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Some Properties of q-Rung Orthopair Fuzzy Derivatives and Indefinite Integrals

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Abstract. In this paper, we first present the chain rule and the form invariance of differential for q-rung orthopair fuzzy functions (q-ROFFs). In addition, we obtain the general formula of indefinite integrals of q-ROFFs by ordinary differential equations. Next, we give the substitution rule of indefinite integrals for q-ROFFs. Meanwhile, we give some examples to verify above-mentioned formula, such as the chain rule and the substitution rule.

Keywords: q-rung orthopair fuzzy function; q-rung orthopair fuzzy derivatives; q-rung orthopair fuzzy indefinite integrals.

AMS Mathematics Subject Classification (2010): 03E72, 26E50, 62B86

1. Introduction

The concept of fuzzy set (FS) was proposed by Zadeh [1] in 1965, which can describe the grade of membership effectively. However, there are some shortcomings in FS, such as the hesitation information cannot be shown. In 1983, Atanassov [2] extended the FS to the intuitionistic fuzzy set (IFS), which contains a non-membership function, a membership function and a hesitancy function. Xu and Yager [3] defined the intuitionistic fuzzy numbers (IFNs) and discussed some aggregation operators, which has been used wildly for a long time. Some theories have been established for fuzzy clustering and decision making [4,5,6,7,8,9]. On the other hand, Xu and Lei [10] defined the intuitionistic fuzzy function (IFF), and studied in detail their continuities, derivatives and differentials. Subsequently, the definite integral and double integral of IFF have been studied [11,12] by Xu and Lei.

But constraints do exist in IFS that membership plus non-membership, positive but less than 1, would not be satisfied if we make membership equal to 0.7 and non-membership equal to 0.6, which means that the theory of IFNs is invalid. To solve this problem, Yager [13] proposed the concept of Pythagorean fuzzy sets (PFSs), which contains the above-mentioned value. Obviously, PFSs is the extension of FSs. Moreover, Yager [14] proposed the q-rung orthopair fuzzy sets (q-ROFSs), which contains more pairs of value.

Liu and Wang [15] defined some basic operations of q-rung orthopair fuzzy functions (q-ROFFs) including addition, multiplication, scalar-multiplication and power operation. Recently, Xu and Gao [16] defined the subtraction and division operations, and they also revealed the partial order relations of q-ROFNs, which are "$\leq$" and "$\trianglelefteq$". Moreover, they studied the derivatives and differentials of the q-ROFFs, and discussed some particular properties of derivatives similar to those in mathematical analysis. They disclosed the chain's rule on derivatives, but there isn't a detailed proof, and they did not show the form invariance of differential in q-rung orthopair fuzzy calculus.

In this paper, we give a detailed proof for chain's rule based on the definition of right and left derivatives. And investigate the inverse operation of the derivative which is indefinite integral.

The paper is organized as the following: in Section 2, some basic concepts and operations of q-ROFNs are given. Two important theorems are presented in Section 3, concluding the chain's rule and the form invariance of differential in q-rung orthopair fuzzy calculus. Moreover, there is an example which can illustrate the chain's rule and show some important results. We also give the general formula of indefinite integrals of q-ROFFs in Section 4, besides, some properties are shown which are similar to mathematical analysis. There are also some examples presented which make these theorems well-understand. In Section 5 we give some conclusion remaking.

2. Preliminaries

In this section, we introduce some basic concepts of q-ROFNs, and some operations which are utilized frequently.

Definition 2.1. [14] Let X be a fixed non-empty set, then

$$A=\left\{\left\langle x,\mu_A(x),v_A(x)\right\rangle \middle| x\in X\right\},$$

is called a q-ROFS, which satisfies

(1) $0\leq \mu_A(x)\leq 1$.

(2) $0\leq v_A(x)\leq 1$.

(3) $0\leq (\mu_A(x))^q+(v_A(x))^q\leq 1$ for any $x\in X$.

Definition 2.2. [15][21] Let $\alpha=\langle \mu_\alpha,v_\alpha\rangle$ and $\beta=\langle \mu_\beta,v_\beta\rangle$ be two q-ROFNs, then the addition, multiplication, subtraction and division operations are defined as follows:

$$\alpha\oplus\beta=\left\langle \left(\mu_\alpha^q+\mu_\beta^q-\mu_\alpha^q\mu_\beta^q\right)^{\frac{1}{q}},v_\alpha v_\beta\right\rangle,$$

$$\alpha\otimes\beta=\left\langle \mu_\alpha\mu_\beta,\left(v_\alpha^q+v_\beta^q-v_\alpha^q v_\beta^q\right)^{\frac{1}{q}}\right\rangle.$$

$$\beta\ominus\alpha=\left\langle \left(\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\frac{1}{q}},\frac{v_\beta}{v_\alpha}\right\rangle,$$

which satisfies

$$0 \le \frac{v_\beta}{v_\alpha} \le \frac{\left(1-\mu_\beta^q\right)^{\frac{1}{q}}}{\left(1-\mu_\alpha^q\right)^{\frac{1}{q}}} \le 1 .$$

And

$$\beta \oslash \alpha = \left\langle \frac{\mu_\beta}{\mu_\alpha}, \left(\frac{v_\beta^q - v_\alpha^q}{1-v_\alpha^q}\right)^{\frac{1}{q}} \right\rangle ,$$

which satisfies

$$0 \le \frac{\mu_\beta}{\mu_\alpha} \le \frac{\left(1-v_\beta^q\right)^{\frac{1}{q}}}{\left(1-v_\alpha^q\right)^{\frac{1}{q}}} \le 1 .$$

Remark 2.1. If $\lambda > 0$, we have

$$\lambda\alpha = \alpha \oplus \alpha \oplus \cdots \oplus \alpha = \left\langle \left(1-\left(1-\mu_\alpha^q\right)^\lambda\right)^{\frac{1}{q}}, v_\alpha^\lambda \right\rangle ,$$

and

$$\alpha^\lambda = \alpha \otimes \alpha \otimes \cdots \otimes \alpha = \left\langle \mu_\alpha^\lambda, \left(1-\left(1-v_\alpha^q\right)^\lambda\right)^{\frac{1}{q}} \right\rangle .$$

Definition 2.3. [21] Let $\alpha_i = \left\langle \mu_{\alpha_i}, v_{\alpha_i} \right\rangle$ ($i = 1,2,3$) be three q-ROFNs, then

$$\alpha_1 \le \alpha_2 \ if \ \mu_{\alpha_1} \le \mu_{\alpha_2} \ and \ v_{\alpha_1} \ge v_{\alpha_2} ,$$
$$\alpha_1 \ge \alpha_2 \ if \ \mu_{\alpha_1} \ge \mu_{\alpha_2} \ and \ v_{\alpha_1} \le v_{\alpha_2} ,$$
$$\alpha_1 = \alpha_2 \ if \ \alpha_1 \le \alpha_2 \ and \ \alpha_1 \ge \alpha_2 .$$

Definition 2.4. [21] If there exists a q-ROFN α_3, such that $\alpha_1 \oplus \alpha_3 = \alpha_2$, then we define that α_1 is less than or equal to α_2, denoted by $\alpha_1 \trianglelefteq \alpha_2$. In particular, $\alpha_1 \triangleleft \alpha_2$ if $\alpha_3 \ne \langle 0,1 \rangle$.

Definition 2.5. [21] Let $\alpha = \langle \mu_\alpha, v_\alpha \rangle$ and $\beta = \langle \mu_\beta, v_\beta \rangle$ be two q-ROFNs, and S be the set $S = \left\{ (\mu, v) \middle| \mu \ge 0, v \ge 0, \mu^q + v^q \le 1 \right\}$, we define

$$S_\oplus(\alpha) = \left\{ \beta \oplus \alpha \in S \middle| \beta \in S \right\} ,$$
$$S_\ominus(\alpha) = \left\{ \alpha \ominus \beta \in S \middle| \beta \in S \right\} ,$$
$$S_\otimes(\alpha) = \left\{ \beta \otimes \alpha \in S \middle| \beta \in S \right\} ,$$
$$S_\oslash(\alpha) = \left\{ \alpha \oslash \beta \in S \middle| \beta \in S \right\} .$$

Definition 2.6. [21] The basic arithmetics are defined as follow:
(1) Let $\alpha_1 \in S_\oplus(\alpha_2)$ and $\alpha_3 \in S_\oplus(\alpha_4)$, then

$$\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1 ,$$
$$(\alpha_1 \oplus \alpha_2) \ominus (\alpha_3 \oplus \alpha_4) = (\alpha_1 \ominus \alpha_3) \oplus (\alpha_2 \oplus \alpha_4) ,$$

$$(\alpha_1 \ominus \alpha_2) \ominus (\alpha_3 \ominus \alpha_4) = (\alpha_1 \ominus \alpha_3) \ominus (\alpha_2 \ominus \alpha_4).$$

(2) Let $\alpha_1 \in S_{\otimes}(\alpha_2)$, $\alpha_1 \in S_{\otimes}(\alpha_3)$, $\alpha_3 \in S_{\otimes}(\alpha_2)$ and $\alpha_2 \in S_{\otimes}(\alpha_4)$, then we have

$$(\alpha_1 \otimes \alpha_2) \oslash (\alpha_2 \otimes \alpha_3) = \alpha_1 \oslash \alpha_3,$$

$$(\alpha_1 \oslash \alpha_2) \oslash (\alpha_3 \oslash \alpha_2) = \alpha_1 \oslash \alpha_3,$$

$$(\alpha_1 \oslash \alpha_2) \otimes (\alpha_2 \oslash \alpha_4) = \alpha_1 \oslash \alpha_4.$$

(3) For $\lambda_i > 0$ ($i = 1, 2$),

$$\lambda_1(\alpha_1 \oplus \alpha_2) = \lambda_1\alpha_1 \oplus \lambda_1\alpha_2,$$

$$\lambda_1(\alpha_1 \ominus \alpha_2) = \lambda_1\alpha_1 \ominus \lambda_1\alpha_2,$$

$$(\lambda_1 + \lambda_2)\alpha_1 = \lambda_1\alpha_1 \oplus \lambda_1\alpha_2,$$

$$(\lambda_1 - \lambda_2)\alpha_1 = \lambda_1\alpha_1 \ominus \lambda_1\alpha_2.$$

Proof: We just prove the third part in (2) and (3). The case of (2):

$$(\alpha_1 \oslash \alpha_2) \otimes (\alpha_2 \oslash \alpha_4) = \left\langle \frac{\mu_1}{\mu_2}, \left(\frac{v_1^q - v_2^q}{1 - v_2^q}\right)^{1/q} \right\rangle \otimes \left\langle \frac{\mu_2}{\mu_4}, \left(\frac{v_2^q - v_4^q}{1 - v_4^q}\right)^{1/q} \right\rangle = \left\langle \frac{\mu_1}{\mu_4}, \left(\frac{v_1^q - v_4^q}{1 - v_4^q}\right)^{1/q} \right\rangle = \alpha_1 \oslash \alpha_4.$$

The case of (3) is shown below:
We can calculate:

$$\lambda_1\alpha_1 = \left\langle \left(1 - \left(1 - \mu_{\alpha_1}^q\right)^{\lambda_1}\right)^{1/q}, v_{\alpha_1}^{\lambda_1} \right\rangle \text{ and } \lambda_2\alpha_1 = \left\langle \left(1 - \left(1 - \mu_{\alpha_1}^q\right)^{\lambda_2}\right)^{1/q}, v_{\alpha_1}^{\lambda_2} \right\rangle,$$

then we get

$$\lambda_1\alpha_1 \oplus \lambda_1\alpha_2 = \left\langle \left(1 - \left(1 - \mu_{\alpha_1}^q\right)^{\lambda_1 + \lambda_2}\right)^{1/q}, v_{\alpha_1}^{\lambda_1 + \lambda_2} \right\rangle = (\lambda_1 + \lambda_2)\alpha_1,$$

which completes this proof.

Definition 2.7. [21] Let $\varphi(\alpha) = \langle f_\alpha, g_\alpha \rangle$ be a q-ROFF, then

$$\frac{d\varphi}{d\alpha}\Big|_\alpha = \lim_{\beta \ominus \alpha \to \langle 0,1 \rangle} \frac{\varphi(\beta) \ominus \varphi(\alpha)}{\beta \ominus \alpha},$$

is called the derivative of φ at α. In particular,

$$\frac{d\varphi}{d\alpha}\Big|_\alpha = \left\langle \left(\frac{1 - \mu_\alpha^q}{1 - f_\alpha^q} \cdot \frac{f_\alpha^{q-1}}{\mu_\alpha^{q-1}} \cdot \frac{\partial f_\alpha}{\partial \mu_\alpha}\right)^{1/q}, \left(1 - \frac{v_\alpha}{g_\alpha} \cdot \frac{\partial g_\alpha}{\partial v_\alpha}\right)^{1/q} \right\rangle.$$

Theorem 2.1. [21] Let $\alpha \oplus \Delta\alpha \in S_{\oplus}^{\varphi}$ with $\alpha \in S$.

If $\frac{d\varphi}{d\alpha}\Big|_\alpha$ exists, then the q-ROFF φ is differentiable. In particular,

$$d\varphi(\alpha) = \frac{d\varphi}{d\alpha}\Big|_\alpha \otimes \Delta\alpha.$$

In order to express conveniently, we use x to denote general variable $\langle \mu, v \rangle$ in the following sections.

3. Some properties of q-rung orthopair fuzzy derivatives

We define the derivatives of q-ROFFs by the right and left derivatives, which is similar to ones in the mathematical analysis.

Definition 3.1. Let $\varphi(x)=\langle f(\mu),g(v)\rangle$ be a q-ROFF in the set $E\subseteq S$, x and y be both q-ROFNs in E. If $x\trianglelefteq y$, $\varphi(x)\trianglelefteq\varphi(y)$ holds, then we call $\varphi(x)$ a monotonically increasing q-ROFF.

Definition 3.2. Let $\varphi(x)=\langle f(\mu),g(v)\rangle$ be a monotonically increasing q-ROFF defined in the set E, x be an accumulation point of E (Maybe there is $x\notin E$). If $\lim\limits_{y\to x^{\oplus},y\in E}\dfrac{\varphi(y)\ominus\varphi(x)}{y\ominus x}$ is still a q-ROFN, then we call it the right derivative of $\varphi(x)$ at x, denoted by $\varphi'_{\oplus}(x)$. Similarly, $\lim\limits_{y\to x^{\ominus},y\in E}\dfrac{\varphi(x)\ominus\varphi(y)}{x\ominus y}$ is the left derivative if it is a q-ROFF, which can be denoted by $\varphi'_{\ominus}(x)$. In addition, if the left and the right derivatives are both q-ROFNs and equal to each other, then we call $\varphi(x)$ is derivable at x and $\lim\limits_{y\to x,y\in E}\left|\dfrac{\varphi(y)\ominus\varphi(x)}{y\ominus x}\right|$ is the derivative of $\varphi(x)$ at x, denoted by $\dfrac{d\varphi(x)}{dx}$.

Remark 3.1. The $\left|\dfrac{\varphi(y)\ominus\varphi(x)}{y\ominus x}\right|$ implies

$$\left|\frac{\varphi(y)\ominus\varphi(x)}{y\ominus x}\right|=\begin{cases}\dfrac{\varphi(y)\ominus\varphi(x)}{y\ominus x} & y\in S_{\oplus}(x)\\ \dfrac{\varphi(x)\ominus\varphi(y)}{x\ominus y} & y\in S_{\ominus}(x)\end{cases}$$

Remark 3.2. The monotonically increasing condition is essential, which can make the left and right derivatives meaningful. If $\varphi(x)$ is derivable at x_0, then there must exist a neighborhood

$$U(x_0,\varepsilon)=\left\{x\,\middle\|\,x\ominus x_0\right|\triangleleft\varepsilon,x\in S_{\oplus}(x_0)\cup S_{\ominus}(x_0)\},$$

and $\varphi(x)$ is monotonically increasing in $U(x_0,\varepsilon)$.

Remark 3.3. It is unnecessary that $\varphi(x)$ is monotonically increasing all over the universe $S=\left\{(\mu,v)\,\middle|\,\mu\geq 0,v\geq 0,\mu^q+v^q\leq 1\right\}$, if $\varphi(x)$ is monotonically increasing in a subset $\mathrm{K}\subseteq S$ and derivable in K, then we call $\varphi(x)$ is derivable in K.

It is worth noting that the Definition 3.2 is reasonable. Because if the left and the right derivatives are both q-ROFNs and equal to each other, we can only calculate the right derivative, whose process is similar to [21], and then we will get the same formula of derivative in q-ROFF.

Based on the Definition 3.2, we can easily prove the chain rule for q-ROFFs.

Theorem 3.1. Let $\varphi(x)$ and $\Phi(x)$ be q-ROFFs and derivable, then

$$\frac{d\Phi(\varphi(t))}{dt} = \frac{d\Phi(\varphi(t))}{d\varphi(t)} \otimes \frac{d\varphi(t)}{dt},$$

which is called the chain's rule in q-rung orthopair fuzzy calculus.

Proof: We consider the right derivative of $\Phi(\varphi(t))$, which is

$$\lim_{\Delta t \to O^{\oplus}} \frac{\Phi(\varphi(t+\Delta t)) \ominus \Phi(\varphi(t))}{\Delta t} = \lim_{\Delta t \to O^{\oplus}} \frac{\Phi(\varphi(t+\Delta t)) \ominus \Phi(\varphi(t))}{\varphi(t+\Delta t) \ominus \varphi(t)} \otimes \lim_{\Delta t \to O^{\oplus}} \frac{\varphi(t+\Delta t) \ominus \varphi(t)}{\Delta t}$$

$$= \frac{d\Phi(\varphi(t))}{d\varphi(t)} \otimes \frac{d\varphi(t)}{dt} = \Phi'_{\oplus}(\varphi(t)) \otimes \varphi'_{\oplus}(t),$$

where $\Phi'_{\oplus}(\varphi(t))$ represent $\Phi'_{\oplus}(x)$ at $x = \varphi(t)$, then we can get the left derivative of $\Phi(\varphi(t))$ in the same way, which is

$$\Phi'_{\ominus}(\varphi(t)) \otimes \varphi'_{\ominus}(t).$$

According to $\varphi(x)$ and $\Phi(x)$ are derivable, we have

$$\Phi'_{\oplus}(\varphi(t)) = \Phi'_{\ominus}(\varphi(t)) \text{ and } \varphi'_{\oplus}(t) = \varphi'_{\ominus}(t),$$

which means

$$\Phi'_{\oplus}(\varphi(t)) \otimes \varphi'_{\oplus}(t) = \Phi'_{\ominus}(\varphi(t)) \otimes \varphi'_{\ominus}(t) = \Phi'(\varphi(t)) \otimes \varphi'(t) = \frac{d\Phi(\varphi(t))}{d\varphi(t)} \otimes \frac{d\varphi(t)}{dt}.$$

Thus, $\Phi(\varphi(t))$ is derivable, and

$$\frac{d\Phi(\varphi(t))}{dt} = \frac{d\Phi(\varphi(t))}{d\varphi(t)} \otimes \frac{d\varphi(t)}{dt}.$$

The proof is completed.

Example 3.1. Let $\Phi(x) = \lambda_1 x \oplus C$ and $\varphi(t) = \lambda_2 t$ be two q-ROFFs, assume that $C = \langle \mu_c, v_c \rangle$. Then we calculate $\frac{d\Phi(\varphi(t))}{dt}$ in two ways:

(1) Based on the basic arithmetics, we have

$$\Phi(\varphi(t)) = \lambda_1\lambda_2 t \oplus C = \left\langle \left(1-\left(1-\mu^q\right)^{\lambda_1\lambda_2}\right)^{1/q}, v^{\lambda_1\lambda_2} \right\rangle \oplus \langle \mu_c, v_c \rangle = \left\langle \left(1-\left(1-\mu_c^q\right)\left(1-\mu^q\right)^{\lambda_1\lambda_2}\right)^{1/q}, v_c v^{\lambda_1\lambda_2} \right\rangle.$$

If we denote $\left(1-\left(1-\mu_c^q\right)\left(1-\mu^q\right)^{\lambda_1\lambda_2}\right)^{1/q}$ and $v_c v^{\lambda_1\lambda_2}$ by f and g respectively, then according to the Definition 2.7, there is

$$\frac{d\Phi(\varphi(t))}{dt}=\left\langle\left(\frac{1-\mu^q}{(1-\mu_c^q)(1-\mu^q)^{\lambda_1\lambda_2}}\cdot\frac{\left(1-(1-\mu_c^q)(1-\mu^q)^{\lambda_1\lambda_2}\right)^{1/q}}{\mu^{q-1}\left(1-(1-\mu_c^q)(1-\mu^q)^{\lambda_1\lambda_2}\right)}\cdot\frac{\partial f}{\partial\mu}\right)^{1/q},\left(1-\frac{v}{v_c v^{\lambda_1\lambda_2}}\cdot\frac{\partial g}{\partial v}\right)^{1/q}\right\rangle,$$

where

$$\frac{\partial f}{\partial\mu}=\frac{1}{q}\cdot\frac{\left(1-(1-\mu_c^q)(1-\mu^q)^{\lambda_1\lambda_2}\right)}{\left(1-(1-\mu_c^q)(1-\mu^q)^{\lambda_1\lambda_2}\right)^{1/q}}\cdot\lambda_1\lambda_2\cdot(1-\mu_c^q)\cdot(1-\mu^q)^{\lambda_1\lambda_2-1},$$

and

$$\frac{\partial g}{\partial v}=\lambda_1\lambda_2 v_c v^{\lambda_1\lambda_2-1}.$$

Thus we get

$$\frac{d\Phi(\varphi(t))}{dt}=\left\langle(\lambda_1\lambda_2)^{1/q},(1-\lambda_1\lambda_2)^{1/q}\right\rangle.$$

(2) We use the chain's rule to calculate it, the same as above, we can obtain two results below:

$$\frac{d\Phi(x)}{dx}=\left\langle\lambda_1^{1/q},(1-\lambda_1)^{1/q}\right\rangle \text{ and } \frac{d\varphi(t)}{dt}=\left\langle\lambda_2^{1/q},(1-\lambda_2)^{1/q}\right\rangle.$$

Then we have

$$\frac{d\Phi(\varphi(t))}{dt}=\frac{d\Phi(\varphi(t))}{d\varphi(t)}\otimes\frac{d\varphi(t)}{dt}=\left\langle\lambda_1^{1/q},(1-\lambda_1)^{1/q}\right\rangle\otimes\left\langle\lambda_2^{1/q},(1-\lambda_2)^{1/q}\right\rangle=\left\langle(\lambda_1\lambda_2)^{1/q},(1-\lambda_1\lambda_2)^{1/q}\right\rangle.$$

Remark 3.4. The above process implies an important conclusion, which is

$$\frac{d(\lambda x\oplus C)}{dx}=\frac{d(\lambda x)}{dx}=\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle.$$

Theorem 3.2. Let $\Phi(\varphi(t))$ and $\varphi(t)$ be two q-ROFFs and both derivable. Then we have

$$d\Phi=\Phi_t'\otimes dt=\Phi_\varphi'\otimes\varphi_t'\otimes dt=\Phi_\varphi'\otimes d\varphi,$$

where $\Phi_t'=\frac{d\Phi(\varphi(t))}{dt}$, $\Phi_\varphi'=\frac{d\Phi(\varphi(t))}{d\varphi(t)}$ and $\varphi_t'=\frac{d\varphi(t)}{dt}$.

Proof: Based on the Theorem 2.1, Definition 2.7 and $dt=E\otimes\Delta t$, we have

$$d\varphi(t)=\frac{d\varphi(t)}{dt}\otimes dt=\varphi_t'\otimes dt,$$

then by the Theorem 3.1, we can get

$$d\Phi(\varphi(t))=\frac{d\Phi(\varphi(t))}{d\varphi(t)}\otimes\frac{d\varphi(t)}{dt}\otimes dt\Rightarrow d\Phi=\Phi_\varphi'\otimes\varphi_t'\otimes dt,$$

thus the $d\Phi=\Phi_\varphi'\otimes d\varphi$ holds.

In addition, if we consider t as the independent variable, we have $d\Phi=\Phi_t'\otimes dt$, which means that

$$d\Phi = \Phi_t' \otimes dt = \Phi_\varphi' \otimes \varphi_t' \otimes dt = \Phi_\varphi' \otimes d\varphi,$$

holds, and the proof is completed.

Remark 3.5. This theorem is called the form invariance of differential in q-rung orthopair fuzzy calculus.

4. Indefinite integrals of q-ROFFs

In this section, we discuss the indefinite integrals of q-ROFFs, which is the inverse operations of the derivatives for q-ROFFs. We will give the general formula and show some properties of indefinite integral for q-ROFFs, such as the substitution rule which is important and practical.

Theorem 4.1. Let $\varphi(x) = \langle f(\mu), g(v) \rangle$ be a q-ROFF, and $\Phi(x)$ is a primitive function of $\varphi(x)$, which satisfies $\frac{d\Phi(x)}{dx} = \varphi(x)$, then the function $\Phi(x)$ must have the following form:

$$\Phi(x) = \left\langle \left(1 - c_1 \exp\left\{ -\int \left[f(\mu) \right]^q \frac{q\mu^{q-1}}{1-\mu^q} du \right\}\right)^{1/q}, c_2 \exp\left\{ \int \frac{1-\left[g(v) \right]^q}{v} dv \right\} \right\rangle.$$

Proof: Let $\Phi(x) = \langle f(\mu), g(v) \rangle$, we need to solve two ordinary differential equations:

$$\begin{cases} \dfrac{1-u^q}{1-\left[F(\mu)\right]^q} \cdot \dfrac{\left[F(\mu)\right]^{q-1}}{u^{q-1}} \cdot \left[F(\mu)\right]' = \left[f(\mu)\right]^q \\ 1 - \dfrac{v}{G(v)} \cdot G'(v) = \left[g(v)\right]^q \end{cases} \Rightarrow \begin{cases} F(\mu) = \left(1 - c_1 \exp\left\{ -\int \left[f(\mu) \right]^q \dfrac{q\mu^{q-1}}{1-\mu^q} du \right\}\right)^{1/q} \\ G(v) = c_2 \exp\left\{ \int \dfrac{1-\left[g(v) \right]^q}{v} dv \right\} \end{cases}$$

which can be solved by the following process:

$$\frac{1-\mu^q}{1-\left[F(\mu)\right]^q} \cdot \frac{\left[F(\mu)\right]^{q-1}}{\mu^{q-1}} \cdot \left[F(\mu)\right]' = \left[f(\mu)\right]^q$$

$$\Rightarrow \frac{-q\left[F(\mu)\right]^{q-1}}{1-\left[F(\mu)\right]^q} \cdot \left[F(\mu)\right]' = \left[f(\mu)\right]^q \cdot \frac{-q\mu^{q-1}}{1-\mu^q}$$

$$\Rightarrow \frac{d\ln\left(1-\left[F(\mu)\right]^q\right)}{du} = \left[f(\mu)\right]^q \cdot \frac{-q\mu^{q-1}}{1-\mu^q}$$

$$\Rightarrow F(\mu) = \left(1 - c_1 \exp\left\{ -\int \left[f(\mu) \right]^q \frac{q\mu^{q-1}}{1-\mu^q} d\mu \right\}\right)^{1/q},$$

and

$$1 - \frac{v}{G(v)} \cdot G'(v) = \left[g(v)\right]^q \Rightarrow \frac{G'(v)}{G(v)} = \frac{1-\left[g(v)\right]^q}{v} \Rightarrow G(v) = c_2 \exp\left\{ \int \frac{1-\left[g(v) \right]^q}{v} dv \right\}.$$

The proof is completed.

We must attention that c_1 and c_2 are two integral constants, which are both real numbers such that $\Phi(x)$ is a q-ROFF. In other words, c_1 and c_2 should make the following (1)-(3) hold.

(1) $0 \le \left(1-c_1 \exp\left\{-\int\left[f(\mu)\right]^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q} \le 1$.

(2) $0 \le c_2 \exp\left\{\int \frac{1-\left[g(v)\right]^q}{v}dv\right\} \le 1$.

(3) $0 \le \left(1-c_1 \exp\left\{-\int\left[f(\mu)\right]^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right) + c_2^q \exp\left\{q\int \frac{1-\left[g(v)\right]^q}{v}dv\right\} \le 1$.

In the following, we demonstrate whether the derivative of $\Phi(x)$ is certainly $\varphi(x)$: Let F denote

$$\left(1-c_1 \exp\left\{-\int\left[f(\mu)\right]^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q},$$

and G denote

$$c_2 \exp\left\{\int \frac{1-\left[g(v)\right]^q}{v}dv\right\},$$

then

$$\frac{d\Phi(x)}{dx} = \left\langle \left(\frac{1-\mu^q}{1-F^q}\frac{F^{q-1}}{\mu^{q-1}}\frac{\partial F}{\partial \mu}\right)^{1/q}, \left(1-\frac{v}{G}\frac{\partial G}{\partial v}\right)^{1/q} \right\rangle$$

$$= \left\langle \left(\frac{\left(1-\mu^q\right)}{\mu^{q-1}c_1 \exp\left\{-\int f^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}} \cdot \frac{1-c_1 \exp\left\{-\int f^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}}{\left(1-c_1 \exp\left\{-\int f^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q}} \cdot \frac{\partial F}{\partial \mu} \right)^{1/q} \right.$$

$$\left., \left(1-\frac{v}{c_2 \exp\left\{\int \frac{1-g^q}{v}dv\right\}} \cdot \frac{\partial G}{\partial v}\right)^{1/q} \right\rangle,$$

where

$$\frac{\partial F}{\partial \mu} = \frac{1}{q}\frac{\left(1-c_1 \exp\left\{-\int f^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q}}{\left(1-c_1 \exp\left\{-\int f^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)} \cdot c_1 \exp\left\{-\int f^q \frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\} \cdot f^q \frac{q\mu^{q-1}}{1-\mu^q},$$

and

$$\frac{\partial G}{\partial v}=c_2\exp\left\{\int\frac{1-g^q}{v}dv\right\}\cdot\frac{1-g^q}{v}.$$

Then it is obvious that

$$\frac{d\Phi(x)}{dx}=\langle f(\mu),g(v)\rangle=\varphi(x).$$

Remark 4.1. According to the uniqueness of solution of ordinary differential equations, all primitive functions of $\varphi(x)$ must have the above-mentioned form. The difference between them is that the integral constants are different. Therefore, we have following theorem.

Theorem 4.2. If $\Phi(x)$ and $\Psi(x)$ are two q-ROFFs, and

$$\Phi(x)=\left\langle\left(1-c_1\exp\left\{-\int[f(\mu)]^q\frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q},c_2\exp\left\{\int\frac{1-[g(v)]^q}{v}dv\right\}\right\rangle,$$

$$\Psi(x)=\left\langle\left(1-\lambda_1c_1\exp\left\{-\int[f(\mu)]^q\frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q},\lambda_2c_2\exp\left\{\int\frac{1-[g(v)]^q}{v}dv\right\}\right\rangle.$$

Then they are both the primitive functions of $\varphi(x)$, which means that

$$\frac{d\Phi(x)}{dx}=\frac{d\Psi(x)}{dx}=\varphi(x),$$

holds.

Proof: Based on Theorem 4.1, we can easily obtain the proof of Theorem 4.2, which is omitted here.

Example 4.1. Calculate $\int\langle 1,0\rangle dx$. In fact, based on the Theorem 4.1, we can easily obtain

$$\int\langle 1,0\rangle dx$$

$$=\left\langle\left(1-c_1\exp\left\{-\int\frac{qu^{q-1}}{1-u^q}du\right\}\right)^{1/q},c_2\exp\left\{\int\frac{1}{v}dv\right\}\right\rangle=\left\langle\left(1-c_1\left(1-\mu^q\right)\right)^{1/q},c_2v\right\rangle$$

$$=\langle\mu,v\rangle\oplus\left\langle(1-c_1)^{1/q},c_2\right\rangle=x\oplus C\ .$$

Remark 4.2. We can see the q-ROFN $\langle 1,0\rangle$ is similar to constant "1" in real integral.

Theorem 4.3. If there are $\lambda_2\trianglelefteq\Phi(x)$ and $\frac{d\Phi(x)}{dx}=\varphi(x)$, then we have

$$\frac{d\Phi(x)}{dx}=\frac{d(\Phi(x)\oplus\lambda_1)}{dx}=\frac{d(\Phi(x)\ominus\lambda_2)}{dx}.$$

Proof: Let $\lambda_1=(u_1,v_1)$, then

$$\Phi(x)\oplus\lambda_1=\left\langle\left(1-(1-\mu_1)^q c_1\exp\left\{-\int\left[f(\mu)\right]^q\frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q}, v_1c_2\exp\left\{\int\frac{1-\left[g(v)\right]^q}{v}dv\right\}\right\rangle.$$

We denote $(1-\mu_1)^q$ by k_1 and v_1 by k_2, then

$$\Phi(x)\oplus\lambda_1=\left\langle\left(1-k_1c_1\exp\left\{-\int\left[f(\mu)\right]^q\frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q}, k_2c_2\exp\left\{\int\frac{1-\left[g(v)\right]^q}{v}dv\right\}\right\rangle.$$

Based on Theorem 4.2, we have

$$\frac{d\left(\Phi(x)\oplus\lambda_1\right)}{dx}=\varphi(x),$$

which completes the proof of this theorem.

Remark 4.3. This theorem shows that $\int\varphi(x)dx=\Phi(x)\oplus C$ is always holds, where C is any q-ROFN and $\Phi(x)$ is any primitive function of $\varphi(x)$.

Next, we present some properties of indefinite integrals of q-ROFFs:

Theorem 4.4. If there is $\Phi(x)=\int\varphi(x)dx$, then

$$\int\varphi(x(t))x'(t)dt=\Phi(x(t)),$$

where $\varphi(x(t))x'(t)$ represents that $\varphi(x(t))\otimes\frac{dx(t)}{dt}$.

Proof: Based on the chain rule of derivatives of the compound q-ROFFs, we have

$$\frac{d\Phi(x(t))}{dt}=\frac{d\Phi(x(t))}{d(x(t))}\otimes\frac{dx(t)}{dt}\Rightarrow\frac{d\Phi(x(t))}{dt}=\varphi(x(t))x'(t).$$

Hence, $\Phi(x(t))$ must be the primitive function of $\varphi(x(t))x'(t)$, which means that $\int\varphi(x(t))x'(t)dt=\Phi(x(t))$ holds.

Remark 4.4. This theorem can be called substitution rule for indefinite integrals.

Theorem 4.5. Let

$\varphi(x)=\langle f(\mu),g(v)\rangle$ and $\varphi_i(x)=\langle f_i(\mu),g_i(v)\rangle$ ($i=1,2,\cdots,n$),

be $n+1$ derivable q-ROFFs, then

(1) $\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\varphi(x)dx=\lambda\int\varphi(x)dx$, where $0\le\lambda\le1$.

(2) $\int\left\langle\left(\sum_{i=1}^{n}f_i^q\right)^{1/q},\left(1-\sum_{i=1}^{n}\left(1-g_i^q\right)\right)^{1/q}\right\rangle dx=\bigoplus_{i=1}^{n}\int\langle f_i,g_i\rangle dx$.

(3) $\int\left\langle\left(f_1^q-f_2^q\right)^{1/q},\left(1-\left(g_2^q-g_1^q\right)\right)^{1/q}\right\rangle dx=\int\langle f_1,g_1\rangle dx\ominus\int\langle f_2,g_2\rangle dx$.

Proof: We can utilize the chain rule of derivatives to prove (1).

$$\frac{d\lambda\int\varphi(x)dx}{dx}=\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\varphi(x).$$

Then we get that

$$\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\varphi(x)dx=\lambda\int\varphi(x)dx,$$

holds. In another way,

$$\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\varphi(x)dx=\left\langle\left(1-c_1\exp\left\{-\int\lambda f^q\frac{q\mu^{q-1}}{1-\mu^q}d\mu\right\}\right)^{1/q},c_2\exp\left\{\lambda\int\frac{1-g^q}{v}dv\right\}\right\rangle$$

$=\lambda\int\varphi(x)dx$.

We can also get that

$$\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\varphi(x)dx=\lambda\int\varphi(x)dx,$$

holds. Moreover, we can prove it based on Theorem 4.4,i.e.,

$$\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\varphi(x)dx=\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle d\left(\int\varphi(x)dx\right)=\lambda\int\varphi(x)dx.$$

The proof is completed.

Especially, if $\varphi(x)=\langle 1,0\rangle$, the result is

$$\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\langle 1,0\rangle dx=\lambda\int\langle 1,0\rangle dx=\lambda x\oplus C.$$

Similarly, (2) and (3) can be proved by the same manner, which are omitted here.

Example 4.2. Calculate $\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\left\langle k^{1/q},(1-k)^{1/q}\right\rangle dt$.

We calculate it in two ways:

(1) By the definition of multiplication we get

$$\int\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle\otimes\left\langle k^{1/q},(1-k)^{1/q}\right\rangle dt=\int\left\langle(\lambda k)^{1/q},(1-\lambda k)^{1/q}\right\rangle dt,$$

then, according to Theorem 4.5, we have

$$\int\left\langle(\lambda k)^{1/q},(1-\lambda k)^{1/q}\right\rangle dt=\lambda k\int\langle 1,0\rangle dt=\lambda kt\oplus C'.$$

(2) We can let

$$\varphi(t)=\left\langle\lambda^{1/q},(1-\lambda)^{1/q}\right\rangle \text{ and } x(t)=kt\oplus C_1,$$

then we get

$$\Phi(t)=\int\varphi(t)dt=\lambda t\oplus C_2.$$

By the Theorem 4.5, we have

$$\Phi(x)=\int\varphi(x(t))x'(t)dt=\Phi(x(t))=\lambda kt\oplus C_3,$$

where C_i ($i=1,2,3$) are three constants of q-ROFNs. Actually, both of two results are correct.

5. Conclusion and future researches

In this paper, we have provided some properties of derivatives for q-ROFFs, which contain the chain's rule and the form invariance of differential in q-rung orthopair fuzzy calculus. We also showed the validity of chain's rule by an example. In addition, we have studied the indefinite integrals of q-ROFFs which are the inverse operations of derivatives. We got the general formula of indefinite integral and investigated some basic properties of indefinite integrals. Besides, some examples were presented to show the process of calculating indefinite integral and the substitution rule's effectiveness and practicability. Moreover, several typical indefinite integrals were included in these examples, which are nonnegligible. The future researches can be focused on two directions: (1) There are more properties in q-rung orthopair fuzzy calculus which are worth exploring, such as the derivative formula for the product of two q-ROFFs and the integral formula for the sum of two q-ROFFs. (2) The definite integrals and double integrals in q-ROFFs can be investigated. Moreover, the relation between definite integral in q-ROFF and q-ROFWA can also be studied.

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