Journal of Mathematics and Informatics
Vol. 17, 2019, 31-54
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 19 July 2019
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.148av17a4

Journal of
Mathematics and Informatics

Double Cubic Singular Blend Bézier Surfaces

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Received 11 June 2019; accepted 9 July 2019

Abstract. Firstly, based on the traditional Bézier surface, the traditional Bézier surface is generalized by using the idea of power and combining with the singular hybrid technology, and a singular hybrid Bézier surface is obtained. It not only contains the original Bézier surface, but also introduces new local control parameters with local control and modification ability. Secondly, the properties of two cubic singular Bézier surfaces are studied, and the geometric meaning of the mixed parameters is studied and its influence on the shape of the surface is discussed. In the end, the splicing conditions of two cubic singular Bézier surfaces in parameter splicing and geometric splicing are presented.

Keywords: Bézier surface; singular mixing; mixing parameters; parameter splicing; collection of joining together

AMS Mathematics Subject Classification (2010): 65D17

1. Introduction

Surface modeling technology is the key technology in many fields such as CAD/CAM, CG, computer animation, computer simulation, artificial intelligence [1], computer visualization, some aspects of computer image processing [2-3], and so on. In surface design, people strive to use methods to ensure that the surface has a higher smooth continuity, but also convenient and flexible, easy to control and modify the designed surface. Bézier surface method plays an important role in CAD/CAM technology and many fields mentioned above, especially in computer aided geometric design. Bézier surface has many good properties [4-9]. However, with the development of modern geometry industry, the traditional Bézier itself has some shortcomings, such as not smooth connection and inflexible local modification, which make it difficult to meet all kinds of requirements in geometric modeling. At the same time, many rational Bézier curves [10-11] have been proposed, which solves the problem of traditional methods. However, the denominator appears in curves and surface equations due to rationalization, and consequently the gradual problem inevitably arises. In addition, improper application of weights in rational methods may lead to bad parameterization and even destroy the surface structure behind [12]. For this reason, many non-rational Bézier curves and surfaces with shape parameters [13-18] have been proposed by scholars, so that the curves can achieve a certain degree of local adjustability and good results, but their parameters are unstable, which will make it difficult to control and calculate the influence of parameters on the surface.

The earliest concept of singular mixing technology was put forward by Loe to improve the flexibility of curves and surfaces and expand their descriptive ability. Unlike physical and chemical methods, singular mixing methods do not cause denominators in curves and surfaces, and therefore do not cause progressive problems. In addition, the mixing factor of singular mixing technology has obvious geometric intuitive significance. Therefore, the research on singular mixing technology has theoretical and practical value. Guicang et al. [12] used the singular mixing technique in reference [19-21] to construct the singular mixing Bézier curve. This novel curve construction method not only retains the good properties of all traditional Bézier curves, but also has good shape adjustability. In this paper, on the basis of reference [12], the traditional Bézier surface is generalized by using the idea of weight and singular mixing technology, and a new form of singular mixed Bézier surface is obtained, and its definition is given. The special properties of bi-cubic singular mixed Bézier surface are emphatically analyzed, the geometric meaning of mixing parameters is understood, and its influence on the shape of surface is discussed Finally, the bi-cubic singular mixed B The splicing conditions of the cubic singular mixed Bézier surface in parameter splicing and geometric splicing.

2. Definition of singular mixed Bézier surfaces

Singular mixed Bézier surface is a generalized form of Bézier surface which is obtained by using the idea of weight and the technique of singular mixing. Before introducing its concept, we first review the definition of Bézier surface.

A set of points $\{P_{i,j}\}|i = 0,1,\cdots,m; j = 0,1,\cdots,n$ in a given space is a surface described by formula (1) called $m \times n$ -order Bézier surface.

$$P(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} J_{m,i}(u) J_{n,j}(v) (0 \leq u, v \leq 1) \tag{1}$$

Type in the $J_{n,i}(t) = C_n^i t^i\ (1-t)^{n-i}, i = 1,2.\cdots,n$

The space grid composed of two adjacent points in the line segment connection point column successively is called the feature grid [19]. In theory and practical use, bicubic

Bézier Surface is of great significance, and bicubic Bézier Surface is mainly used in the actual shape design [22]. Therefore, this paper focuses on the study of cubic Bicubic Bézier. Surface is determined by 16 control vertices $\{P_{i,j}\}|i,j = 0,1,2,3$, and its expression is (1).

$$P(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} P_{i,j}J_{3,i}(u)J_{3,j}(v) \qquad (0 \le u,v \le 1) \tag{2}$$

type in the $J_{3,i}(t) = C_3^i t^i \ (1-t)^{3-i}, i = 0,1,2,3.$

Definition 1. Let $f(t)$ be a continuous function defined on [0,1], if $f(t)$ satisfies the following conditions

$$f(0) = 1, f(1) = 0, f^{(k)}(0) = f^{(k)}(1) = 0, k = 1,2,\cdots,n \tag{3}$$

$f(t)$ is called the singular mixed function of order n.

The types of singular mixed functions can be varied naturally, and polynomials are the simplest and most commonly used form of functions. The polynomial singular mixed function with the smallest number of times is called the smallest singular mixed function.

With the concept of singular mixing function, the definition of bi-cubic singular mixing Bezier surfaces can be obtained by introducing the corresponding mixing parameter $\{\alpha_{i,j}\}|i,j = 1,2$ at the corresponding control vertex $\{P_{i,j}\}|i,j = 1,2$ of Bézier surfaces.

Definition 2. Let $f(t)$ be the smallest singular mixed function and $\bar{f}(t) = 1 - f(t)$, then the bicubic singular mixed Bézier surface is

$$Q(u,v) = \alpha(u,v)P(u,v) + (1-\alpha(u,v))L(u,v) \qquad (0 \le u,v \le 1) \tag{4}$$

Among themL$(u,v) =$

$$P_{1,1}f(u)f(v) + P_{1,2}f(u)\bar{f}(v) + P_{2,1}\bar{f}(u)f(v) + P_{2,2}\bar{f}(u)\bar{f}(v) \qquad (0 \le u,v \le 1) \tag{5}$$

$$\alpha(u,v) = \alpha_{1,1}f(u)f(v) + \alpha_{1,2}f(u)\bar{f}(v) + \alpha_{2,1}\bar{f}(u)f(v) + \alpha_{2,2}\bar{f}(u)\bar{f}(v)$$

$$(0 \le u,v \le 1, \{\alpha_{i,j} \text{is any real number}\}|i,j = 1,2) \tag{6}$$

The shape of the surface can be adjusted by changing the value of the blending parameter $\{\alpha_{i,j}\}|i,j = 1,2$. See Section 3 for details.

3. Properties of double cubic singular mixed Bézier surfaces

By defining 2,bi-cubic singular mixed Bézier surfaces can be written

$$Q(u,v) = \alpha(u,v)P(u,v) + (1-\alpha(u,v))L(u,v)$$

$$= \alpha(u,v)\sum_{i=0}^{3}\sum_{j=0}^{3} P_{i,j}J_{3,i(u)}J_{3,j}(v) + (1-\alpha(u,v))$$

$$(P_{1,1}f(u)f(v) + P_{1,2}f(u)\bar{f}(v) + P_{2,1}\bar{f}(u)f(v) + P_{2,2}\bar{f}(u)\bar{f}(v) \tag{7}$$

It can be seen from the formula that the surface patch is composed of 16 control vertices $\{P_{i,j}\}|i,j = 0,1,2,3$ of the control mesh. Therefore, it can be written as the following basic representation:

$$Q(u,v) = \sum_{k=1}^{16} D_k(u,v)P_k \qquad (0 \le u,v \le 1) \tag{8}$$

where $D_k(u,v)$ is made up o$J_{3,j}(v)$ f,$\alpha(u,v)$,$f(u)$,$f(v)$.

Nature 1. The basis of double cubic singular mixed Bézier surfaces is normative.

Proof: $\sum_{k=1}^{16} D_k(u,v) = \alpha(u,v)\sum_{i=0}^{3}\sum_{j=0}^{3} J_{3,i}(u)J_{3,j}(v) + (1-\alpha(u,v))$

$$(f(u)f(v) + f(u)\bar{f}(v) + \bar{f}(u)f(v) + \bar{f}(u)\bar{f}(v) \tag{9}$$

$$= \alpha(u,v)(\sum_{i=0}^{3} J_{3,i}(u))(\sum_{j=0}^{3} J_{3,j}(v)) + (1-\alpha(u,v))\,[f(u)(f(v)+\bar{f}(v)) +$$

$$\bar{f}(u)(f(v)+\bar{f}(v))] = \alpha(u,v) + (1-\alpha(u,v)) \;=1\;\; (0 \le u,v \le 1)$$

From equation (7), it can be concluded that this is a normative basis [20].

Nature 2. When $0 \le \alpha_{i,j} \le 1|i,j = 1,2$, the value range of the basis of double cubic singular Bézier surface is $[0,1]$.

Proof: When $0 \le \alpha_{i,j} \le 1|i,j = 1,2$, since $f(t)$ is the minimum singular function, $0 \le f(t), \bar{f}(t) \le 1, 0 \le t \le 1$ can be proved by equation (6)

$$0 \le \alpha(u,v) \le 1 \tag{10}$$

Because when the range of mixing parameters varies from 0 to 1, (6) each item on the right side of the equation is greater than 0, there is

$$0 \le \alpha(u,v) \tag{11}$$

It may also be assumed that the largest of these mixed parameters is $\alpha_{1,1}$

$$\alpha(u,v) = \alpha_{1,1}f(u)f(v) + \alpha_{1,2}f(u)\bar{f}(v) + \alpha_{2,1}\bar{f}(u)f(v) + \alpha_{2,2}\bar{f}(u)\bar{f}(v) \tag{12}$$

$$\le \alpha_{1,1}(f(u)f(v) + f(u)\bar{f}(v) + \bar{f}(u)f(v) + \bar{f}(u)\bar{f}(v)) = \alpha_{1,1}[f(u)(f(v)+\bar{f}(v)) +$$

$$\bar{f}(u)(f(v)+\bar{f}(v))] = \alpha_{1,1}[f(u)+\bar{f}(u)] = \alpha_{1,1} \le 1$$

So corresponding to control vertex $P_{1,1}$,$P_{1,2}$,$P_{2,1}P_{2,1}$,$P_{2,2}$, other than $D_k(u,v)$

$$D_k(u,v) = \alpha(u,v)J_{3,i}(u)J_{3,j}(v) \tag{13}$$

Because of $0 \le J_{3,i}(u), J_{3,j}(v) \le 1$, we have $\quad 0 \le D_k(u,v) \le 1$ (14)

Corresponding to the $P_{1,1} D_k(u,v) = \alpha(u,v) J_{3,1}(u) J_{3,1}(v) + (1-\alpha(u,v)) f(u) f(v)$

Corresponding to the $P_{1,2} D_k(u,v) = \alpha(u,v) J_{3,1}(u) J_{3,2}(v) + (1-\alpha(u,v)) f(u) \bar{f}(v)$

Corresponding to the$P_{2,1} D_k(u,v) = \alpha(u,v) J_{3,2}(u) J_{3,1}(v) + (1-\alpha(u,v)) \bar{f}(u) f(v)$

Corresponding to the

$$P_{2,2} D_k(u,v) = \alpha(u,v) J_{3,2}(u) J_{3,2}(v) + (1-\alpha(u,v)) \bar{f}(u) \bar{f}(v) \tag{15}$$

Because the range of $\alpha(u,v)$,$J_{3,i}$,$J_{3,j}$,$f(u)$,$\bar{f}(u)$ is between 0 and 1, there is

$$0 \le D_k(u,v) \le 1 \tag{16}$$

To sum up, it is evident.

.Since the basis function corresponds to the control vertex one to one, in order to unify formally, the tensor product form of singular mixed Bézier surface equation can be obtained by taking the subscript of the basis function as the subscript of the control vertex.

Definition 3. The tensor product form of the equation of bicubic singular mixed Bézier surface is

$$Q(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} D_{i,j}(u,v,\alpha) P_{i,j} \qquad (0 \le u, v \le 1) \tag{17}$$

Nature 3. Bicubic singular mixed Bézier surface patches (7) have geometric invariance. It can be seen from the formulas (8) and (9) that the surface patch can be expressed as a normative basis, so it has geometric invariance [23].

Nature 4. When $0 \le \alpha_{i,j} \le 1 | i,j = 1,2$, the surface patch (7) is convex. That is, the surface patch (7) is located in the convex hull determined by 16 control vertices $\{P_{i,j}\} | i,j = 0,1,2,3$.

This can be obtained by formula (8), (9), (16).

Nature 5. Bicubic Bézier surface patches (2) is .contained by bicubic singularly mixed Bézier surface patches (7)

If command $\alpha_{i,j} = 1 | i,j = 1,2$, then there are

$$\alpha(u,v) = \alpha_{1,1} f(u) f(v) + \alpha_{1,2} f(u) \bar{f}(v) + \alpha_{2,1} \bar{f}(u) f(v) + \alpha_{2,2} \bar{f}(u) \bar{f}(v)$$

$$= f(u)(f(v) + \bar{f}(v)) + \bar{f}(u)(f(v) + \bar{f}(v)) = f(u) + \bar{f}(u) = 1 \tag{18}$$

Formula 6 has

$$Q(u, v) = \alpha(u, v)P(u, v) + (1 - \alpha(u, v))L(u, v) = P(u, v) \tag{19}$$

Therefore, singular hybrid Bézier surface patches include Bézier surface patches, which are its generalization. Because this singular hybrid Bézier surface patch equation contains freely adjustable mixed parameters, its expressive ability is stronger than the original Bézier surface patches, and it has more flexible ability to control the shape of the surface, as well as the potential ability to modify the shape locally.

Nature 6. Bicubic singular mixed Bézier surface patches (7) have approximation property. That is, when A, the surface approximates the surface of its control polyhedron.

When $\alpha_{i,j} \to 0 | i, j = 1,2$, because $\alpha(u, v) \to 0$, there is

$$Q(u, v) = \alpha(u, v)P(u, v) + (1 - \alpha(u, v))L(u, v) \to L(u, v) \tag{20}$$

$L(u, v)$ is a bilinear singular surface patch determined by four space points $\{P_{i,j}\} | i, j = 1,2$. When these four points are coplanar, the bilinear singular surface patch is a plane patch determined by them. Therefore, surface patches have approximation. Figure 6 can be seen in detail.

Nature 7. Bicubic Singular Mixed Bézier Surface Patches have the following endpoint properties.

□The four boundary lines are

$$Q(0, v) = (\alpha_{1,1} f(v) + \alpha_{1,2}\bar{f}(v)) \textstyle\sum_{j=0}^{3} J_{3,j}(v)P_{0,j} + (1 - \alpha_{1,1} f(v) - \alpha_{1,2}\bar{f}(v))(P_{1,1} f(v) + P_{1,2}\bar{f}(v))(0 \le v \le 1) \tag{21}$$

$$Q(1, v) = (\alpha_{2,1} f(v) + \alpha_{2,2}\bar{f}(v)) \sum_{j=0}^{3} J_{3,j}(v)P_{3,j} + (1 - \alpha_{2,1} f(v) - \alpha_{2,2}\bar{f}(v))(P_{2,1} f(v) + P_{2,2}\bar{f}(v))(0 \le v \le 1) \tag{22}$$

$$Q(u, 0) = (\alpha_{1,1} f(u) + \alpha_{2,1}\bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u)P_{i,0} + (1 - \alpha_{1,1} f(u)$$

$$-\alpha_{1,2}\bar{f}(u))(P_{1,1}f(u)+P_{2,1}\bar{f}(u))(0\le u\le 1) \tag{23}$$

$$Q(u,1)=(\alpha_{1,2}f(u)+\alpha_{2,2}\bar{f}(u))\sum_{i=0}^{3}J_{3,i}(u)P_{i,3}+(1-\alpha_{1,2}f(u)$$

$$-\alpha_{2,2}\bar{f}(u))(P_{1,2}f(u)+P_{2,2}\bar{f}(u))(0\le u\le 1) \tag{24}$$

②The four corners are

$$Q(0,0)=\alpha_{1,1}P_{0,0}+\left(1-\alpha_{1,1}\right) \tag{25}$$

$$Q(0,1)=\alpha_{1,2}P_{0,3}+(1-\alpha_{1,2})P_{1,2} \tag{26}$$

$$Q(1,0)=\alpha_{2,1}P_{3,0}+(1-\alpha_{2,1})P_{2,1} \tag{27}$$

$$Q(1,1)=\alpha_{2,2}P_{3,3}+(1-\alpha_{2,2})P_{2,2} \tag{28}$$

The transboundary tangent vectors of four edges are respectively

a) First order tangent vector

$$\frac{\partial Q(0,v)}{\partial u}=(\alpha_{1,1}f(v)+\alpha_{1,2}\bar{f}(v))(\textstyle\sum_{j=0}^{3}J_{3,j}(v)(-3P_{0,j}+3P_{1,j}) \tag{29}$$

$$=3(\alpha_{1,1}f(v)+\alpha_{1,2}\bar{f}(v))(\sum_{j=0}^{3}J_{3,j}(v)(P_{1,j}-P_{0,j})\qquad(0\le v\le 1)$$

$$\frac{\partial Q(1,v)}{\partial u}=(\alpha_{2,1}f(v)+\alpha_{2,2}\bar{f}(v))(\textstyle\sum_{j=0}^{3}J_{3,j}(v)(-3P_{2,j}+3P_{3,j}) \tag{30}$$

$$=3(\alpha_{2,1}f(v)+\alpha_{2,2}\bar{f}(v))(\sum_{j=0}^{3}J_{3,j}(v)(P_{3,j}-P_{2,j})\qquad(0\le v\le 1)$$

$$\frac{\partial Q(u,0)}{\partial v}=(\alpha_{1,1}f(u)+\alpha_{1,2}\bar{f}(u))(\textstyle\sum_{i=0}^{3}J_{3,i}(u)(-3P_{i,0}+3P_{i,1}) \tag{31}$$

$$=3(\alpha_{1,1}f(u)+\alpha_{1,2}\bar{f}(u))(\sum_{i=0}^{3}J_{3,i}(u)(P_{i,1}-P_{i,0})\qquad(0\le u\le 1)$$

$$\frac{\partial Q(u,1)}{\partial v}=(\alpha_{2,1}f(u)+\alpha_{2,2}\bar{f}(u))(\textstyle\sum_{i=0}^{3}J_{3,i}(u)(-3P_{i,2}+3P_{i,3}) \tag{32}$$

$$=3(\alpha_{2,1}f(u)+\alpha_{2,2}\bar{f}(u))(\sum_{i=0}^{3}J_{3,i}(u)(P_{i,3}-P_{i,2})\qquad(0\le u\le 1)$$

b) The second order vector

$$\frac{\partial^2 Q(0,v)}{\partial u^2} = (\alpha_{1,1} f(v) + \alpha_{1,2} \bar{f}(v))(\sum_{j=0}^{3} J_{3,j}(v))(6P_{0,j} - 12P_{1,j} + 6P_{2,j}) \quad (33)$$

$$= 6(\alpha_{1,1} f(v) + \alpha_{1,2} \bar{f}(v))(\sum_{j=0}^{3} J_{3,j}(v))(P_{0,j} - 2P_{1,j} + P_{2,j}) \quad (0 \le v \le 1)$$

$$\frac{\partial^2 Q(1,v)}{\partial u^2} = (\alpha_{2,1} f(v) + \alpha_{2,2} \bar{f}(v))(\sum_{j=0}^{3} J_{3,j}(v)(6P_{1,j} - 12P_{2,j} + 6P_{3,j}) \quad (34)$$

$$= 6(\alpha_{2,1} f(v) + \alpha_{2,2} \bar{f}(v))(\sum_{j=0}^{3} J_{3,j}(v)(P_{1,j} - 2P_{2,j} + P_{3,j}) \quad (0 \le v \le 1)$$

$$\frac{\partial^2 Q(u,0)}{\partial v^2} = (\alpha_{1,1} f(u) + \alpha_{1,2} \bar{f}(u))(\sum_{i=0}^{3} J_{3,i}(u)(6P_{i,0} - 12P_{i,1} + 6P_{i,2}) \quad (35)$$

$$= 6(\alpha_{1,1} f(u) + \alpha_{1,2} \bar{f}(u))(\sum_{i=0}^{3} J_{3,i}(u)(P_{i,0} - 2P_{i,1} + P_{i,2}) \quad (0 \le u \le 1)$$

$$\frac{\partial^2 Q(u,1)}{\partial v^2} = (\alpha_{2,1} f(u) + \alpha_{2,2} \bar{f}(u))(\sum_{i=0}^{3} J_{3,i}(u)(6P_{i,1} - 12P_{i,2} + 6P_{i,3}) \quad (36)$$

$$= 6(\alpha_{2,1} f(u) + \alpha_{2,2} \bar{f}(u))(\sum_{i=0}^{3} J_{3,i}(u)(P_{i,1} - 2P_{i,2} + P_{i,3}) \quad (0 \le u \le 1)$$

The torsion vector of four angles is

$$\frac{\partial^2 Q(0,0)}{\partial v \partial u} = 9\alpha_{1,1}(P_{0,0} - P_{0,1} - P_{1,0} + P_{1,1}) \quad (37)$$

$$\frac{\partial^2 Q(0,1)}{\partial v \partial u} = 9\alpha_{1,2}(P_{0,2} - P_{1,2} - P_{0,3} + P_{1,3}) \quad (38)$$

$$\frac{\partial^2 Q(1,0)}{\partial v \partial u} = 9\alpha_{2,1}(P_{2,0} - P_{2,1} - P_{3,0} + P_{3,1}) \quad (39)$$

$$\frac{\partial^2 Q(1,1)}{\partial v \partial u} = 9\alpha_{2,2}(P_{2,2} - P_{2,3} - P_{3,2} + P_{3,3}) \quad (40)$$

Using the properties of bilinear singular surface patches and minimal singular mixed functions, we can get property 7 by simple calculation.

4. Effect of mixed parameters on curved surface shape

4.1. Geometric meaning of mixed parameters

Singular mixed Bézier surface is obtained by introducing parameters on the basis of Bézier surface.

Double Cubic Singular Blend Bézier Surfaces

Singular mixed Bézier surface is obtained by introducing parameters on the basis of Bézier surface. The biggest difference between it and Bézier surface is the parameters introduced. This parameter is similar to the weight factor in rational surface. It is very important to understand the geometric meaning of this parameter both in theory and in application.

By property 7 and the deformations of pair A are as follows

$$Q(0,0) - P_{1,1} = \alpha_{1,1}(P_{0,0} - P_{1,1}) \tag{41}$$

$$Q(0,1) - P_{1,2} = \alpha_{1,2}(P_{0,3} - P_{1,2}) \tag{42}$$

$$Q(1,0) - P_{2,1} = \alpha_{2,1}(P_{3,0} - P_{2,1}) \tag{43}$$

$$Q(1,1) - P_{2,2} = \alpha_{2,2}(P_{3,3} - P_{2,2}) \tag{44}$$

It can be seen from this that point $Q(0,0)$ and point $\overrightarrow{P_{1,1}P_{0,0}}$ are the fixed points of point $\alpha_{1,1}$, it can be seen from this that point $Q(0,1)$ and point $\overrightarrow{P_{1,2}P_{0,3}}$ are the fixed points of point $\alpha_{1,2}$, and it can be seen from this that point $Q(1,1)$ and point $\overrightarrow{P_{2,2}P_{3,3}}$ are the fixed points of point $\alpha_{2,2}$.If the four corners of a surface are $Q_{0,0}$, $Q_{0,1}$, $Q_{1,0}$ and $Q_{1,1}$, and the directed distance between the two points is recorded by P_1P_0, and $\overrightarrow{P_1P_0} = -\overrightarrow{P_0P_1}$, then

$$\alpha_{1,1} = \frac{Q_{0,0}P_{1,1}}{P_{0,0}P_{1,1}} = \frac{P_{1,1}Q_{0,0}}{P_{1,1}P_{0,0}} \tag{45}$$

$$\alpha_{1,2} = \frac{Q_{0,1}P_{1,2}}{P_{0,3}P_{1,2}} = \frac{P_{1,2}Q_{0,1}}{P_{1,2}P_{0,3}} \tag{46}$$

$$\alpha_{2,1} = \frac{Q_{1,0}P_{2,1}}{P_{3,0}P_{2,1}} = \frac{P_{2,1}Q_{1,0}}{P_{2,1}P_{3,0}} \tag{47}$$

$$\alpha_{2,2} = \frac{Q_{1,1}P_{2,2}}{P_{3,3}P_{2,2}} = \frac{P_{2,2}Q_{1,1}}{P_{2,2}P_{3,3}} \tag{48}$$

Theorem 1. If $\boldsymbol{Q_{0,0}}$, $\boldsymbol{Q_{0,1}}$, $\boldsymbol{Q_{1,0}}$ and $\boldsymbol{Q_{1,1}}$ are the four corners of singular mixed Bézier surface patches (7), then there are

$$\alpha_{1,1}:(1-\alpha_{1,1}) = P_{1,1}Q_{0,0}:Q_{0,0}P_{0,0} = Q_{0,0}P_{1,1}:P_{0,0}Q_{0,0} \tag{49}$$

$$\alpha_{1,2}:(1-\alpha_{1,2}) = P_{1,2}Q_{0,1}:Q_{0,1}P_{0,3} = Q_{0,1}P_{1,2}:P_{0,3}Q_{0,1} \tag{50}$$

$$\alpha_{2,1}:(1-\alpha_{2,1}) = P_{2,1}Q_{1,0}:Q_{1,0}P_{3,0} = Q_{1,0}P_{2,1}:P_{3,0}Q_{1,0} \tag{51}$$

$$\alpha_{2,2}:(1-\alpha_{2,2}) = P_{2,2}Q_{1,1}:Q_{1,1}P_{3,3} = Q_{1,1}P_{2,2}:P_{3,3}Q_{1,1} \tag{52}$$

The proof of the theorem can be obtained from the previous deduction.

4.2. Effect of mixed parameters on curved surface shape

The shape of a surface patch depends not only on the control vertex, but also on the mixing parameters and singular mixing functions. Generally, when we choose a singular mixing function, it will not change. Therefore, we will focus on the influence of the mixing parameters on the shape of the surface. Let's discuss it in several ways.

1. $\alpha_{1,1} = \alpha_{1,2} = \alpha_{2,1} = \alpha_{2,2} = 1$

From property 5, we can see that the singular mixed surface is the Bézier surface, and the four corners of the surface patch are the four control vertices of the feature mesh, as shown in Figure 1.

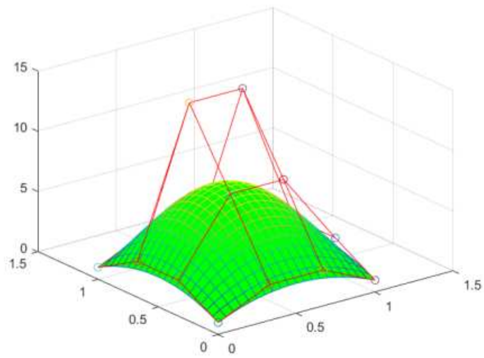


Figure 1:

(2) $\alpha_{1,1} = 0, \alpha_{1,2} = \alpha_{2,1} = \alpha_{2,2} = 1$

When a mixed parameter is 0, the surface will be drawn to the control vertex corresponding to the mixed parameter as shown in Figure 2. When the parameter is 1, the surface will be drawn to the control vertex corresponding to the parameter at the outmost part of the control mesh. That is to say, the surface behavior of the area corresponding to the parameter is similar to that of the Bézier surface as shown in Figure 3 in case 3.

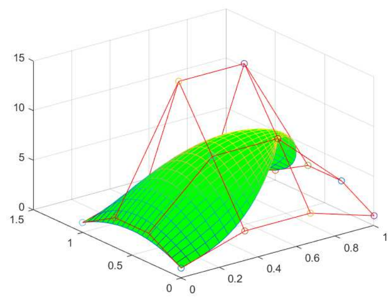


Figure 2:

(3) $\alpha_{1,\ 1} = 1, \alpha_{1,\ 2} = \alpha_{2,\ 1} = \alpha_{2,\ 2} = 0$

In this case, the surface passes through four control points, as shown in Figure 3.

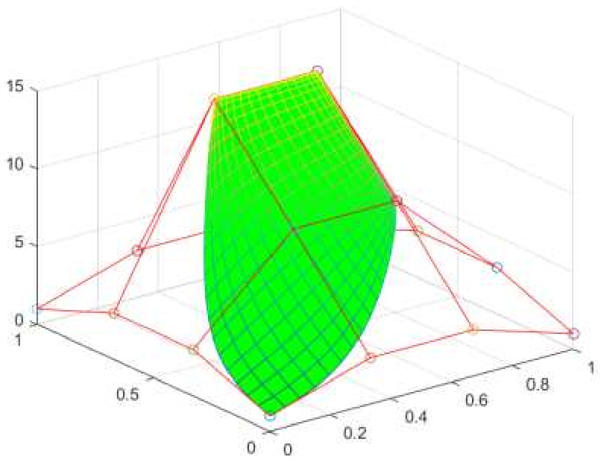


Figure 3:

(4) $\alpha_{1,\ 1} = -1, \alpha_{1,\ 2} = \alpha_{2,\ 1} = \alpha_{2,\ 2} = 0$

At this point, the surface is pulled out of the convex hull of the control mesh and a region opposite to the original direction is formed as shown in Figure 4.

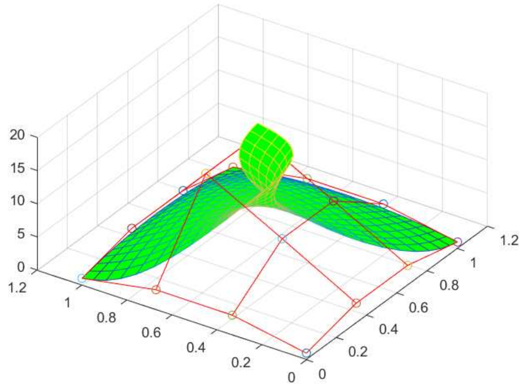


Figure 4:

(5) $\alpha_{1,1}=2,\alpha_{1,2}=\alpha_{2,1}=\alpha_{2,2}=0$

At this time, the surface is also pulled out of the convex hull of the control mesh, because the parameters are positive at this time, so the direction of the surface being stretched is the same as that of the original surface,that is, the direction of the control grid is the same, as shown in Figure 5.

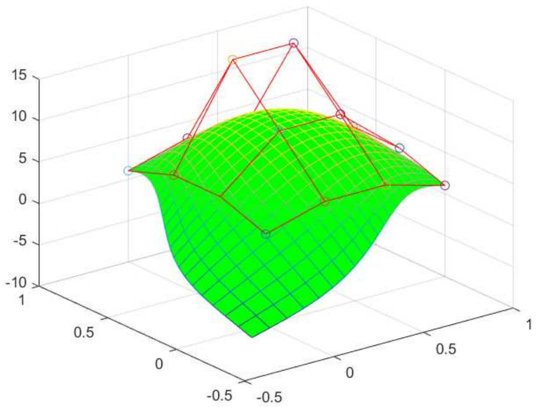


Figure 5:

(6) $\alpha_{1,1}=\alpha_{1,2}=\alpha_{2,1}=\alpha_{2,2}=0$

In this case, the singular mixed Bézier surface becomes a bilinear singular surface patch. The surface approximates the surface of the polyhedron formed by the control mesh. See Properties 6 and Figure 6.

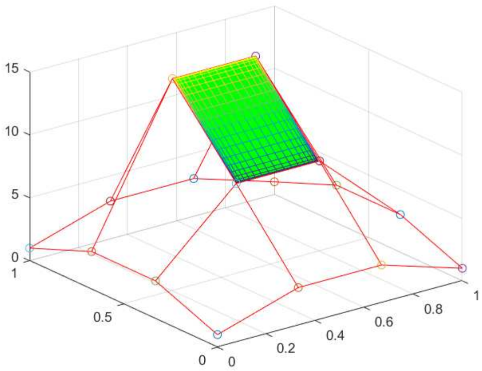


Figure 6:

(7) $\alpha_{1,1} = \alpha_{1,2} = \alpha_{2,1} = \alpha_{2,2} = \alpha$

At this time, according to formula (6), there are

$$\alpha(u,v) = \alpha_{1,1} f(u)f(v) + \alpha_{1,2} f(u)\bar{f}(v) + \alpha_{2,1}\bar{f}(u)f(v) + \alpha_{2,2}\bar{f}(u)\bar{f}(v)$$

$$= \alpha(f(u)f(v) + f(u)\bar{f}(v) + \bar{f}(u)f(v) + \bar{f}(u)\bar{f}(v))$$

$$= \alpha[f(u)(f(v) + \bar{f}(v)) + \bar{f}(u)(f(v) + \bar{f}(v))] = \alpha(f(u) + \bar{f}(u)) = \alpha \qquad (53)$$

According to Formula (4)

$$Q(u,v) = \alpha P(u,v) + (1-\alpha)L(u,v) \qquad (54)$$

It can be seen from the above formula that the surface at this time is a partitioned surface that divides bilinear singular surface patches A and B into a partitioned surface with a fixed ratio of C as shown in Figure 7.Q0 is the point on$L(u,v)$, Q1 and Q2 are the corresponding points on $P(u,v)$ and mixed surface, where the corresponding points are the points with the same parameters, then there are

$$\alpha = \frac{Q_0Q_2}{Q_0Q_1} = \frac{Q_0Q_2}{Q_0Q_2+Q_2Q_1} \qquad (55)$$

So there are

$$\alpha:(1-\alpha) = Q_0Q_2:Q_2Q_1 \qquad (56)$$

Therefore, we call the singular mixed Bézier surface at this time a bi-linear singular surface patch $L(u,v)$ and surface patch $P(u,v)$as a fixed ratio α of the partitioned surface.

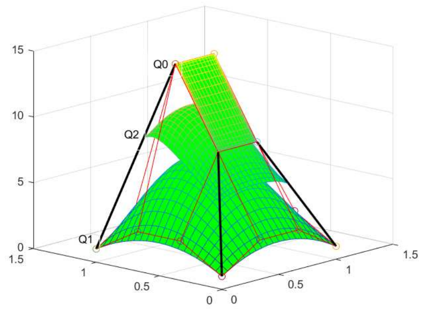
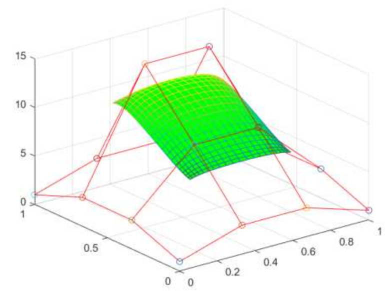


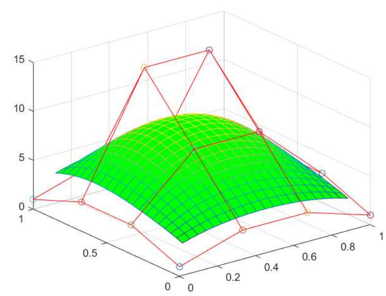
Figure 7:

When the segmentation parameters vary with the points on the surface and change according to regular pattern in formula (6), the usual singular mixed Bézier surface patches are obtained. From here we can see the true meaning of mixed surface.

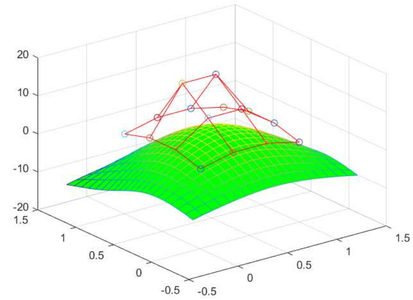
In this case, α becomes a whole parameter, and the shape of the surface can be adjusted as a whole by changing its value as shown in Figure 8.



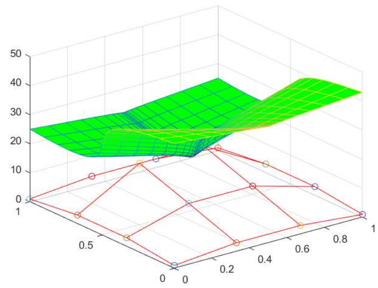
(a) $\alpha = 0.3$



(b) $\alpha = 0.8$



(b) $\alpha = 2$



(d) $\alpha = -2$

Figure 8:

Other cases can be discussed similarly, but not here.

5. Splicing of surface patches

5.1. Parameter splicing of singular mixed Bézier surfaces

If the two curved surfaces patches $Q_1(u,v)$ and $Q_2(u,v)$ are determined by the control vertices and $\bar{P}|i,j = 0,1,2,3$ and the mixed parameters $\alpha_{i,j}|i,j = 1,2$ and $\bar{\alpha}_{i,j}|i,j = 1,2$ respectively, then their equations are as follows:

$$Q_1(u,v) = \alpha(u,v)\sum_{i=0}^{3}\sum_{j=0}^{3} P_{i,j}J_{3,i}(u)J_{3,j}(v) + (1-\alpha(u,v))\,L(u,v) \qquad (0 \le u,v \le 1) \tag{57}$$

$$Q_2(u,v) = \bar{\alpha}(u,v)\sum_{i=0}^{3}\sum_{j=0}^{3} \bar{P}_{i,j}J_{3,i}(u)J_{3,j}(v) + (1-\bar{\alpha}(u,v))\,\bar{L}(u,v) \qquad (0 \le u,v \le 1) \tag{58}$$

Because of the directionality of the curved surface, there are three forms of stitching:

u -direction and u -direction splicing, u -direction and v -direction splicing, v -direction and v -direction splicing.Limited to space, this question only discusses the splicing of directio$u$nd direction uu. The other two splicing methods can be discussed similarly.

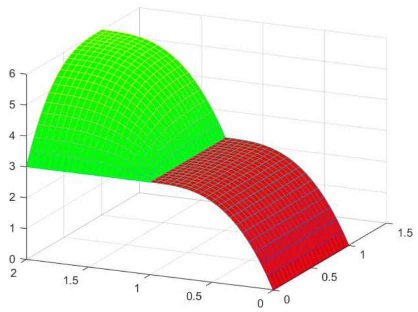
Therefore, if the two surfaces are to be continuous in position or C^0, there should be a public boundary, namely

$$Q_1(u,1) = Q_2(u,0) \tag{59}$$

Formulas (23) and (24) can be sorted out.

$$(\alpha_{2,1} f(u) + \alpha_{2,2}\bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u) P_{i,3} + (1 - \alpha_{2,1} f(u) - \alpha_{2,2}\bar{f}(u))(P_{2,1} f(u) + P_{2,2}\bar{f}(u)) \tag{60}$$

$$= (\bar{\alpha}_{1,1} f(u) + \bar{\alpha}_{1,2}\bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u) \bar{P}_{i,0} + (1 - \bar{\alpha}_{1,1} f(u) - \bar{\alpha}_{1,2}\bar{f}(u))(\bar{P}_{1,1} f(u) + \bar{P}_{1,2}\bar{f}(u))$$



$(0 \le u, v \le 1)$

Figure 9: C^0 continuous stitching curved surface

In order to achieve C^1, C^1 continuity of this surface, In addition to satisfying the above formula, it should also satisfy

$$\frac{\partial}{\partial v} Q_1(u,1) = \frac{\partial}{\partial v} Q_2(u,0) \tag{61}$$

Namely

$$(\alpha_{2,1} f(u) + \alpha_{2,2}\bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u)(P_{i.3} - P_{i,2}) \tag{62}$$

$$= (\bar{\alpha}_{1,1} f(u) + \bar{\alpha}_{1,2}\bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u)(\bar{P}_{i,1} - \bar{P}_{i,0}) \qquad (0 \le u, v \le 1)$$

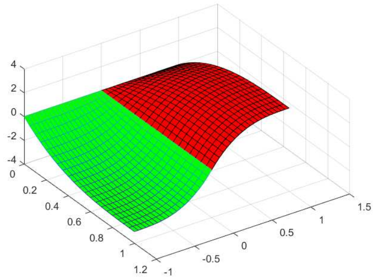


Figure 10: C^1 continuous stitching curved surface

In order to achieve C^2-continuity of this Bézier surface, In addition to the satisfactions (59) and (61), they should also be satisfied.

$$\frac{\partial^2}{\partial v^2}Q_1(u,1)=\frac{\partial^2}{\partial v^2}Q_2(u,0) \qquad (0\le u,v\le 1) \ (63)$$

Namely

$$(\alpha_{2,1}f(u)+\alpha_{2,2}\bar{f}(u))\sum_{i=0}^{3}J_{3,i}(u)(P_{i,1}-2P_{i,2}+P_{i,3})$$

$$=$$

$$(\bar{\alpha}_{1,1}f(u)+\bar{\alpha}_{1,2}\bar{f}(u))\sum_{i=0}^{3}J_{3,i}(u)(\bar{P}_{i,0}-2\bar{P}_{i,1}+\bar{P}_{i,2}) \qquad (0\le u,v\le 1)(64)$$

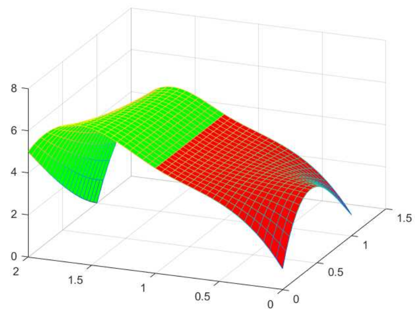


Figure 11: C^1C^1 continuous stitching curved surface

From these stitching conditions, we can see that the stitching conditions of singular hybrid surfaces are much looser than those of Bézier surfaces, that is to say, the stitching conditions of singular hybrid surfaces are much more flexible and have more degrees of freedom than the original surfaces [22].

6.2. Geometric stitching of singular hybrid Bézier surfaces

Parametric continuity is always related to parameter selection, because there is no intrinsic arc length parameter on the surface like a curve, make the parametric continuity of a surface always hold only under certain parameters. If the common joints of two surfaces are non-regular; although they are A along this connection line, it is possible that there are common tangent planes not everywhere along the curves, so they are not smooth. It is known from experience and intuition that the smooth connection of two surfaces along the lowest order of the common connection line, i.e. the first order smooth, requires only a common tangent plane exists along the junction, without requiring them to be C^1 .It can be seen that parameter continuity can not accurately measure the smoothness of surface connection, which is too restrictive and unnecessary. The following is a description of the geometric continuity that can accurately measure the smoothness of surface joints.

The zero order geometric continuity of two surfaces, i.e. G^0 continuity, is consistent with C^0 continuity.

The first-order geometric continuity of two surfaces, i.e. G^0-continuity, is also called tangent plane continuity, it is specifically defined as:

Definition 3. Two surfaces have G^1 continuity along their common connecting lines, if and only if they have common tangent plane or common surface normal everywhere along the common connecting lines.

Therefore, for two surfaces $P(s,t)$, $Q(u,v)$ with common isoparametric lines, P_s and Q_u are parallel at any point on the common isoparametric lines, so the common tangent plane requirement becomes the

$$P_t(P_t, Q_u, Q_v) = 0 \tag{65}$$

Or represent one vector as a linear combination of the other two vectors

$$P_t = \square(u)Q_v + g(u)Q_u \qquad \square(u) > 0 \tag{66}$$

G continuity is also called curvature continuity. Veron , Kanmann and Zhang Guicang [24-25] all show that G continuity requires common normal curvature in all directions along the public connection line. It can be specifically defined as

Definition 4. Two surfaces have G^2 continuity along their common connection line, if and only if they have tangent planes along the common connection line, and have common principal curvature, and if the two principal curvatures are not equal, they have common principal direction.

$$P_{ts} = gQ_{uv} + \square Q_{vv} + aQ_u + bQ_v \tag{67}$$

$$P_{tt} = g^2\, Q_{uu} + 2g\square Q_{uv} + \square^2\, Q_{vv} + cQ_u + dQ_v \tag{68}$$

Therefore, for two surfaces (57) and (58), the G^0 connection with C^0 is the satisfiable formula (60), for them to be G^1-connected, it is not only necessary to satisfy formula (60) , and according to Definition 3 and Formula (66), it should be satisfied.

$$\frac{\partial Q_2(u,0)}{\partial v} = \square(u)\frac{\partial Q_1(u,1)}{\partial v} + g(u)\frac{\partial Q_1(u,1)}{\partial u} \tag{69}$$

Namely

$$(\bar{\alpha}_{1,1} f(u) + \bar{\alpha}_{1,2}\bar{f}(u))\sum_{i=0}^{3} J_{3,i}(u)(\bar{P}_{i,1} - \bar{P}_{i,0}) \tag{70}$$

$$= \quad (u)(\alpha_{2,1} f(u) + \alpha_{2,2}\bar{f}(u))\sum_{i=0}^{3} J_{3,i}(u)(P_{i.3} - P_{i,2}) + g(u)[(\alpha_{2,1} - \alpha_{2,2})f^{'}(u)\sum_{i=0}^{3} J_{3,i}(u)(P_{i,3} - P_{i,2}) + (\alpha_{2,1} f(u) + \alpha_{2,2}\bar{f}(u))\sum_{i=0}^{3} J_{3,i}^{'}(u)(P_{i,3} - P_{i,2})]\ (0 \le u \le 1)$$

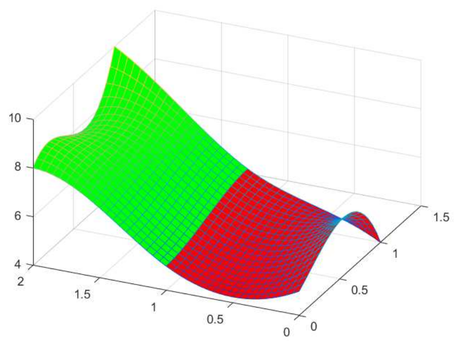


Figure 12: G^1 continuous stitching curved surface

The conditions for G^2-connection of two surfaces (57) and (58) are based on satisfying formulas (60) and (69), and then according to formulas (67) and (68), they should also be satisfied.

$$\frac{\partial^2 Q_2(u,0)}{\partial u \partial v} = g(u)\frac{\partial^2 Q_1(u,1)}{\partial u \delta v} + \quad (u)\frac{\partial^2 Q_1(u,1)}{\partial v^2} + a(u)\frac{\partial Q_1(u,1)}{\partial u} + b(u)\frac{\partial Q_1(u,1)}{\partial v}\ (0 \le u \le 1) \tag{71}$$

$$\frac{\partial^2 Q_2(u,0)}{\partial v^2} g^2(u) \frac{\partial^2 Q_1(u,1)}{\partial u^2} + 2g(u)\square(u) \frac{\partial^2 Q_1(u,1)}{\partial u \partial v} + \square^2(u) \frac{\partial^2 Q_1(u,1)}{\partial v^2}$$

$$+c(u)\frac{\partial Q_1(u,1)}{\partial u} d(u) \frac{\partial Q_1(u,1)}{\partial v} (0 \le u \le 1) \quad (72)$$

The two formulas above are sorted out as follows

$$3\left(\alpha_{1,1} f(u) + \alpha_{1,2} f(u)\right) \sum_{i=0}^{3} J'_{3,i}(u)(P_{i,1} - P_{i,0})$$

$$+3\left(\bar{\alpha}_{1,1} + \bar{\alpha}_{1,2}\right) f'(u) \sum_{i=0}^{3} J_{3,i}(u)(\bar{P}_{i,1} - \bar{P}_{i,0})$$

$$= g(u)(3(\alpha_{2,1} f(u) + \alpha_{2,2}\bar{f}(u)) \sum_{i=0}^{3} J'_{3,i}(u)(P_{i,3} - P_{i,2}) + 3(\alpha_{2,1}$$

$$- \alpha_{2,2}) f'(u) \sum_{i=0}^{3} J_{3,i}(u)(P_{i,3} - P_{i,2}))$$

$$+6\quad(u)(\alpha_{2,1} f(u) + \alpha_{2,2}\bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u)(P_{i,1} - 2P_{i,2} + P_{i,3}) + (\alpha_{2,1} f(u)$$

$$+ \alpha_{2,2}\bar{f}(u)) \sum_{i=0}^{3} J'_{3,i}(u) P_{i,3}$$

$$+a(u)(\alpha_{2,1} f(u) + \alpha_{2,2}\bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u)(P_{i,3} - P_{i,2}) + b(u)((\alpha_{2,1}$$

$$- \alpha_{2,2}) f'(u) \sum_{i=0}^{3} J_{3,i}(u) P_{i,3})$$

$$-\left(\alpha_{2,1} - \alpha_{2,2}\right) f'(u)(P_{2,1} f(u) + P_{2,2} f(u)) - (1 - \alpha_{2,1} f(u) - \alpha_{2,2} f(u))(P_{2,1} - P_{2,2}) f'(u)) \quad (0 \le u \le 1) \quad (73)$$

$$6\ (\alpha_{1,1} f(u) + \alpha_{1,2}\ f\ (u)) \sum_{i=0}^{3} J_{3,i}(u)(P_{i,0} - 2P_{i,1} + P_{i,2})$$

$$= g^2 (u)(3(\alpha_{2,1} - \alpha_{2,2}) f''(u) \sum_{i=0}^{3} J_{3,i}(u) P_{i,3} + 6(\alpha_{2,1} - \alpha_{2,2}) f'(u) \sum_{i=0}^{3} J'_{3,i}(u) P_{i,3}$$

$$+ 3\ (\alpha_{2,1} f(u) + \alpha_{2,2}\ f\ (u)) \sum_{i=0}^{3} J''_{3,i}(u) P_{i,3}$$

$$-\ (\alpha_{2,1} - \alpha_{2,2}) f''(u)(P_{2,1} f(u) + P_{2,2} \bar{f}(u))$$

$$-2\ (\alpha_{2,1} - \alpha_{2,2})(f'(u))^2 (P_{2,1} - P_{2,2}) - (1 - \alpha_{2,1} f(u) - \alpha_{2,2} \bar{f}(u))(P_{2,1} - P_{2,2}) f''(u))$$

$$+2g(u)\quad (u)(3(\alpha_{2,1} f(u) + \alpha_{2,2} \bar{f}(u)) \sum_{i=0}^{3} J'_{3,i}(u)(P_{i,3} - P_{i,2}) + 3(\alpha_{2,1} - \alpha_{2,2}) f'(u) \sum_{i=0}^{3} J_{3,i}(u)(P_{i,3} - P_{i,2}))$$

$$+6\ \ {}^2 (u)(\alpha_{2,1} f(u) + \alpha_{2,2} \bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u)(P_{i,1} - 2P_{i,2} + P_{i,3})$$

$$+c(u)(\alpha_{2,1} f(u) + \alpha_{2,2} \bar{f}(u)) \sum_{i=0}^{3} J_{3,i}(u)(P_{i,3} - P_{i,2}) + d(u)((\alpha_{2,1} - \alpha_{2,2}) f'(u) \sum_{i=0}^{3} J_{3,i}(u) P_{i,3})$$

$$+(\alpha_{2,1} f(u) + \alpha_{2,2} \bar{f}(u)) \sum_{i=0}^{3} J'_{3,i}(u) P_{i,3}$$

$$-\ (\alpha_{2,1} - \alpha_{2,2}) f'(u)(P_{2,1} f(u) + P_{2,2}\ f\ (u)) - (1 - \alpha_{2,1} f(u) - \alpha_{2,2}\ f\ (u))(P_{2,1} - P_{2,2}) f'(u)) \qquad (0 \le u \le 1)$$

(74)

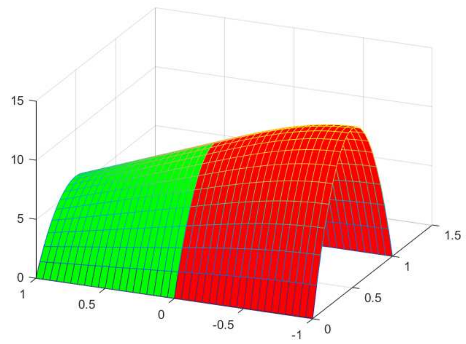


Figure 13: G^2 continuous stitching curved surface

From the formulas (70), (73) and (74) above, it can be seen that the stitching conditions of geometric continuity are much looser than those of parametric continuity, and there are many parameters that can be freely selected, that certainly increases the degree of freedom of surface design control. However, too many free parameters in the actual surface design work also make people confused, increasing trouble, so in the actual application, we often adopt a simplified way. The $g(u)$、 (u)、$a(u)$、$b(u)$、$c(u)$ and$d(u)$ functions can be taken as constants. When $(u) = 1, g(u) = a(u) = b(u) = c(u) = d(u) = 0$, the condition of geometric continuity is the condition of parameter continuity.

7. Conclusion

In this paper, a singular mixed Bézier surface is constructed. The special properties of bicubic singular mixed Bézier surface are analyzed importantly, the geometric meaning of the mixed parameters is understood, and the influence of the mixed parameters on the shape of the surface is discussed. Finally, the splicing conditions of bicubic singular mixed Bézier surface in parameter splicing and geometric splicing are given.

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