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On Positive Solution for a Class of Singular Elliptic System involving Critical Coupling Terms

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Abstract. This paper deals with a class of singular semilinear elliptic system involving Hardy terms, critical coupling terms and negative-exponent terms. By using Ekeland variational principle and the Nehari set, we prove the existence of positive ground state weak solution.

Keywords: Semilinear elliptic system; Hardy term; Critical Sobolev exponents; The Ekeland variational principle

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1. Introduction

The aim of this paper is to establish the existence of nontrivial weak solution to the following semilinear elliptic system

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{\lambda \alpha_1}{\alpha_1+\beta_1} u^{\alpha_1-1} v^{\beta_1} + \dfrac{\alpha_2}{\alpha_2+\beta_2} f(x) u^{\alpha_2-1} v^{\beta_2}, & x \in \Omega, \\ -\Delta v - \mu \dfrac{v}{|x|^2} = \dfrac{\lambda \beta_1}{\alpha_1+\beta_1} u^{\alpha_1} v^{\beta_1-1} + \dfrac{\beta_2}{\alpha_2+\beta_2} f(x) u^{\alpha_2} v^{\beta_2-1}, & x \in \Omega, \\ u, v > 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \ge 3)$ is a smooth bounded domain, $0 \in \Omega$, Δ is the Laplacian operator, $0 \le \mu < \bar{\mu}$, $\bar{\mu} \triangleq \left(\frac{N-2}{2}\right)^2$ is the best Hardy constant, $\lambda > 0$ is a real parameter, $\alpha_1, \beta_1 > 1$, $\alpha_1 + \beta_1 = 2^*$, $2^* \triangleq \frac{2N}{N-2}$ is the critical Sobolev exponent, $\alpha_2, \beta_2 > 0, \alpha_2 + \beta_2 < 2$ and $f : \Omega \to \mathbb{R}$ is a given non-negative nontrivial function.

By setting α_2, β_2 larger than 0 rather 1 and $\alpha_2+\beta_2<2$, we find that at least one of the exponents of u and v is negative, namely $\min\{\alpha_2-1, \beta_2-1\}<0$. Hence problem (1) contains both negative exponent terms and critical Sobolev exponent terms.

Set $u=v, \alpha_1=\beta_1=\frac{2^*}{2}, \alpha_2=\beta_2>0, \alpha_2+\beta_2<1$, then system (1) reduces to the following equation

$$\begin{cases} -\Delta u-\mu\frac{u}{|x|^2}=\lambda u^{2^*-1}+f(x)u^q, & x\in\Omega, \\ u>0, & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{2}$$

where $0<q<1$. The existence of solution for problems like (2) are studied extensively and many results have been found [1-3].

In recent years, much attention has been paid to critical elliptic systems, but the exponents in these systems are usually positive (see [4-7]). Hsu and Li [8] studied the following semilinear elliptic system:

$$\begin{cases} -\Delta u-\mu\frac{u}{|x|^2}=\lambda|u|^{q-2}u+\frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^s}, & \text{in } \Omega\setminus\{0\}, \\ -\Delta v-\mu\frac{v}{|x|^2}=\sigma|v|^{q-2}v+\frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s}, & \text{in } \Omega\setminus\{0\}, \\ u=v=0, & \text{on } \partial\Omega, \end{cases}$$

where $\lambda,\sigma>0$, $0\le\mu<\bar{\mu}=\left(\frac{N-2}{2}\right)^2$, $1\le q<2$, $0\le s<2$, and $\alpha,\beta>1$ satisfy $\alpha+\beta=2^*(s)=\frac{2(N-s)}{N-2}$. They obtained the multiplicity of positive solutions by variational methods when the parameters λ,σ satisfy $0<\lambda^{\frac{2}{2-q}}+\sigma^{\frac{2}{2-q}}<\Lambda$, where Λ is a certain positive constant.

The existence and nonexistence of solutions of elliptic systems with negative exponents are also studied in [9-11] and the papers therein. However, to the best of our knowledge, there are few results for the elliptic system (1) involving both critical Sobolev exponents and negative exponents. Therefore, it is meaningful for us to study the elliptic systems (1).

However, the first difficulty to study the existence of solutions of (1) is that the functional corresponding with (1) is no longer Frechet differentiable. Therefore, there exists no $(PS)_c$ sequence for the functional. The second difficulty is due to the lack of compactness of the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^{2^*}(\Omega)$. By using the Ekeland variational principle and the Vitali theorem, we overcome the above difficulties and obtain some new results.

The main result of this paper is the following theorem.

Theorem 1.1. Suppose $f\in L^{\infty}(\Omega)$ and $\Omega\subset\mathbb{R}^N$ is bounded, $0\in\Omega$, then there exists constants $\lambda^*>0$ and $\overline{C}$ such that for all $\lambda\in(0,\lambda^*)$ and $|f|_{\infty}<\overline{C}$, problem (1) has a positive ground state weak solution.

2. Preliminaries and some technical lemmas

Throughout this paper, the space $W_0^{1,2}(\Omega)$ represents the completion of $C_0^{\infty}(\Omega)$ with the norm $\left(\int_\Omega|\nabla\cdot|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$. Problem (1) is related to the well-known Hardy inequality

$$\int_\Omega\frac{|u|^2}{|x|^2}\mathrm{d}x\le\frac{1}{\overline{\mu}}\int_\Omega|\nabla u|^2\,\mathrm{d}x,\quad\forall u\in W_0^{1,2}(\Omega).$$

In space $W_0^{1,2}(\Omega)$, we employ the following norm

$$\|u\|_\mu=\left[\int_\Omega\left(|\nabla u|^2-\mu\frac{|u|^2}{|x|^2}\right)\mathrm{d}x\right]^{\frac{1}{2}},\quad\forall\mu\in[0,\overline{\mu}).$$

By Hardy inequality, the above norm is equivalent to the usual norm $\left(\int_\Omega|\nabla\cdot|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$ in $W_0^{1,2}(\Omega)$. In this paper, we work on the product space $E=W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ with the norm

$$\|(u,v)\|=\left(\|u\|_\mu^2+\|v\|_\mu^2\right)^{\frac{1}{2}}.$$

Besides, the space $L^p(\Omega),\ 1\le p<\infty$ denotes the usual Lebesgue space.

For simplicity, we write $q=\alpha_2+\beta_2$. A pair of functions $(u,v)\in E$ is said to be a weak solution of problem (1) if

$$\int_\Omega\left(\nabla u\nabla\varphi_1+\nabla v\nabla\varphi_2-\mu\frac{u\varphi_1}{|x|^2}-\mu\frac{v\varphi_2}{|x|^2}\right)\mathrm{d}x-\frac{\lambda}{2^*}\int_\Omega\left(\alpha_1u^{\alpha_1-1}v^{\beta_1}\varphi_1+\beta_1u^{\alpha_1}v^{\beta_1-1}\varphi_2\right)\mathrm{d}x$$
$$-\int_\Omega\frac{f(x)}{q}\left(\alpha_2u^{\alpha_2-1}v^{\beta_2}\varphi_1+\beta_2u^{\alpha_2}v^{\beta_2-1}\varphi_2\right)\mathrm{d}x=0,\quad\forall(\varphi_1,\varphi_2)\in E.$$

The corresponding energy functional $I:E\to\mathbb{R}$ associated with the problem (1) is defined by

$$I(u,v)=\frac{1}{2}\int_\Omega\left(|\nabla u|^2+|\nabla v|^2-\mu\frac{|u|^2}{|x|^2}-\mu\frac{|v|^2}{|x|^2}\right)dx$$
$$-\frac{\lambda}{2^*}\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}\,dx-\frac{1}{q}\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,dx.$$

For any $\mu\in[0,\bar{\mu})$, $\alpha,\beta>1$ and $\alpha+\beta=2^*$, we define the following constants:

$$S=S(\mu)\triangleq\inf_{u\in W_0^{1,2}(\Omega)\setminus\{0\}}\frac{\int_\Omega\left(|\nabla u|^2-\mu\frac{|u|^2}{|x|^2}\right)dx}{\left(\int_\Omega|u|^{2^*}\,dx\right)^{\frac{2}{2^*}}},$$

$$S_{\alpha,\beta}\triangleq\inf_{u,v\in W_0^{1,2}(\Omega)\setminus\{0\}}\frac{\|(u,v)\|^2}{\left(\int_\Omega|u|^{\alpha}|v|^{\beta}\,dx\right)^{\frac{2}{2^*}}},$$

$$\bar{C}\triangleq\frac{2^*-2}{2^*-q}\left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}}S_{\alpha_1,\beta_1}^{\frac{2^*-q}{2^*-2}}|\Omega|^{\frac{q-2^*}{2^*}}\left[\left(\frac{\alpha_2}{q}\right)^{\frac{2}{2-q}}+\left(\frac{\beta_2}{q}\right)^{\frac{2}{2-q}}\right]^{\frac{q-2}{2}}.$$

For each $(u,v)\in E$, we define $J(t)=I((tu,tv))$, namely

$$J(t)=\frac{t^2}{2}\|(u,v)\|^2-\lambda\frac{t^{2^*}}{2^*}\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}\,dx-\frac{t^q}{q}\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,dx,$$
$$J'(t)=t\|(u,v)\|^2-\lambda t^{2^*-1}\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}\,dx-t^{q-1}\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,dx,$$
$$J''(t)=\|(u,v)\|^2-\lambda(2^*-1)t^{2^*-2}\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}\,dx+(1-q)t^{q-2}\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,dx.$$

Meanwhile, we define Nehari set

$$\Lambda=\{(u,v)\in E\setminus\{(0,0)\}\mid J'(1)=0\}$$
$$=\left\{(u,v)\in E\setminus\{(0,0)\}\,\middle|\,\|(u,v)\|^2-\lambda\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}\,dx-\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,dx=0\right\},$$

which contains all solutions of problem (1). We make the following splitting for Λ: $\Lambda^{\pm}=\{(u,v)\in\Lambda\mid J''(1)>(<)0\}$, $\Lambda^0=\{(u,v)\in\Lambda\mid J''(1)=0\}$, i.e.

$$\Lambda^+=\left\{(u,v)\in\Lambda\mid(2-q)\|(u,v)\|^2-\lambda(2^*-q)\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}\,dx>0\right\},$$
$$\Lambda^0=\left\{(u,v)\in\Lambda\mid(2-q)\|(u,v)\|^2-\lambda(2^*-q)\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}\,dx=0\right\},$$
$$\Lambda^-=\left\{(u,v)\in\Lambda\mid(2-q)\|(u,v)\|^2-\lambda(2^*-q)\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}\,dx<0\right\}.$$

In the following, some technical lemmas will be elaborated as follows.

Lemma 2.1. The energy functional $I(u,v)$ is coercive and bounded below on Λ.

Proof: By the Young inequality, $\alpha_2, \beta_2 < q$ and the Holder inequality, we have

$$\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,\mathrm{d}x \le |f|_\infty \left(\frac{\alpha_2}{q}|u|_q^q + \frac{\beta_2}{q}|v|_q^q\right)$$

$$\le |f|_\infty |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \left(\left(\frac{\alpha_2}{q}\right)^{\frac{2}{2-q}} + \left(\frac{\beta_2}{q}\right)^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|(u,v)\|^q \triangleq C_1 \|(u,v)\|^q, \tag{3}$$

where $|\Omega|$ is the Lebesgue measure of Ω. Based on above inequality, we have

$$I(u,v) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\|(u,v)\|^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,\mathrm{d}x$$

$$\ge \left(\frac{1}{2} - \frac{1}{2^*}\right)\|(u,v)\|^2 - C_1\left(\frac{1}{q} - \frac{1}{2^*}\right)\|(u,v)\|^q, \quad \forall (u,v) \in \Lambda.$$

Since $0 < q < 2$, this implies that the conclusion is true. □

Lemma 2.2. There exists a constant $\lambda_1 > 0$, such that $\Lambda^0 = \varnothing$ for all $\lambda < \lambda_1$.

Proof: To the contrary, we suppose that there exists $(\bar{u}, \bar{v}) \in \Lambda^0 \subset \Lambda$, which means

$$(2-q)\|(\bar{u},\bar{v})\|^2 - \lambda(2^* - q)\int_\Omega |\bar{u}|^{\alpha_1}|\bar{v}|^{\beta_1}\,\mathrm{d}x = 0. \tag{4}$$

Based on the definition of Λ, it is easy to find that $\bar{u}, \bar{v} \not\equiv 0$. From (4) and Sobolev inequality it follows that

$$\frac{\|(\bar{u},\bar{v})\|^2}{\int_\Omega |\bar{u}|^{\alpha_1}|\bar{v}|^{\beta_1}\,\mathrm{d}x} = \frac{\lambda(2^*-q)}{2-q} \ge \frac{\|(\bar{u},\bar{v})\|^2}{S_{\alpha_1,\beta_1}^{-\frac{2^*}{2}}\|(\bar{u},\bar{v})\|^{2^*}}.$$

Consequently,

$$\|(\bar{u},\bar{v})\| \ge \left[\frac{S_{\alpha_1,\beta_1}^{\frac{2^*}{2}}(2-q)}{\lambda(2^*-q)}\right]^{\frac{1}{2^*-2}}. \tag{5}$$

On the other hand, from $(\bar{u},\bar{v}) \in \Lambda$, we have

$$\lambda\int_\Omega |\bar{u}|^{\alpha_1}|\bar{v}|^{\beta_1}\,\mathrm{d}x = \|(\bar{u},\bar{v})\|^2 - \int_\Omega f(x)|\bar{u}|^{\alpha_2}|\bar{v}|^{\beta_2}\,\mathrm{d}x,$$

combining with (4) and (3), we have

$$\frac{2^*-2}{2^*-q}\|(\bar{u},\bar{v})\|^2 - C_1\|(\bar{u},\bar{v})\|^q \le 0.$$

Consequently,

$$\|(\bar{u},\bar{v})\| \le \left(\frac{C_1(2^*-q)}{2^*-2}\right)^{\frac{1}{2-q}}. \tag{6}$$

Let

$$\lambda_1 = S_{\alpha_1,\beta_1}^{\frac{2^*}{2}} \frac{2-q}{2^*-q}\left(\frac{2^*-2}{C_1(2^*-q)}\right)^{\frac{2^*-2}{2-q}},$$

then, by (5) and (6), we have $\lambda > \lambda_1$ which is a contradiction, thus $\Lambda^0 = \varnothing$. □

Lemma 2.3. There exists a positive number $\lambda_2 > 0$ such that if $\lambda < \lambda_2$,

(i) $I(u,v) < 0$ for any $(u,v) \in \Lambda^+$;

(ii) $I(u,v) > 0$ for any $(u,v) \in \Lambda^-$.

Proof: First, we prove (i). Arguing by contradiction, we assume that there exists $(u,v) \in \Lambda^+$ which satisfies $I(u,v) \ge 0$, i.e.

$$\frac{1}{2}\|(u,v)\|^2 - \frac{\lambda}{2^*}\int_\Omega |u|^{\alpha_1}|v|^{\beta_1}\,\mathrm{d}x - \frac{1}{q}\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,\mathrm{d}x \ge 0.$$

From $(u,v) \in \Lambda$, it follows that

$$\|(u,v)\| \ge \left[S_{\alpha_1,\beta_1}^{\frac{2^*}{2}} \frac{2^*(2-q)}{2\lambda(2^*-q)}\right]^{\frac{1}{2^*-2}}. \tag{7}$$

On the other hand, from $(u,v) \in \Lambda^+ \subset \Lambda$, we have

$$(2^*-2)\|(u,v)\|^2 - (2^*-q)\int_\Omega f(x)|u|^{\alpha_2}|v|^{\beta_2}\,\mathrm{d}x < 0,$$

combining with (3), we have

$$\|(u,v)\| < \left(\frac{2^*-q}{2^*-2}C_1\right)^{\frac{1}{2-q}}. \tag{8}$$

Let

$$\bar{\lambda}_2 = S_{\alpha_1,\beta_1}^{-\frac{2^*}{2}} \frac{2(2^*-q)}{2^*(2-q)}\left(\frac{2^*-q}{2^*-2}C_1\right)^{\frac{2^*-2}{2-q}}.$$

Then, by (7) and (8), we have $\lambda > \bar{\lambda}_2$ which is a contradiction, thus $I(u,v) < 0$.

In the following, we prove (ii). Let

$$\tilde{\lambda}_2 = S_{\alpha_1,\beta_1}^{\frac{2^*}{2}} \frac{2-q}{2^*-q}\left[\frac{q(2^*-2)}{2(2^*-q)C_1}\right]^{\frac{2^*-2}{2-q}},$$

then by the same method, it can be proved that when $\lambda < \tilde{\lambda}_2$, we have $I(u,v) > 0$ for any $(u,v) \in \Lambda^-$. By taking $\lambda_2 = \min\{\bar{\lambda}_2, \tilde{\lambda}_2\}$, the conclusion is proved. □

Lemma 2.4. If $|f|_\infty < \bar{C}$, there exists a positive number $\lambda_3 \in (0,1)$ such that Λ^+ is nonempty and closed for any $\lambda < \lambda_3$.

Proof: Let

$$\phi(t) = t^{2-q}\|(u,v)\|^2 - \lambda t^{2^*-q}\int_\Omega |u|^{\alpha_1}|v|^{\beta_1}\,\mathrm{d}x, \quad \forall (u,v)\in E\setminus\{(0,0)\},$$

and $t_{\max} > 0$ satisfy $\phi'(t_{\max}) = 0$. By calculation we find

$$t_{\max} = \left[\frac{(2-q)\|(u,v)\|^2}{\lambda(2^*-q)\int_\Omega |u|^{\alpha_1}|v|^{\beta_1}\,\mathrm{d}x}\right]^{\frac{1}{2^*-2}}.$$

From $2^* > 2$, it follows that $\phi(t)$ increases monotonically on $(0,t_{\max})$ and decreases monotonically on $(t_{\max},+\infty)$. So $\phi(t)$ gets its maximum at $t_{\max}$ and

$$\phi(t_{\max}) = \frac{2^*-2}{2^*-q}\left[\frac{2-q}{\lambda(2^*-q)}\right]^{\frac{2-q}{2^*-2}}\left[\frac{\|(u,v)\|^{2(2^*-q)}}{(\int_\Omega |u|^{\alpha_1}|v|^{\beta_1}\,\mathrm{d}x)^{2-q}}\right]^{\frac{1}{2^*-2}}.$$

If $\lambda < 1$, combining with $|f|_\infty < \bar{C}$ and (9), we have

$$0 < \int_\Omega f(x)|u|^{\alpha_1}|v|^{\beta_1}\,\mathrm{d}x < \phi(t_{\max}).$$

From $\phi(t)$ increases monotonically on $(0,t_{\max})$, we know there exists only one $t^+ = t^+_{u,v} < t_{\max}$ such that

$$\phi(t^+) = \int_\Omega f(x)|u|^{\alpha_1}|v|^{\beta_1}\,\mathrm{d}x \text{ and } \phi'(t^+) > 0,$$

which implies $t^+u \in \Lambda^+$. Therefore $\Lambda^+ \neq \emptyset$. □

Lemma 2.5. If $(u,v)\in\Lambda^+$, then there exists a positive number $\epsilon > 0$ and a differentiable function $h = h(t) > 0$, where $t < \epsilon$ such that $h(0) = 1$ and $h(t)(u+t\varphi_1, v+t\varphi_2)\in\Lambda^+$ for all $(\varphi_1,\varphi_2)\in E$.

Proof: By applying the method in [12, 13], we define $H:\mathbb{R}\times E\to\mathbb{R}$ by

$$H(t,\varphi_1,\varphi_2) = t^{2-q}\|(u+\varphi_1, v+\varphi_2)\|^2 - \int_\Omega f(x)|u+\varphi_1|^{\alpha_2}|v+\varphi_2|^{\beta_2}\,\mathrm{d}x$$
$$-\lambda t^{2^*-q}\int_\Omega |u+\varphi_1|^{\alpha_1}|v+\varphi_2|^{\beta_1}\,\mathrm{d}x.$$

Since $u\in\Lambda^+\subset\Lambda$, we have $H(1,0,0) = 0$ and

$$H_t(1,0,0) = (2-q)\|(u,v)\|^2 - \lambda(2^*-q)\int_\Omega \lambda|u|^{\alpha_1}|v|^{\beta_1}\,\mathrm{d}x > 0.$$

According to the implicit function theorem at the point $(1,0,0)$, there exist an $\bar{\epsilon} > 0$ and a continuous function $h = h(t) > 0$, where $t < \bar{\epsilon}$, such that

$$h(0)=1, \quad h(t)(u+t\varphi_1, v+t\varphi_2)\in \Lambda.$$

Clearly, we can take $\epsilon<\overline{\epsilon}$ satisfying

$$h(t)(u+t\varphi_1, v+t\varphi_2)\in \Lambda^+, \quad \forall t<\epsilon, (\varphi_1,\varphi_2)\in E. \ \square$$

Lemma 2.6. There exists a weak solution (u_0, v_0) of the problem (1).

Proof: From Lemma 2.4 and Lemma 2.1, we have $\Lambda^+\neq\emptyset$ and $m=\inf_{(u,v)\in\Lambda} I(u,v)>-\infty$. By the Ekeland variational principle [14], there exists a minimizing sequence $\{(u_k, v_k)\}\subset\Lambda^+$ which satisfies

$$I(u_k,v_k)<m+\frac{1}{k}, \quad I(u_k,v_k)\le I(u,v)+\frac{1}{k}\|(u-u_k, v-v_k)\|, \quad \forall (u,v)\in\Lambda^+. \tag{9}$$

Since $I(u,v)=I(|u|,|v|)$, we assume $u_k, v_k\ge 0$ in Ω and there is a subsequence which is still denoted by $\{(u_k, v_k)\}$ and $u_0, v_0\ge 0$ such that

$$\begin{cases} u_k \rightharpoonup u_0, \quad v_k \rightharpoonup v_0, & \text{weakly in } W_0^{1,2}(\Omega), \\ u_k \to u_0, \quad v_k \to v_0, & \text{strongly in } L^p(\Omega), p\in[1,2^*), \\ u_k(x)\to u_0(x), \quad v_k(x)\to v_0(x), & \text{a.e. in } \Omega. \end{cases}$$

From Hardy inequality and Sobolev embedding theorem, it follows that

$$\begin{cases} u_k^{2^*-1} \rightharpoonup u_0^{2^*-1}, \quad v_k^{2^*-1} \rightharpoonup v_0^{2^*-1}, & \text{weakly in } L^{\frac{2^*}{2^*-1}}(\Omega), \\ \dfrac{u_k}{|x|} \rightharpoonup \dfrac{u_0}{|x|}, \quad \dfrac{v_k}{|x|} \rightharpoonup \dfrac{v_0}{|x|}, & \text{weakly in } L^2. \end{cases}$$

By applying Lemma 2.5, there exists a differentiable function $h_k(t)$ which satisfies

$$h_k(0)=1, \quad h_k(t)(u_k+t\varphi_1, v_k+t\varphi_2)\in\Lambda^+,$$

where $\varphi_1,\varphi_2\in W_0^{1,2}(\Omega)$, $t>0$ small enough. For the sake of simplicity, we write

$$A(u,v)=\nabla u\nabla\varphi_1+\nabla v\nabla\varphi_2-\mu\frac{u\varphi_1}{|x|^2}-\mu\frac{v\varphi_2}{|x|^2},$$

$$B_i(u,v)=\alpha_i u^{\alpha_i-1}v^{\beta_i}\varphi_1+\beta_i u^{\alpha_i}v^{\beta_i-1}\varphi_2, \ i=1,2.$$

From the definition of Λ, we have

$$\|(u_k,v_k)\|^2-\lambda\int_\Omega |u_k|^{\alpha_1}|v_k|^{\beta_1}\,dx-\int_\Omega f(x)|u_k|^{\alpha_2}|v_k|^{\beta_2}\,dx=0. \tag{10}$$

By applying (9), we deduce that

$$\frac{1}{k}\left[|h_k(t)-1|\|(u_k,v_k)\|+th_k(t)\|(\varphi_1,\varphi_2)\|\right]\ge\frac{1}{k}\|h_k(t)(u_k+t\varphi_1,v_k+t\varphi_2)-(u_k,v_k)\|$$

$$\ge I(u_k,v_k)-I\left(h_k(t)(u_k+t\varphi_1),h_k(t)(v_k+t\varphi_2)\right)$$

$$=\frac{1}{2}\|(u_k,v_k)\|^2-\frac{\lambda}{2^*}\int_\Omega|u_k|^{\alpha_1}|v_k|^{\beta_1}\,\mathrm{d}x-\frac{1}{q}\int_\Omega f(x)|u_k|^{\alpha_2}|v_k|^{\beta_2}\,\mathrm{d}x$$

$$-\left[\frac{h_k^2(t)}{2}\|(u_k+t\varphi_1,v_k+t\varphi_2)\|^2-\frac{\lambda}{2^*}h_k^{2^*}(t)\int_\Omega|u_k+t\varphi_1|^{\alpha_1}|v_k+t\varphi_2|^{\beta_1}\,\mathrm{d}x\right.$$

$$\left.-\frac{h_k^q(t)}{q}\int_\Omega f(x)|u_k+t\varphi_1|^{\alpha_2}|v_k+t\varphi_2|^{\beta_2}\,\mathrm{d}x\right]$$

$$=\frac{1-h_k^2(t)}{2}\|(u_k,v_k)\|^2-\lambda\frac{1-h_k^{2^*}(t)}{2^*}\int_\Omega|u_k|^{\alpha_1}|v_k|^{\beta_1}\,\mathrm{d}x-\frac{1-h_k^q(t)}{q}\int_\Omega f(x)|u_k|^{\alpha_2}|v_k|^{\beta_2}\,\mathrm{d}x$$

$$+\frac{h_k^2(t)}{2}\left(\|(u_k,v_k)\|^2-\|(u_k+t\varphi_1,v_k+t\varphi_2)\|^2\right)$$

$$-\frac{\lambda h_k^{2^*}(t)}{2^*}\int_\Omega|u_k|^{\alpha_1}|v_k|^{\beta_1}-|u_k+t\varphi_1|^{\alpha_1}|v_k+t\varphi_2|^{\beta_1}\,\mathrm{d}x$$

$$-\frac{h_k^q(t)}{q}\int_\Omega f(x)\left(|u_k|^{\alpha_2}|v_k|^{\beta_2}-|u_k+t\varphi_1|^{\alpha_2}|v_k+t\varphi_2|^{\beta_2}\right)\mathrm{d}x.$$

From Lemma 2.2 we know $u_k,v_k\not\equiv 0$. Dividing the above inequality by $t>0$ and letting $t\to 0^+$, by (10) we find,

$$\frac{1}{k}\left[|h_k'(0)|\|(u_k,v_k)\|+\|(\varphi_1,\varphi_2)\|\right]$$
$$\ge-\int_\Omega A(u_k,v_k)\mathrm{d}x+\lambda\int_\Omega B_1(u_k,v_k)\mathrm{d}x+\int_\Omega f(x)B_2(u_k,v_k)\mathrm{d}x.$$

Namely,

$$\int_\Omega f(x)B_2(u_k,v_k)\mathrm{d}x$$
$$\le\frac{1}{k}\left[|h_{k'}(0)|\|(u_k,v_k)\|+\|(\varphi_1,\varphi_2)\|\right]+\int_\Omega A(u_k,v_k)\mathrm{d}x-\lambda\int_\Omega B_1(u_k,v_k)\mathrm{d}x.$$

From [12], there exists a constant $C_2>0$ such that $|h_k'(0)|\le C_2$. Let $k\to\infty$ and by the Fatou Lemma, we have

$$\int_\Omega f(x)B_2(u_0,v_0)\mathrm{d}x\le\liminf_{k\to\infty}\int_\Omega f(x)B_2(u_k,v_k)\mathrm{d}x\le\int_\Omega A(u_0,v_0)\mathrm{d}x-\lambda\int_\Omega B_1(u_0,v_0)\mathrm{d}x.$$

Since (φ_1,φ_2) is arbitrary, holds

$$\int_\Omega A(u_0,v_0)\mathrm{d}x-\lambda\int_\Omega B_1(u_0,v_0)\mathrm{d}x-\int_\Omega f(x)B_2(u_0,v_0)\mathrm{d}x=0,\tag{11}$$

thus (u_0,v_0) is a weak solution of problem (1). □

Lemma 2.7. For any $\lambda>0$, $(u_0,v_0)\in\Lambda^+$.

Proof: Since $\{(u_k,v_k)\}$ is bounded in $W_0^{1,2}(\Omega)$, by Sobolev embedding theorem, there exists $C_3>0$ such that $|u_k|_{2^*}^{\alpha_2}|v_k|_{2^*}^{\beta_2}\le C_3$. By the absolute continuity of integral, for all $\epsilon>0$ there exists $\delta>0$ such that if $E\subset\Omega$ and $|E|<\delta$, holds

$$\int_E f^{\frac{2^*}{2^*-q}}\mathrm{d}x<\epsilon^{\frac{2^*}{2^*-q}}.$$

By the Holder inequality, it follows that:

$$\int_E f(x)u_k^{\alpha_2}v_k^{\beta_2}\mathrm{d}x\le\left(\int_E u_k^{2^*}\mathrm{d}x\right)^{\frac{\alpha_2}{2^*}}\left(\int_E v_k^{2^*}\mathrm{d}x\right)^{\frac{\beta_2}{2^*}}\left(\int_E f^{\frac{2^*}{2^*-q}}\mathrm{d}x\right)^{\frac{2^*-q}{2^*}}$$

$$\le|u_k|_{2^*}^{\alpha_2}|v_k|_{2^*}^{\beta_2}\left(\int_E f^{\frac{2^*}{2^*-q}}\mathrm{d}x\right)^{\frac{2^*-q}{2^*}}<C_3\epsilon.$$

Therefore, $\left\{\int_\Omega f(x)|u_k|^{1-s}\,\mathrm{d}x\,\middle|\,k=1,2,\cdots\right\}$ is equi-absolutely continuous. By applying the Vitali theorem [15], we have

$$\lim_{k\to\infty}\int_\Omega f(x)u_k^{\alpha_2}v_k^{\beta_2}\mathrm{d}x=\int_\Omega f(x)u_0^{\alpha_2}v_0^{\beta_2}\mathrm{d}x. \tag{12}$$

Taking $\phi_1=u_0,\phi_2=v_0$ in (11), we know that $(u_0,v_0)\in\Lambda$. Then, by the weak lower semi continuity of norm and (12), we have

$$\begin{aligned}&\inf_{(u,v)\in\Lambda^+}I(u,v)=\lim_{k\to\infty}I(u_k,v_k)\\&=\lim_{k\to\infty}\left[\left(\frac{1}{2}-\frac{1}{2^*}\right)\|(u_k,v_k)\|^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega f(x)|u_k|^{\alpha_2}|v_k|^{\beta_2}\,\mathrm{d}x\right]\\&\ge\left(\frac{1}{2}-\frac{1}{2^*}\right)\|(u_0,v_0)\|^2-\left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega f(x)|u_0|^{\alpha_2}|v_0|^{\beta_2}\,\mathrm{d}x\\&=I(u_0,v_0)\ge\inf_{(u,v)\in\Lambda}I(u,v).\end{aligned}\tag{13}$$

It is easy to know $\inf_{(u,v)\in\Lambda^+}I(u,v)=\inf_{(u,v)\in\Lambda}I(u,v)$ from Lemma 2.2 and Lemma 2.3. Therefore, the inequality signs in (13) should be equal signs. We can deduce that $\lim_{k\to\infty}\|(u_k,v_k)\|=\|(u_0,v_0)\|$. Namely, $u_k\to u_0,v_k\to v_0$ in $W_0^{1,2}(\Omega)$. From Lemma 2.4, we know Λ^+ is closed, thus $(u_0,v_0)\in\Lambda^+$. □

3. The proof of Theorem 1.1

Let $\lambda^* = \min\{\lambda_1, \lambda_2, \lambda_3\}$, then Lemmas 2.1-2.7 hold for all $\lambda < \lambda^*$. Then we have $(u_0, v_0) \in \Lambda^+$ is a weak solution of problem (1) and $I(u_0, v_0) = \inf_{(u,v)\in\Lambda} I(u,v)$ which means (u_0, v_0) is a ground state solution.

Now we prove $u_0, v_0 > 0$ in Ω. Since $u_0, v_0 \geq 0$, for any $\varphi_1, \varphi_2 \in W_0^{1,2}(\Omega)$, $\varphi_1, \varphi_2 \geq 0$, $t > 0$, we have

$$
\begin{aligned}
0 &\leq I(u_0 + t\varphi_1, v_0 + t\varphi_2) - I(u_0, v_0) \\
&= \frac{1}{2}\int_\Omega \left(|\nabla u_0 + t\nabla\varphi_1|^2 + |\nabla v_0 + t\nabla\varphi_2|^2 - \mu\frac{|u_0 + t\varphi_1|^2}{|x|^2} - \mu\frac{|v_0 + t\varphi_2|^2}{|x|^2} \right)\mathrm{d}x \\
&\quad - \frac{\lambda}{2^*}\int_\Omega |u_0 + t\varphi_1|^{\alpha_1} |v_0 + t\varphi_2|^{\beta_1}\, \mathrm{d}x - \frac{1}{2}\int_\Omega \left(|\nabla u_0|^2 + |\nabla v_0|^2 - \mu\frac{|u_0|^2}{|x|^2} - \mu\frac{|v_0|^2}{|x|^2} \right)\mathrm{d}x \\
&\quad - \frac{1}{q}\int_\Omega f(x) |u_0 + t\varphi_1|^{\alpha_2} |v_0 + t\varphi_2|^{\beta_2}\, \mathrm{d}x + \frac{\lambda}{2^*}\int_\Omega |u_0|^{\alpha_1} |v_0|^{\beta_1}\, \mathrm{d}x \\
&\quad + \frac{1}{q}\int_\Omega f(x) |u_0|^{\alpha_2} |v_0|^{\beta_2}\, \mathrm{d}x \\
&\leq \frac{1}{2}\int_\Omega \left(|\nabla u_0 + t\nabla\varphi_1|^2 + |\nabla v_0 + t\nabla\varphi_2|^2 - \mu\frac{|u_0 + t\varphi_1|^2}{|x|^2} - \mu\frac{|v_0 + t\varphi_2|^2}{|x|^2} \right)\mathrm{d}x \\
&\quad - \frac{1}{2}\int_\Omega \left(|\nabla u_0|^2 + |\nabla v_0|^2 - \mu\frac{|u_0|^2}{|x|^2} - \mu\frac{|v_0|^2}{|x|^2} \right)\mathrm{d}x.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\int_\Omega \left(|\nabla u_0 + t\nabla\varphi_1|^2 + |\nabla v_0 + t\nabla\varphi_2|^2 - \mu\frac{|u_0 + t\varphi_1|^2}{|x|^2} - \mu\frac{|v_0 + t\varphi_2|^2}{|x|^2} \right)\mathrm{d}x \\
&- \int_\Omega \left(|\nabla u_0|^2 + |\nabla v_0|^2 - \mu\frac{|u_0|^2}{|x|^2} - \mu\frac{|v_0|^2}{|x|^2} \right)\mathrm{d}x \geq 0,
\end{aligned}
$$

dividing by $t > 0$ and letting $t \to 0$, we get

$$
\int_\Omega \left(\nabla u_0 \nabla\varphi_1 + \nabla v_0 \nabla\varphi_2 - \mu\frac{u_0\varphi_1 + v_0\varphi_2}{|x|^2} \right)\mathrm{d}x \geq 0.
$$

Then, by the strong maximum principle, we have $u_0, v_0 > 0$ in Ω. This completes the proof of Theorem 1.1.

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