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A Generalization of Hyperbolic Special Mean and its Schur Power Convexity

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Abstract. By combining hyperbolic special mean with power average, a more general mean is studied. The necessary and sufficient conditions for the determination of Schur are also given.

Keywords: Hyperbolic function; Schur convexity; Schur power convexity

AMS Mathematics Subject Classification (2010): 52A55

1. Introduction

Let x, y be two positive numbers, then

$$M_p(x,y) = \begin{cases} (\frac{x^p}{2} + \frac{y^p}{2})^{1/p}, & p \neq 0 \\ \sqrt{xy}, & p = 0 \end{cases},$$

$$G(x,y) = M_0(x,y) = \sqrt{xy},$$

$$A(x,y) = M_1(x,y) = \frac{x+y}{2}$$

are called Power mean, geometric mean and arithmetic mean of x, y, respectively.

In 2003, American mathematical monthly, problem 11031, proposed a strong mean and Inequality conjecture as below.

Problem 11031: Let $x, y > 0$,define $M(x,y) = \ln N(x,y)$,where

$$N = N(x,y) = \frac{1+\ln(\sqrt{1+f}+\sqrt{f})}{1-\ln(\sqrt{1+f}-\sqrt{f})},\ f = f(x,y) = \frac{1}{4}(e^{\frac{2(e^x-1)}{e^x+1}} - 1)(e^{\frac{2(e^y-1)}{e^y+1}} - 1)e^{-(\frac{e^x-1}{e^x+1}+\frac{e^y-1}{e^y+1})},$$

to prove or disprove $M(x,y) \leq G(x,y)$.

Zhang [14] studied the Schur- geometric convexity of $m(x,y)=\sqrt{1+f(x,y)}+\sqrt{f(x,y)}$, and give the positive proof of the above problem.

Li and Shi [6] adapted $M(x,y)$ as $M(x,y)=2\mathrm{th}^{-1}\mathrm{sh}^{-1}\sqrt{\mathrm{sh}(\mathrm{th}\frac{x}{2})\mathrm{sh}(\mathrm{th}\frac{y}{2})}$, then by geometric convexity of $\mathrm{sh}(\mathrm{th}x)$, they also solved the above Inequality conjecture.

Shi [7] discussed the Schur- convexity and Schur- geometric convexity of $M(x,y)$

He [5] further defined $H(x,y)=2\tan^{-1}\sin^{-1}\sqrt{\sin(\tan\frac{x}{2})\sin(\tan\frac{y}{2})}$ ($x,y\in(0,2\tan^{-1}\pi/2)$), by polynomial discriminant system [9-10], discussed the Schur power convexity of $M(x,y)$ and $H(x,y)$ [2,11-13,17].

Similarly, Chen et. al. [1] defined

$$M^*(x,y)=2\mathrm{sh}^{-1}\mathrm{th}^{-1}\sqrt{\mathrm{th}(\mathrm{sh}\frac{x}{2})\mathrm{th}(\mathrm{sh}\frac{y}{2})}\ (\ x,y\in(0,+\infty)\ ),$$

$$H^*(x,y)=2\sin^{-1}\tan^{-1}\sqrt{\tan(\sin\frac{x}{2})\tan(\sin\frac{y}{2})}\ (\ x,y\in(0,\pi)\ ),$$

and then discussed their Schur power convexity.

In this paper, we generalize the above means and define

$$\begin{aligned}M_p^*(x,y)&=2\mathrm{sh}^{-1}\mathrm{th}^{-1}\left[M_p\left(\mathrm{th}(\mathrm{sh}\frac{x}{2}),\mathrm{th}(\mathrm{sh}\frac{y}{2})\right)\right]\\&=\begin{cases}2\mathrm{sh}^{-1}\mathrm{th}^{-1}\left\{[\frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{x}{2})+\frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{y}{2})]^{1/p}\right\}, & p>0\\ 2\mathrm{sh}^{-1}\mathrm{th}^{-1}\sqrt{\mathrm{th}(\mathrm{sh}\frac{x}{2})\mathrm{th}(\mathrm{sh}\frac{y}{2})}, & p=0\end{cases}\quad (\ x,y\in(0,+\infty)\ ),\end{aligned}$$

then discussed its Schur power convexity.

2. Definition and lemma

For $x=(x_1,x_2,\cdots,x_n)\in R^n$, We rearrange its components in descending order, and denote $x_{[1]}\ge x_{[2]}\ge\cdots\ge x_{[n]}$. When $x_i\le y_i$ $(i=1,\cdots,n)$, we write $x\le y$ for short.

Definition 1. [8] Suppose $x,y\in R^n$ satisfy:

(i) $\sum_{i=1}^{k}x_{[i]}\le\sum_{i=1}^{k}y_{[i]}\ (k=1,2,\cdots,n-1)$,

(?) $\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}y_i$,

then we say x is controlled by y, denoted by $x\prec y$.

Definition 2. [8] Suppose $\Omega\subset R^n$, $\varphi:\Omega\to R$,

(i) If for any $x, y \in \Omega$, $x \le y \Rightarrow \varphi(x) \le \varphi(y)$, then φ is called increasing function on Ω; if $-\varphi$ is a increasing function on Ω, then φ is called reduction function on Ω. φ is called Schur convex function on Ω

(ii) If for any $x, y \in \Omega$, $x \prec y \Rightarrow \varphi(x) \le \varphi(y)$, then φ is called Schur-convex function on Ω; if $-\varphi$ is called Schur-convex function on Ω, then φ is called Schur-concave function on Ω.

Lemma 1. [8] Let $E\left(\subseteq R^n\right)$ be a symmetric convex set with certain interior points, $f: E \to R$ is continuous and differentiable in $\operatorname{int} E$,then f is called a Schur- convex (concave) function on E if and only iff f is symmetrical on E and for all $x \in \operatorname{int} E$,

$$\left(x_1 - x_2\right)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \ge 0\left(\le 0\right). \tag{1}$$

Definition 3. [14] Let $E \subseteq R_{++}^n$,For any two-vector $x, y \in E$, when $\left(\ln x_1, \ln x_2, \cdots, \ln x_n\right) \prec \left(\ln y_1, \ln y_2, \cdots, \ln y_n\right)$, there are $f\left(x\right) \le f\left(y\right)$. Then f are the Schur-geometric convex function on E; f is the Schur- geometry concave function on E, if and only if-f is schur-geometric convex function.

Lemma 2. [16] Let $E(\subseteq R^n)$ is a symmetric set with interior points, $\left\{\left(\ln x_1, \ln x_2, \cdots, \ln x_n\right) \middle| x \in E\right\}$ is convex set, $f: E \to R$ continuation, and differentiable in the **int** E. Then the necessary and sufficient condition for f to be a convex (concave) function of Schur- geometry is f is symmetric on E, and for all $x \in \operatorname{int} E$,both of

$$\left(x_1 - x_2\right)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \ge (\le) 0. \tag{2}$$

Definition 4. [3,4] Let $E \subset R_{++}^n$, $f: E \to R$,If you take it at will $x, y \in E$,when

$$\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right) \prec \left(\frac{1}{y_1}, \frac{1}{y_2}, \cdots, \frac{1}{y_n}\right),$$

there are $f\left(x\right) \le f\left(y\right)$,Then f are the Schur- harmonic convex function on E; If -f is harmonic convex function on E,then f are the Schur- harmonic concave function on E.

Lemma 3. [4] Let $E(\subset R_{++}^n)$ is a symmetric set with interior points, $\left\{\left(1/x_1, 1/x_2, \cdots, 1/x_n\right) \middle| x \in E\right\}$ is convex set, $f: E \to R$ continuation, and differentiable in the **int** E, Then the necessary and sufficient condition for f to be a convex (concave) function of Schur- harmonic is f is symmetric on E, and for all $x \in \operatorname{int} E$,both of

$$\left(x_1 - x_2\right)\left(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}\right) \ge 0\left(\le 0\right). \tag{3}$$

Definition 5. [11-13] (i) Let $f: R_{++} \to R$ is strictly monotone function, $\Omega \subset R^n$. If for any $x, y \in \Omega$, always $f^{-1}(\alpha f(x) + \beta f(y)) \in \Omega$, call Ω is f-convex set, among $\alpha, \beta \in [0,1]$ and $\alpha + \beta = 1$.

(ii) Let $\Omega \subset R^n$, Ω internal is not empty. $\varphi: \Omega \to R$, for any $x, y \in \Omega$, when $f(x) \prec f(y)$ there are $\varphi(x) \le \varphi(y)$, Then $-\varphi$ are Ω the Schur- **f** convex function on E. If $-\varphi$ is Schur- f convex function on Ω, then φ are the Schur- f concave function on Ω.

According to the definition of Schur- f convex function. If g is monotonously increasing (decrease), $g(\varphi(x))$ make sense, then φ is Schur- f convex function, if and only if $g \circ \varphi$ is Schur- f convex (concave) function.

Definition 6. [11-13] In definition 5, we take

$$f: x \in (0, +\infty) \to \begin{cases} \dfrac{x^m - 1}{m}, & m \ne 0; \\ \ln x, & m = 0. \end{cases}$$

then φ are the Schur- m order power convex function on Ω; If $-\varphi$ is Schur- m order power convex function on Ω, then φ are the Schur- m order power concave function on Ω.

Lemma 4. [11-13] Let $f: R \to R$ is strictly monotone differentiable functions, $\Omega(\subset R_{++}^n)$ is symmetry with interior points f-convex set, $\varphi: \Omega \to R$ on Ω is continuation, Inside of Ω, Ω^0 is differentiable, then the necessary and sufficient condition for φ to be a Schur- f bulge (Schur- f concave) is φ is symmetric on Ω, and for $\forall x \in \Omega^0$, we have

$$\Delta := (f(x_1) - f(x_2)) \left(\frac{1}{f'(x_1)} \frac{\partial \varphi}{\partial x_1} - \frac{1}{f'(x_2)} \frac{\partial \varphi}{\partial x_2} \right) \ge 0 (\le 0). \tag{4}$$

For Schur- m order power convex function, if $m \ne 0$, the corresponding Schur condition is

$$\frac{x_1^m - x_2^m}{m} \left(x_1^{1-m} \frac{\partial \varphi}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi}{\partial x_2} \right) \ge 0 (\le 0). \tag{5}$$

It's not hard to find, formula (4) Synthetic formulae (1-5).

Remarks: Owing to $\operatorname{sgn}(\frac{x_1^m - x_2^m}{m}) = \operatorname{sgn}(x_1 - x_2)$, so the above Schur condition is equivalent to

$$(x_1 - x_2) \left(x_1^{1-m} \frac{\partial \varphi}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi}{\partial x_2} \right) \ge 0 (\le 0) \, (m \in R).$$

Lemma 5. $g_1(x) = x^{-1} \operatorname{th}(\operatorname{sh} \frac{x}{2})$ on $(0, +\infty)$ is monotone decreasing, and $0 < g_1(x) < \frac{1}{2}$.

Proof: $g_1'(x) = -\frac{1}{x^2}\left\{\mathrm{th}(\mathrm{sh}\frac{x}{2}) - \frac{1}{2}x[1-\mathrm{th}^2(\mathrm{sh}\frac{x}{2})]\mathrm{ch}\frac{x}{2}\right\} = -\frac{2\mathrm{sh}(\mathrm{sh}\frac{x}{2})\mathrm{ch}(\mathrm{sh}\frac{x}{2}) - x\mathrm{ch}\frac{x}{2}}{2x^2\mathrm{ch}^2(\mathrm{sh}\frac{x}{2})}$,

owing to $p_1(t) = \mathrm{sh}t$, $p_2(t) = \mathrm{ch}t$ all about t on $(0,+\infty)$ is monotone increment, and $\mathrm{sh}t > t(t>0)$ t ,then

$$2\mathrm{sh}(\mathrm{sh}\frac{x}{2})\mathrm{ch}(\mathrm{sh}\frac{x}{2}) - x\mathrm{ch}\frac{x}{2} > 2\mathrm{sh}\frac{x}{2}\mathrm{ch}\frac{x}{2} - x\mathrm{ch}\frac{x}{2} > 2\times\frac{x}{2}\mathrm{ch}\frac{x}{2} - x\mathrm{ch}\frac{x}{2} = 0 .$$

Thus $g_1'(x) < 0$, $g_1(x)$ about x on $(0,+\infty)$ is monotone decrease. Obvious $g_1(x) > 0$,again

$$\lim_{x\to 0^+} g_1(x) = \lim_{x\to 0^+}\frac{\mathrm{th}(\mathrm{sh}\frac{x}{2})}{x} = \lim_{x\to 0^+}\frac{\mathrm{sh}\frac{x}{2}}{x} = \lim_{x\to 0^+}\frac{\frac{x}{2}}{x} = \frac{1}{2} .$$

Then for any $x\in(0,+\infty)$,we have $g_1(x) < g_1(0) = \frac{1}{2}$.

Lemma 6. [1] $g_2(x) = 2x\mathrm{ch}\frac{x}{2}[\mathrm{sh}(2\mathrm{sh}\frac{x}{2})]^{-1}$ on $(0,+\infty)$ is monotone decrease, and $0 < g_2(x) < 2$.

3. Main results and proof

Theorem 1. $M_p^*(x,y)$ about (x,y) on $(0,+\infty)^2$ Schur-m order concave, if and only if $m \ge p$.

Proof: When $p=0$, $M_p^*(x,y) = M^*(x,y)$,on document[12] certified $M^*(x,y)$ about (x,y) on $(0,+\infty)^2$ Schur-m order concave, if and only if $m \ge p$.

When $p>0$,calculated

$$\frac{\partial M_p^*(x,y)}{\partial x} = \frac{f_1(x,y)}{f_2(x,y)}\mathrm{th}^{p-1}(\mathrm{sh}\frac{x}{2})[1-\mathrm{th}^2(\mathrm{sh}\frac{x}{2})]\mathrm{ch}\frac{x}{2} ,$$

$$\frac{\partial M_p^*(x,y)}{\partial y} = \frac{f_1(x,y)}{f_2(x,y)}\mathrm{th}^{p-1}(\mathrm{sh}\frac{y}{2})[1-\mathrm{th}^2(\mathrm{sh}\frac{y}{2})]\mathrm{ch}\frac{y}{2} .$$

Among

$$f_1(x,y) = \frac{[\frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{x}{2}) + \frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{y}{2})]^{1/p-1}}{2\sqrt{1+\left\{\mathrm{th}^{-1}\left[\left(\frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{x}{2}) + \frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{y}{2})\right)^{1/p}\right]\right\}^2}} ,$$

$$f_2(x,y) = 1 - \left[\left(\frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{x}{2}) + \frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{y}{2})\right)^{1/p}\right]^2 .$$

Obvious $f_1(x,y) > 0$,owing to $\mathrm{th}(\mathrm{sh}\frac{x}{2}), \mathrm{th}(\mathrm{sh}\frac{y}{2}) \in (0,1)$,then

$$\mathrm{th}^p(\mathrm{sh}\frac{x}{2}), \mathrm{th}^p(\mathrm{sh}\frac{y}{2}) \in (0,1), \left(\frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{x}{2}) + \frac{1}{2}\mathrm{th}^p(\mathrm{sh}\frac{y}{2})\right)^{1/p} \in (0,1),$$

Thus $f_2(x,y) > 0$.

$\Delta_{M_p^*}(x,y) = (x-y)\left(x^{1-m}\frac{\partial M_p^*(x,y)}{\partial x} - y^{1-m}\frac{\partial M_p^*(x,y)}{\partial y}\right)$, then

$$\begin{aligned}
&\Delta_{M_p^*}(x,y) \\
&= \frac{(x-y)f_1(x,y)}{f_2(x,y)}\left\{x^{1-m}\mathrm{th}^{p-1}(\mathrm{sh}\frac{x}{2})[1-\mathrm{th}^2(\mathrm{sh}\frac{x}{2})]\mathrm{ch}\frac{x}{2} - y^{1-m}\mathrm{th}^{p-1}(\mathrm{sh}\frac{y}{2})[1-\mathrm{th}^2(\mathrm{sh}\frac{y}{2})]\mathrm{ch}\frac{y}{2}\right\} \\
&= \frac{(x-y)^2 f_1(x,y)}{f_2(x,y)} \cdot \frac{x^{1-m}\mathrm{th}^{p-1}(\mathrm{sh}\frac{x}{2})[1-\mathrm{th}^2(\mathrm{sh}\frac{x}{2})]\mathrm{ch}\frac{x}{2} - y^{1-m}\mathrm{th}^{p-1}(\mathrm{sh}\frac{y}{2})[1-\mathrm{th}^2(\mathrm{sh}\frac{y}{2})]\mathrm{ch}\frac{y}{2}}{x-y}
\end{aligned}$$

owing to

$$\begin{aligned}
x^{1-m}\mathrm{th}^{p-1}(\mathrm{sh}\frac{x}{2})[1-\mathrm{th}^2(\mathrm{sh}\frac{x}{2})]\mathrm{ch}\frac{x}{2} &= x^{p-m}[x^{-1}\mathrm{th}(\mathrm{sh}\frac{x}{2})]^p \frac{x[1-\mathrm{th}^2(\mathrm{sh}\frac{x}{2})]\mathrm{ch}\frac{x}{2}}{\mathrm{th}(\mathrm{sh}\frac{x}{2})} \\
&= x^{p-m}[x^{-1}\mathrm{th}(\mathrm{sh}\frac{x}{2})]^p \frac{2x\mathrm{ch}\frac{x}{2}}{\mathrm{sh}(2\mathrm{sh}\frac{x}{2})} \\
&= x^{p-m}[g_1(x)]^p g_2(x).
\end{aligned}$$

When $m \ge p$, apparently $x^{p-m} > 0$,and $y_1 = x^{p-m}$ about x on $(0,+\infty)$ is monotone decrease. By Lemma 5 and $p > 0$, we have $[g_1(x)]^p > 0$ and $y_2 = [g_1(x)]^p$ about x on $(0,+\infty)$ is monotone decrease. Then according to lemma 6, obvious $g_2(x) > 0$, and $g_2(x)$ about x on $(0,+\infty)$ is monotone decrease. Comprehensive, function $h(x) = x^{p-m}[g_1(x)]^p g_2(x)$ about x on $(0,+\infty)$ is monotone decrease, then

$$\Delta_{M_p^*}(x,y) = \frac{(x-y)^2 f_1(x,y)}{f_2(x,y)} \cdot \frac{h(x)-h(y)}{x-y} \le 0.$$

According to lemma 6， we can get $M_p^*(x,y)$ about (x,y) on $(0,+\infty)^2$ is Schur-m order concave.

When $p > 0$ and $m < p$,owing to

$$\lim_{x\to+\infty}[g_1(x)]^p = [\lim_{x\to+\infty} g_1(x)]^p = (\lim_{x\to+\infty}\frac{1}{x})^p = 0,$$

$$0 < \frac{2x^{1+p-m}\mathrm{ch}\frac{x}{2}}{\mathrm{sh}(2\mathrm{sh}\frac{x}{2})} < \frac{2x^{1+p-m}\mathrm{ch}\frac{x}{2}}{\mathrm{sh}x} = \frac{x^{1+p-m}}{\mathrm{sh}\frac{x}{2}} < \frac{2x^{1+p-m}}{e^{\frac{x}{2}}-1} \to 0 (x \to +\infty).$$

Then $\lim\limits_{x\to+\infty} h(x)=\lim\limits_{x\to+\infty}[g_1(x)]^p\cdot\lim\limits_{x\to+\infty}\dfrac{2x^{1+p-m}\mathrm{ch}\frac{x}{2}}{\mathrm{sh}(2\mathrm{sh}\frac{x}{2})}=0\times 0=0$.

So $\lim\limits_{x\to+\infty}[h(x)-h(1)]=\lim\limits_{x\to+\infty}\left[h(x)-\mathrm{th}^p(\mathrm{sh}\frac{1}{2})\dfrac{2\mathrm{ch}\frac{1}{2}}{\mathrm{sh}(2\mathrm{sh}\frac{1}{2})}\right]=-\mathrm{th}^p(\mathrm{sh}\frac{1}{2})\dfrac{2\mathrm{ch}\frac{1}{2}}{\mathrm{sh}(2\mathrm{sh}\frac{1}{2})}<0$.

Then $\exists x_0\in(1,+\infty)$ make $\dfrac{h(x_0)-h(1)}{x_0-1}<0$, $\Delta_{M_p^*}(x_0,1)<0$.

Associative Lemma 5 and Lemma 6, we can get

$$0<h(y)=y^{p-m}[g_1(y)]^p g_2(y)<y^{p-m}(\frac{1}{2})^p\times 2\to 0(y\to 0^+)\,.$$

Then $\lim\limits_{y\to 0^+}h(y)=0$, so

$$\lim_{y\to 0^+}[h(1)-h(y)]=\lim_{y\to 0^+}[\mathrm{th}^p(\mathrm{sh}\frac{1}{2})\frac{2\mathrm{ch}\frac{1}{2}}{\mathrm{sh}(2\mathrm{sh}\frac{1}{2})}-h(y)]=\mathrm{th}^p(\mathrm{sh}\frac{1}{2})\frac{2\mathrm{ch}\frac{1}{2}}{\mathrm{sh}(2\mathrm{sh}\frac{1}{2})}>0$$

Then $\exists y_0\in(0,1)$ make $\dfrac{h(1)-h(y_0)}{1-y_0}>0$, $\Delta_{M_p^*}(1,y_0)>0$.

Because at this point, $\Delta_{M_p^*}(x,y)$ on $(0,+\infty)^2$ is symbol uncertainty, thus $M_p^*(x,y)$ is not $(0,+\infty)^2$ Schur-m power concave (convex) function.

In summary, the theorem can be proved.

4. Two unresolved issues

Question 1. To average $M_p^*(x,y)$,Where the range of values of p can be extended to R try to give $M_p^*(x,y)$ ($p\in R$) about (x,y) on $(0,+\infty)^2$ necessary and sufficient conditions for power convexity of Schur-m order.

Question 2. Similarly the form of $H^*(x,y)$. More general averages involving trigonometric functions can be defined as follows

$$H_p^*(x,y)=2\sin^{-1}\tan^{-1}\left[M_p\left(\tan(\sin\frac{x}{2}),\tan(\sin\frac{y}{2})\right)\right]$$
$$=\begin{cases}2\sin^{-1}\tan^{-1}\left\{[\frac{1}{2}\tan^p(\sin\frac{x}{2})+\frac{1}{2}\tan^p(\sin\frac{y}{2})]^{1/p}\right\}, & p\neq 0\\ 2\sin^{-1}\tan^{-1}\sqrt{\tan(\sin\frac{x}{2})\tan(\sin\frac{y}{2})}, & p=0\end{cases}\quad (x,y\in(0,\pi)),$$

try to give $H_p^*(x,y)$ about (x,y) on $(0,\pi)^2$ necessary and sufficient conditions for power convexity of Schur-m order.

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REFERENCES

1. D.S.Chen, C.Y.Wang and S.D.Tao, Two new "strange" averages and their Schur power convexity, Mathematical Practice and Cognition (In press).
2. Y.P.Deng, S.H.Wu and D.He, On the Schur power convexity of generalized muirhead mean, Practice and Cognition of Mathematics (Chinese), 44(5) (2014) 255-268.
3. K.Z.Guan, Schur-convexity of the complete symmetric function, *Math. Inequal. Appl*., 9 (2006) 567-576.
4. K.Z.Guan, Some properties of a class of symmetric functions, *J. Math. Anal. Appl.,* 336 (2007) 70-80.
5. D.He, On the Schur power convexity of two "peculiar" averages, *Journal of Guangdong second normal University* (Chinese), 36(3) (2016) 30-38.
6. D.M.Li and H.N.Shi. A New proof of the conjecture of a binary mean value inequality, *Mathematical Practice and Cognition* (Chinese), 36(4) (2006) 278-283.
7. H.N.Shi, Controlled theory and analytic inequalities, *Harbin: Harbin University of Technology Press* (Chinese) (2012) 283-287.
8. B.Y.Wang, Basis of control inequality, *Beijing: Beijing normal University Press* (Chinese), 1990.
9. L.Yang, J.Z.Zhang and X.R.Hou, Nonlinear algebraic equations and machine proof of theorems, *Shanghai: Shanghai Science and Technology Education Press* (Chinese), (1996) 137-166.
10. L.Yang and B.C.Xia, Inequality machine proof and automatic discovery, *Beijing: Science Publishing House* (Chinese), (2008) 33-46.
11. Z.H.Yang, Schur power convexity of Stolarsky means, *Publ. Math. Debrecen,* 80 (2012) 43-66.
12. Z.H.Yang, Schur power convexity of Gini means, *Bull. Korean Math. Soc.*, 50(2) (2013) 485–498.
13. Z.H.Yang, Schur power convexity of Daróczy means, *Math. Inequal. Appl.*, 16(3) (2013) 751-762.
14. X.M.Zhang, Geometric convex function, *Hefei: Anhui University Press* (Chinese), (2004) 107-121.
15. X.M.Zhang and T.Q.Xu, Definition and Application of Generalized SCHUR-geometric Convex function, *Journal of Qingdao Vocational and Technical College* (Chinese), 18(4) (2005) 60-63.
16. X.M.Zhang and Y.M Chu, New theory of analytic inequalities, *Harbin: Harbin Gonglai Publishing House* (Chinese), (2009) 93-94.
17. X.M.Zhang. The Schur-product of several n- p element averages order power convexity, *Journal of Hunan Institute of Technology* (Chinese), 24 (2011) 1-7.