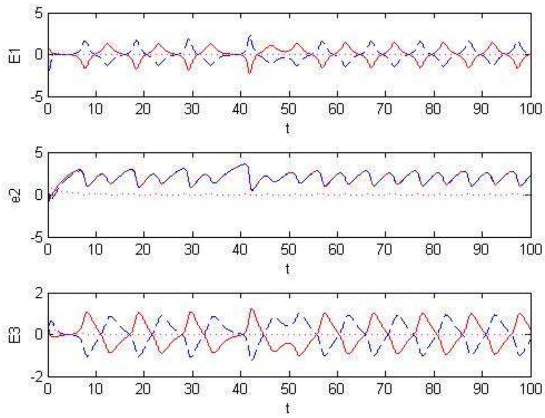

Figure 6: Hybrid synchronization with parameters $k_1=2,\ k_2=1.8,\ k_3=0$

However, in order to realize hybrid synchronization, we can construct a controller applied with adaptive control, which makes the construction of the controller be more simple.

Thus, the controller u can be designed as $u_2=-k_2e_2$, and the controlled error system is given as

$$D^\alpha E_1 = h_1(E,x) = -aE_1 - x_1e_2 + x_2E_1 + E_1e_2 + E_3$$
$$D^\alpha E_2 = h_2(e,x) + u_2 = -be_2 - E_1^2 + 2x_1E_1 + u_2$$
$$D^\alpha E_3 = h_3(E,x) = -E_1 - cE_3$$

According to lemma 2, we have

$$E_1h_1(e,E,x) \le m_1E_1^2$$
$$e_2h_2(e,E,x) \le m_2e_2^2$$
$$E_3h_3(e,E,x) \le m_3E_3^2$$

Let $M=\max\{m_i, i=1,2,3\}$.

Theorem 3. The two systems (3) and systems (5) with different initial values will realize hybrid synchronization when the controller $u=(0,k_1e_2,0)^T$.

Proof: We can let the Lyapunov function is $V=\frac{1}{2}(E_1^2+e_2^2+E_3^2)$, and easy to know V is positive definite function. From Lemma1, we can prove fractional-order function $D^\alpha V$ is negative definite function as follows. Then, we have

$$\begin{aligned} D^{\alpha}V &= D^{\alpha}(\frac{1}{2}(E_1^2 + e_2^2 + E_3^2 + (k_2 + L)^2)) \\ &\le E_1 D^{\alpha} E_1 + e_2 D^{\alpha} e_2 + E_3 D^{\alpha} E_3 + (k_2 + L) D^{\alpha} k_2 \\ &= E_1 h_1(e,E,x) + E_3 h_3(e,E,x) + e_2(h_2(e,E,x) + k_2) - (k_2 + L)e_2^2 \\ &\le M(E_1^2 + e_2^2 + E_3^2) - Le_2^2 \end{aligned}$$

When $L \ge M\frac{E_1^2 + e_2^2 + E_3^2}{e_2^2}$, system is stable; further, if the above inequality is a strict inequality, then system(4) will be asymptotically stable. This moment, the three state variables tend to fixed points as shown as Figure 7.

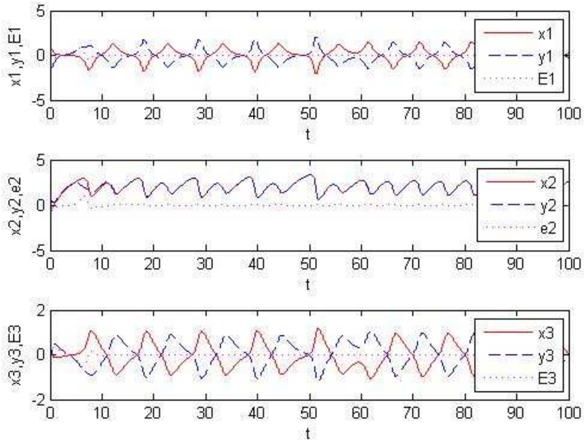

Figure 7: Hybrid synchronization of system (3) and (5)

6. Conclusion

The paper study synchronization of a class of fractional-order chaotic financial systems. Firstly, the paper transforms the fractional-order chaotic financial system and realizes control of system. Then, using theory of adaptive control, synchronization of two systems is realized. For the problem, the condition of the hybrid synchronization should not be related to the value of the state variable of the drive system, can be solved by a simple synchronous controller. Further, another hybrid synchronization controller with a simpler structure through theory of adaptive control is proposed. However, the paper choose the same structure system be response system, more general methods cannot be given, and time delay also is not considered. These may be future work to study.

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