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Impulsive Measure Functional Differential Equations: Differentiability of Solutions with Respect to Parameters

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Abstract. In this paper, we establish the differentiability of solutions with respect to parameters for impulsive measure functional differential equations by using the differentiability of solutions with respect to parameters for generalized ordinary differential equations.

Keywords: Impulsive measure functional differential equations; Differentiability of solutions; Generalized ordinary differential equations; Kurzweil integralézier

AMS Mathematics Subject Classification (2010): 47E05

1. Introduction

In the classical theory of ordinary differential equations, it is well known that under certain assumptions, solutions of the problem

$$\begin{cases} \dot{x}(t) = f(x(t),t), & t \in [a,b] \\ x(t_0) = x_0 \end{cases}$$

are differentiable with respect to the initial condition; that is, if $x(t,x_0)$ denotes the value of the solution at $t \in [a,b]$, then the function $x_0 \mapsto x(t,x_0)$ is differentiable. The key requirement is that the right-hand side f should be differentiable with respect to x. Moreover, the derivative as a function of t is known to satisfy the so-called variational equation, which might be helpful in determining the value of the derivative.

Similarly, under suitable assumptions solutions of a differential equation whose right-hand side depends on a parameter are differentiable with respect to that parameter.

In 1957, Kurzweil. introduced a class of integral equations called generalized ordinary differential equations (see [1]). His original motivation was to use them in the study of continuous dependence of solutions with respect to parameters. However, it became clear that generalized equations encompass various other types of equations, including equations with impulses (see [2]) dynamic equations on time scales (see [3]), or measure differential equations (see [2])

In 2013, Alavikhas given the differentiability theorems for generalized ordinary differential equations. Despite the fact that solutions of generalized equations need not be differentiable or even continuous with respect to t, we show that differentiability of the right-hand side with respect to x (and possibly with respect to parameters) still guarantees that the solutions are differentiable with respect to initial conditions (and parameters, respectively) (see [4]). Moreover, in the other equation with generalized ordinary differential equation under the condition of the theorem of equivalent was very good application, such as measure differential equations (see [5]). In [6], the authors have proved that measure functional differential equations (1) are equivalent to the generalized ordinary differential equations

$$\begin{cases} Dx = f(x_t, t)Dg, t \in [t_0, t_0+\sigma], \\ x_{t_0} = \phi. \end{cases} \tag{1}$$

And give a one-to-one relation between the solutions of equation (1) and impulsive measure functional differential equation (2)

$$\begin{aligned} & x(t) - x(t_0) = \int_{t_0}^{t} f(x_s, s)dg(s), t \in [t_0, t_0+\sigma] \\ & \Delta(x(t_k)) = I_k(x(t_k)), k = \{1, 2, \cdots, m\} \\ & x(t_0) = \phi. \end{aligned} \tag{2}$$

And (2) can be written as follows

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)dg(s) + \sum\limits_{t_0 < t_k < t} I(x(t_k)), t \in [t_0, t_0+\sigma] \\ x_{t_0} = \phi. \end{cases} \tag{3}$$

Therefore, the equivalence relation between the impulsive measure functional differential equation and the generalized ordinary differential equation is established. So we can apply some theories of generalized differential equations to the following impulsive measure functional differential equation,

$$\begin{cases} x(t,\lambda) - x(t_0,\lambda) = \int_{t_0}^{t} f(x_s, s, \lambda)dg(s), t \in [t_0, t_0+\sigma] \\ \Delta(x(t_k),\lambda) = I_k(x(t_k),\lambda), k = \{1, 2\cdots, m\} \\ (x_{t_0},\lambda)(\nu) = x^0(\lambda), \lambda > 0. \end{cases} \tag{4}$$

where function $f : P \times [t_0, t_0+\sigma] \to R^n$,and $g : [t_0, t_0+\sigma] \to R$ is a left-continuous function, $\lambda \in \Lambda = \{\lambda \in R^l ; \|\lambda - \lambda_0\| < \rho\}$.

$P = \{y_t : y \in O, t \in [t_0, t_0+\sigma]\} \subset \overline{G}([-r, 0], R^n)$, and $O \subset \overline{G}([t_0 - r, t_0+\sigma], R^n)$. Here $\overline{G}([t_0 - r, t_0 + \sigma], R^n)$ denotes the set of all bounded regulated functions from $[t_0 - r, t_0+\sigma]$ to R^n.

We assume that the function $f : P \times [t_0, t_0+\sigma] \times \Lambda \to R^{n+l}$ satisfies the following conditions:

(A1) The integral $\int_{t_0}^{t_0+\sigma} f(x_t, t, \lambda)dg(t)$ exists for every $(x, \lambda) \in O \times \Lambda$.

(A2) There exists a constant M > 0 such that the inequality
$\left\| \int_a^b f(x_t, t, \lambda)dg(t) \right\| \le M \int_a^b dg(t)$ holds for every $(x, \lambda) \in O \times \Lambda$, and $a, b \in [t_0, t_0 + \sigma]$.

(A3) There exists a constant L>0 such that the inequality
$\left\| \int_a^b (f(x_t, t, \lambda_1) - f(y_t, t, \lambda_2))dg(t) \right\| \le \int_a^b L\omega(\|x - y\| + \|\lambda_1 - \lambda_2\|)dg(t)$. holds for every $(x, \lambda_1), (y, \lambda_2) \in O \times \Lambda$, and every $a, b \in [t_0, t_0 + \sigma]$. where $\omega : [0, +\infty) \to R$ is an increasing function with $\omega(0) = 0$. (We assume that the right-hand side integral exists).

For impulse operator $I_k : P \times \Lambda \to R^{n+l}, k = \{1, 2, \cdots, m\}$ the following conditions are assumed:

(A4) There exists a constant $K_1 > 0$, such that $\|I_k(y, \lambda)\| \le K_1$.
for every $k = \{1, 2, \cdots, m\}$ and $(x, \lambda) \in O \times \Lambda$.

(A5) There exists a constant $K_2 > 0$, such that
$\|I_k(x, \lambda_1) - I_k(y, \lambda_2)\| \le K_1(\|x - y\| + \|\lambda_1 - \lambda_2\|)$. for every $k = \{1, 2, \cdots, m\}$
and $(x, \lambda_1), (y, \lambda_2) \in O \times \Lambda$.

In this assumption, equation (4) is equivalent to the following integral equation

$$x(t, \lambda) = x^0(\lambda) + \int_{t_0}^{t} D_t F(x(\tau, \lambda), t),\ \lambda \in \Lambda, t \in [t_0, t_0 + \sigma]. \tag{5}$$

Here $F(x, t, \lambda) = \int_{t_0}^{t} f(x_t, t, \lambda)dg(t) + \sum_{t_0 < t_k < t} I_k(x, \lambda)H_{t_k}(t)$, where $H_\nu(t)$ denotes the eigen function on the interval $[t_0, t_0 + \sigma]$, and $\nu \in [t_0, t_0 + \sigma]$. When $\nu \ge t$, $H_\nu(t) = 0$; when $\nu < t$, $H_\nu(t) = 1$. then $\sum_{t_0 < t_k < t} I_k((x, \lambda)(t_k)) = \sum_{k=1}^{m} I_k(x, \lambda)H_{t_k}(t)$ and

$\lambda_0 \in R^l, \rho > 0, \Lambda = \{\lambda \in R^l; \|\lambda - \lambda_0\| < \rho\}$. In addition, assume that $S = O \times \Lambda$ and define a new norm

$$\|(x, \lambda)\| = \sum_{i=1}^{n} |x_i| + \sum_{j=1}^{l} |\lambda_j| = \|x\| + \|\lambda\|. \text{where } (x, \lambda) \in S .$$

The aim of this paper is to obtain the differentiability of solutions with respect to parameters for impulsive measure function differential equations (4) by using the differentiability theorem of generalized ordinary differential equation.

This paper mainly includes three parts: the second part mainly introduces the basic concepts and lemma used in this paper. In the third part, by using the equivalence relation between the functional differential equation with impulse measure and the generalized ordinary differential equation, the differentiability theorem of the relative parameters of the solution of the functional differential equation with impulse measure (4) is established.

2. Preliminaries

In this section, we mainly introduce the relevant knowledge of generalized ordinary differential equation and Kurzweil integral.

Definition 1. A matrix-valued function $U:[a,b]\times[a,b]\to R^{n\times m}$ is called Kurzweil integrable on [a,b], if there is a matrix $I\in R^{n\times m}$, such that for every $\varepsilon>0$, there is a gauge δ on [a,b] such that

$$\left\|\sum_{i=1}^{k}(U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1}))-I\right\|\le\varepsilon \tag{6}$$

for every δ -fine partition D. In this case, we define $\int_a^b D_t U(\tau,t)=I$.

An important special case is the Kurzweil-Stieljes integral of a function $f:[a,b]\to R^n$ with respect to a function $g:[a,b]\to R$, which corresponds to the choice $U(\tau,t)=f(\tau)g(t)$ and will be denoted by $\int_a^b D_t U(\tau,t)=\int_a^b f(t)dg(t)$.

Definition 2. $G\subset R^n\times R,(x(t),t)\in G,\ F:\ G\times[a,b]^2\to R^n$,

A function $x:[a,b]\to G$ is called a solution of the generalized ordinary differential equation

$$\frac{dx}{dt}=D_t F(x,t) \tag{7}$$

whenever

$$x(s)=x(a)+\int_s^a D_t F(x(\tau),t).$$

Definition 3. A function $F:\ G\to R^n$ belongs to the class $\mathscr{F}(G,h,\omega)$, if there exists a non-decreasing function $h:[a,b]\to R$ and a continuous, increasing function $\omega:[0,+\infty)\to R$ with $\omega(0)=0$ such that

$$\|F(x,s_2)-F(x,s_1)\|\le|h(s_2)-h(s_1)| \tag{8}$$

for all $(x,s_2),(x,s_1)\in G$ and

$$\|F(x,s_2)-F(x,s_1)-F(y,s_2)+F(y,s_1)\|\le\omega(\|x-y\|)|h(s_2)-h(s_1)| \tag{9}$$

For all $(x,s_2),(x,s_1),(y,s_2),(y,s_1)\in G$.

Remark 1. In the definition 2, let $F(x,t)$ be replaced by $F(x,t,\lambda)$, then the equation (7) is equivalent to

$$\frac{dx}{dt} = D_t F(x,t,\lambda)$$

Now, definition 2 can be redescribed as follows

A function $x:[a,b]\times\Lambda \to R^n$ is called a solution of the generalized ordinary differential equation

$$\frac{dx}{dt} = D_t F(x,t,\lambda)$$

whenever

$x(s,\lambda) = x(a,\lambda) + \int_a^s D_t F(x(\tau),t,\lambda)$. Then definition 3 can be redefined as follows

Let $P = \{y_t : y \in O, t \in [t_0, t_0+\sigma]\} \subset \overline{G}([-r,0], R^n)$, and $O \subset \overline{G}([t_0 - r, t_0+\sigma], R^n)$.

$F: O\times[t_0,t_0+\sigma]\times\Lambda \to R^n$ belongs to the class $\mathscr{F}(O\times[t_0,t_0+\sigma]\times\Lambda, h, \omega)$, if the following conditions is satisfied:

There exists a nondecreasing function $h:[t_0,t_0+\sigma] \to R$, and increasing function $\omega:[0,+\infty) \to R$ with $\omega(0)=0$ such that

$$\|F(x,s_2,\lambda) - F(x,s_1,\lambda)\| \le |h(s_2) - h(s_1)|. \tag{10}$$

for all (x,s_2,λ), $(x,s_1,\lambda) \in O\times[t_0,t_0+\sigma]\times\Lambda$ and~

$$\|F(x,s_2,\lambda_1) - F(x,s_1,\lambda_1) - F(y,s_2,\lambda_2) + F(y,s_1,\lambda_2)\| \le \omega(\|x-y\| + \|\lambda_1 - \lambda_2\|)|h(s_2) - h(s_1)|$$

for all $(x,s_2,\lambda_1),(x,s_1,\lambda_1)(y,s_2,\lambda_2),(y,s_1,\lambda_2) \in O\times[t_0,t_0+\sigma]\times\Lambda$

Lemma 1. If $f:[a,b]\to R^n$ is a regulated function and $g:[a,b]\to R$ is a nondecreasing function, then the integral $\int_a^b f(s)dg(s)$ exists.

Lemma 2. Let $U:[a,b]\times[a,b]\to R^{n\times n}$ be a Kurzweil integrable function. Assume there exists a pair of functions $f:[a,b]\to R^n$ and $g:[a,b]\to R$, such that f is regulated, g is nondecreasing, and

$$\|U(\tau,t) - U(\tau,s)\| \le f(\tau)|g(t) - g(s)|,\ \tau,t,s \in [a,b]. \tag{11}$$

Then $\left\|\int_a^b D_t U(\tau,t)\right\| \le \int_a^b f(\tau)dg(\tau)$.

Lemma 3. Assume that $U:[a,b]\times[a,b]\to R^{n\times n}$ is Kurzweil integrable and $u:[a,b]\to R^{n\times m}$ is its primitive, i.e.

$$u(s) = u(a) + \int_a^s D_t U(\tau,t),\ s\in[a,b] \tag{12}$$

If U is regulated in the second variable, then u is regulated and satisfies $u(\tau+)=u(\tau)+U(\tau,\tau+)-U(\tau,\tau),\quad u(\tau-)=u(\tau)+U(\tau,\tau-)-U(\tau,\tau),$ Moreover, if there exists a nondecreasing function $h:[a,b]\to R$ such that

$$\|U(\tau,t)-U(\tau,s)\|\le |h(t)-h(s)|,\ \tau,t,s\in[a,b].$$

then $\|u(t)-u(s)\|\le |h(t)-h(s)|.$

Lemma 4. Let $h:[a,b]\to[0,+\infty)$ be a nondecreasing left-continuous function, $k>0, l\ge 0,$ Assume that $\psi:[a,b]\to[0,+\infty)$ is bounded and satisfies

$$\psi(\xi)\le k+l\int_a^{\xi}\psi(\tau)dh(\tau), \text{Then } \psi(\xi)\le ke^{l(h(\xi)-h(a))},\ \text{ for every } \xi\in[a,b].$$

Lemma 5. Assume that matrix-valued function $A:[a,b]\times[a,b]\to R^{n\times n}$ and there exists a nondecreasing function $h:[a,b]\to R$ satisfies

$$\|A(\tau,t)-A(\tau,s)\|\le |h(t)-h(s)|,\ \tau,t,s\in[a,b]. \tag{13}$$

Let $y,z:[a,b]\to R^n$ be a pair of functions such that

$z(s)=z(a)+\int_a^s D_t[A(\tau,t)y(\tau)]$. Then z is regulated on [a,b].

Lemma 6. Assume that matrix-valued function $A:[a,b]\times[a,b]\to R^{n\times n}$ is Kurzweil integrable and satisfies (12) with a left-continuous function h, Then for every $z_0\in R^n$, the initial value problem $\dfrac{dz}{dt}=D_t[A(\tau,t)z],\ z(a)=z_0.$ (14)

has a unique solution $z:[a,b]\to R^n$.

Lemma 7. Let $m\in N,\ t_0\le t_1<t_2<\cdots<t_m<t_0+\sigma,$ consider a pair of functions $f:P\times[t_0,t_0+\sigma]\to R^n$ and $g:[t_0,t_0+\sigma]\to R$, where g is regulated, left continuous on $[t_0,t_0+\sigma]$, and continuous at $t_1,t_2,\cdots t_m$. Let $\tilde f:P\times[t_0,t_0+\sigma]\to R^n$ and $\tilde g:[t_0,t_0+\sigma]\to R,$ such that $\tilde f(t)=f(t)$ for every $t\in[t_0,t_0+\sigma]/\{t_1,t_2,\cdots t_m\}$ and $\tilde g-g$ is constant on each of the intervals $[t_0,t_1],(t_1,t_2],\cdots(t_{m-1},t_m],(t_m,t_0+\sigma]$. Then the integral $\int_{t_0}^t\tilde f(y_s,s)d\tilde g(s)$ exists, if and only if, the integral $\int_{t_0}^t f(y_s,s)dg(s)$ exists. In this case, we have

$$\int_{t_0}^{\tilde s}\tilde f(t)d\tilde g(t)=\int_{t_0}^{\tilde s}f(t)dg(t)+\sum_{t_0\le t_k<t}\tilde f(t_k)\Delta^+\tilde g(t_k)$$

Lemma 8. Let $m\in N,\ t_0\le t_1<t_2<\cdots<t_m<t_0+\sigma,I_1,I_2,\cdots I_m:R^n\to R^n,$

$P=\{y_t : y\in O, t\in[t_0,t_0+\sigma]\}\subset \overline{G}([-r,0],R^n)$, $\quad O\subset \overline{G}([t_0-r,t_0+\sigma],R^n)$, $\quad r>0$, $f: S\times[t_0,t_0+\sigma]\to R^n$ and $g:[t_0,t_0+\sigma]\to R$ is a regulated left-continuous function which is continuous at $t_1,t_2,\cdots t_m$. For every $y\in S$, define

$$\tilde{f}(y,t)=\begin{cases} f(y,t), t\neq t_k \\ I_k(y(0)), t=t_k, k\in\{1,2,\cdots,m-1\}. \end{cases}$$

Moreover, let $d_1,\cdots,d_m\in R$ be constants such that the function $\tilde{g}:[t_0,t_0+\sigma]\to R$ given by

$$\tilde{g}(t)=\begin{cases} g(t), t\in[t_0,t_1], \\ g(t)+d_k, t\in[t_k,t_{k+1}], k\in\{1,2,\cdots m-1\} \\ g(t)+d_m, t\in(t_m,+\infty), \end{cases}$$

satisfies $\Delta^+\tilde{g}(t_k)=1$ for every $k\in\{1,2,\cdots,m\}$. Then $y\in\overline{G}([t_0-r,t_0+\sigma],R^n)$ is a solution of the problem

$$\begin{cases} y(t)=y(t_0)+\int_{t_0}^t f(y_s,s)dg(s)+\sum_{t_0\le t_k<t} I_k(y(t)), t\in[t_0,t_0+\sigma], \\ y_{t_0}=\phi. \end{cases}$$

if and only if, y is also a solution of the problem

$$\begin{cases} y(t)=y(t_0)+\int_{t_0}^t \tilde{f}(y_s,s)d\tilde{g}(s), t\in[t_0,t_0+\sigma], \\ y_{t_0}=\phi. \end{cases}$$

2. Main result

In this part, we discuss differentiability theorems of solutions with respect to parameters for equation (4).

Theorem 1. Assume that $O\subset\overline{G}$ is a closed subset, $P=\{x_t : x\in O, t\in[t_0,t_0+\sigma]\}$, $\phi\in P\times\Lambda$, and $g:[t_0,t_0+\sigma]\to R$ is a nondecreasing function, function $f: P\times[t_0,t_0+\sigma]\times\Lambda\to R^{n+l}$ satisfies conditions (A1)-(A3), operator $I_k: P\times\Lambda\to R^{n+l}$ is continuous differentiable and satisfies (A4), (A5) and G: $O\times[t_0,t_0+\sigma]\times\Lambda\to\overline{G}([t_0-r,t_0+\sigma],R^n)$ given by (17) has values in $\overline{G}$.

If $(y,\lambda)\in O\times\Lambda$ is a solution of the impulsive measure functional differential equation,

$$\begin{cases} (y(t,\lambda),\lambda)=(y(t_0,\lambda),\lambda)+\int_{t_0}^t f(y_s,s,\lambda)dg(s)+\sum_{t_0<t_k<t} I_k((y,\lambda)(t_k)), t\in[t_0,t_0+\sigma], \\ x_{t_0}=\phi, \end{cases} \tag{15}$$

Then the function $(x,\lambda):[t_0,t_0+\sigma]\times R^l \to O\times\Lambda$ given by

$$(x(t,\lambda),\lambda)(\nu)=\begin{cases}(y(\nu,\lambda),\lambda),\nu\in[t_0-r,t],\\(y(t,\lambda),\lambda),\nu\in[t,t_0+\sigma]\end{cases}$$

is a solution of the generalized ordinary differential equation

$$\frac{dx}{dt}=D_tG(x,t,\lambda),t\in[t_0,t_0+\sigma]. \tag{16}$$

where $x\in O$ and G: $O\times[t_0,t_0+\sigma]\times\Lambda\to\overline{G}([t_0-r,t_0+\sigma],R^n)$ is given by

$$G(x,t,\lambda)=F(x,t,\lambda)+I(x,t,\lambda) \tag{17}$$

where

$$F(x,t,\lambda)(\nu)=\begin{cases}0,t_0-r<\nu\le t_0,\\\int_{t_0}^{\nu}f(y_s,s,\lambda)dg(s),t_0\le\nu\le t\le t_0+\sigma,\text{ and}\\\int_{t_0}^{t}f(y_s,s,\lambda)dg(s),t\le\nu\le t_0+\sigma.\end{cases}$$

$$I(x,t,\lambda)(\nu)=\begin{cases}0,t_0-r<\nu\le t_0,\\\sum_{t_0<t_k<t}I_k((y,\lambda)(t_k)),\quad t_0\le\nu\le t\le t_0+\sigma,\text{ for every }x\in O\text{ and}\\\sum_{t_0<t_k<t}I_k((y,\lambda)(t_k)),\quad t\le\nu\le t_0+\sigma.\end{cases}$$

$$t\in[t_0,t_0+\sigma],\lambda\in\Lambda.$$

Proof: Let $\tilde{f}((y_s,\lambda),s)=f(y_s,s,\lambda)$ and $\tilde{y}(t_0,\lambda)=(y(t_0,\lambda),\lambda)$

Then the function $\tilde{y}\in O$ is a solution of the impulsive measure functional differential equation

$$\begin{cases}\tilde{y}(t,\lambda)=\tilde{y}(t_0,\lambda)+\int_{t_0}^{t}\tilde{f}(\tilde{y}_s,s)dg(s)+\sum_{t_0<t_k<t}I_k((y,\lambda)(t_k)),t\in[t_0,t_0+\sigma],\\\tilde{y}_{t_0}=\phi,\end{cases}$$

Let $\tilde{F}((x,\lambda),t)=G(x,t,\lambda)$ and $\tilde{x}(t,\lambda)=(x(t,\lambda),\lambda)$, then, by using lemma7 and lemma 8, the following function $\tilde{x}:[t_0,t_0+\sigma]\times\Lambda\to O$ given by

$$\tilde{x}(t,\lambda)(\nu)=\begin{cases}\tilde{y}(\nu,\lambda),\nu\in(-\infty,t],\\\tilde{y}(t,\lambda),\nu\in[t,t_0+\sigma]\end{cases}$$

is a solution of the generalized ordinary differential equation

$$\frac{dx}{dt}=D_t\tilde{F}(\tilde{x},t,\lambda),t\in[t_0,t_0+\sigma].$$

(see Lemma 4.15 and Theorem 4.16 in [6] for the detailed proof of process).

Remark 2. By the theorem1, equation (4) is equivalent to generalized ordinary differential equation. thus, according to Theorem 4.1, Theorem 4.4 and Theorem 5.1 in the literature [4], it seems reasonable to expect that the derivative $Z(t)=x_{\lambda}(t,\lambda_0)$ is a solution of the linear equation

$$\frac{dx}{dt}=D_t G(x,t,\lambda), z(a)=x^0_{\lambda_0}(\lambda_0).$$

where $G(x,t,\lambda)=[F_x(x(\tau,\lambda_0),\tau,t,\lambda)+I^x_k(x(t_k,\lambda_0),\tau,t,\lambda)]z$
This provides a motivation for the following theorem.

Theorem 2. Let $f:P\times[t_0,t_0+\sigma]\times\Lambda\to R^{n+l}$ be a continuous function which derivative f_x, f_λ exist and continuous on $P\times[t_0,t_0+\sigma]\times\Lambda$ and satisfies (A1)-(A3), function $I_k:P\times\Lambda\to R^{n+l}$ is continuous differentiable and satisfies the condition (A4), (A5), if $g:[t_0,t_0+\sigma]\to R$ is a left continuous function and $\lambda_0\in R^l$, $\rho>0$,
$\Lambda=\{\lambda\in R^l;\|\lambda-\lambda_0\|<\rho\}$, $x^0:\Lambda\times[t_0,t_0+\sigma]\to O$, for every $\lambda\in\Lambda$,
the initial value problem of the impulsive measure functional differential equations (4) is equivalent to the initial value problem

$$\frac{dx}{dt}=D_t F(x,t,\lambda),\quad t\in[t_0,t_0+\sigma],\quad x(t_0)=x^0(\lambda). \tag{18}$$

And (18) has a solution defined on $\Lambda\times[t_0,t_0+\sigma]$. Let $x(t,\lambda)$ be the value of that solution at $t\in[t_0,t_0+\sigma]$.
Moreover, let the following conditions be satisfied:
1. For every fixed $t\in[t_0,t_0+\sigma]$, the function $(x,\lambda)\mapsto F(x,t,\lambda)$ is continuously differentiable on $O\times\Lambda$.
2. The function x^0 is differentiable at λ_0.

Then the function $\lambda\mapsto x(t,\lambda)$ is differentiable at λ_0, uniformly for all $t\in[t_0,t_0+\sigma]$.
Moreover its derivative
$Z(t)=x_\lambda(t,\lambda_0)$, $t\in[t_0,t_0+\sigma]$ is the unique solution of the generalized differential equation

$$Z(s)=x^0_\lambda(\lambda_0)+\int_{t_0}^{s}D_t[F_x(x(\tau,\lambda_0),t,\lambda)Z(\tau)+F_\lambda(x(\tau,\lambda_0),t,\lambda)],\quad s\in[t_0,t_0+\sigma]. \tag{19}$$

Proof: According to the assumptions, there exist positive constants A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 such that

$\|f_x(x_t,t,\lambda)\|\le A_1, \|f_x(x_t,t,\lambda)-f_x(y_t,t,\lambda)\|\le A_3\omega(\|x-y\|)$

$\| f_\lambda(x_t,t,\lambda)\| \le A_2, \| f_\lambda(x_t,t,\lambda) - f_\lambda(y_t,t,\lambda)\| \le A_4\omega(\|x-y\|)$

$\|I_k^x(x,\lambda)\| \le B_1, \|I_k^x(x,\lambda) - I_k^x(y,\lambda)\| \le B_3\|x-y\|$

$\|I_k^\lambda(x,\lambda)\| \le B_2, \|I_k^\lambda(x,\lambda) - I_k^\lambda(y,\lambda)\| \le B_4\|x-y\|$ where I_k^x, I_k^λ is the derivative with respect to x, λ, for every $x, y \in S$, and $t_0 \le t_1 < t_2 \le t_0+\sigma$ .Let

$F^1 = \int_{t_0}^t f(x_s,s,\lambda)dg(s), F^2 = \sum_{k=1}^m I_k(x,\lambda)H_{t_k}(t)$ then

$F_x^1(x,t,\lambda) = \int_{t_0}^t f_x(x_s,s,\lambda)dg(s), F_\lambda^1(x,t,\lambda) = \int_{t_0}^t f_\lambda(x_s,s,\lambda)dg(s)$

$F_x^2(x,t,\lambda) = \sum_{k=1}^m I_k^x(x,\lambda)H_{t_k}(t), F_\lambda^2(x,t,\lambda) = \sum_{k=1}^m I_k^\lambda(x,\lambda)H_{t_k}(t)$ if $t_1, t_2 \in [t_0, t_0+\sigma]$

we have

$$\|F_x^1(x,t_2,\lambda) - F_x^1(x,t_1,\lambda)\| = \sup_{\tau\in[t_0,t_0+\sigma]} \|F_x^1(x,t_2,\lambda)(\tau) - F_x^1(x,t_1,\lambda)(\tau)\|$$

$$\le \sup_{\tau\in[t_1,t_2]} \left\|\int_{t_1}^\tau f_x(x_t,t,\lambda)dg(t)\right\| \quad \le A_1\int_{t_1}^{t_2} dg(t) \le h_1(t_2) - h_1(t_1)$$

In the similar way as above we can show that

$$\|F_\lambda^1(x,t_2,\lambda) - F_\lambda^1(x,t_1,\lambda)\| \le A_2\int_{t_1}^{t_2} dg(t) \le h_2(t_2) - h_2(t_1).$$

In addition,

$$\|F_x^1(x,t_2,\lambda_1) - F_x^1(x,t_1,\lambda_1) - F_x^1(y,t_2,\lambda_2) + F_x^1(y,t_1,\lambda_2)\|$$

$$\le \sup_{\tau\in[t_0,t_0+\sigma]} \|F_x^1(x,t_2,\lambda_1)(\tau) - F_x^1(x,t_1,\lambda_1)(\tau) - F_x^1(y,t_2,\lambda_2)(\tau) + F_x^1(y,t_1,\lambda_2)(\tau)\|$$

$$\le \sup_{\tau\in[t_0,t_0+\sigma]} \left\|\int_{t_1}^\tau f_x(x_t,t,\lambda_1) - f_x(y_t,t,\lambda_2)dg(t)\right\| \qquad = A_3\omega\int_{t_1}^{t_2} (\|x_t-y_t\| + \|\lambda_1-\lambda_2\|)dg(t)$$

$$= A_3\omega\int_{t_1}^{t_2} (\|x-y\| + \|\lambda_1-\lambda_2\|)dg(t) \qquad = A_3\omega(\|x-y\| + \|\lambda_1-\lambda_2\|)\int_{t_1}^{t_2} dg(t)$$

$$= (\|x-y\| + \|\lambda_1-\lambda_2\|)(h_1(t_2) - h_1(t_1))$$

By using the same way as above we have

$$\|F_\lambda^1(x,t_2,\lambda_1) - F_\lambda^1(x,t_1,\lambda_1) - F_\lambda^1(y,t_2,\lambda_2) + F_\lambda^1(y,t_1,\lambda_2)\|$$

$$\le \sup_{\tau\in[t_0,t_0+\sigma]} \|F_\lambda^1(x,t_2,\lambda_1)(\tau) - F_\lambda^1(x,t_1,\lambda_1)(\tau) - F_\lambda^1(y,t_2,\lambda_2)(\tau) + F_\lambda^1(y,t_1,\lambda_2)(\tau)\|$$

$$= A_4\omega(\|x-y\| + \|\lambda_1-\lambda_2\|)\int_{t_1}^{t_2} dg(t) = (\|x-y\| + \|\lambda_1-\lambda_2\|)(h_2(t_2) - h_2(t_1))$$

i.e, $\quad F_x^1 \in \mathscr{F}(O\times[t_0,t_0+\sigma]\times\Lambda, h_1, \omega) \qquad F_\lambda^1 \in \mathscr{F}(O\times[t_0,t_0+\sigma]\times\Lambda, h_2, \omega)$

$h_1(t) = (A_1+A_3)t, h_2(t) = (A_2+A_4)t, \ \omega(t) = t.$

Analogous, let

$h_3(t) = (B_1 + B_3)\sum_{i=1}^{m} H_{t_i}(t), h_4(t) = (B_2 + B_4)\sum_{i=1}^{m} H_{t_i}(t)$ then

$$\|F_x^2(x,t_2,\lambda) - F_x^2(x,t_1,\lambda)\| = \left\|\sum_{i=1}^{m} I_i^x(x,\lambda)H_{t_i}(t_2) - \sum_{i=1}^{m} I_i^x(x,\lambda)H_{t_i}(t_1)\right\|$$

$$\leq \|h_3(t_2) - h_3(t_1)\|$$

$$\|F_\lambda^2(x,t_2,\lambda) - F_\lambda^2(x,t_1,\lambda)\| = \left\|\sum_{i=1}^{m} I_i^\lambda(x,\lambda)H_{t_i}(t_2) - \sum_{i=1}^{m} I_i^\lambda(x,\lambda)H_{t_i}(t_1)\right\|$$

$$\leq \|h_4(t_2) - h_4(t_1)\|$$

$$\|F_x^2(x,t_2,\lambda_1) - F_x^2(x,t_1,\lambda_1) - F_x^2(y,t_2,\lambda_2) + F_x^2(y,t_1,\lambda_2)\|$$

$$\leq \omega(\|x-y\| + \|\lambda_1 - \lambda_2\|)\|h_3(t_2) - h_3(t_1)\|$$

$$\|F_\lambda^2(x,t_2,\lambda_1) - F_\lambda^2(x,t_1,\lambda_1) - F_\lambda^2(y,t_2,\lambda_2) + F_\lambda^2(y,t_1,\lambda_2)\|$$

$\leq \omega(\|x-y\| + \|\lambda_1 - \lambda_2\|)\|h_4(t_2) - h_4(t_1)\|$ i.e, $F_x^2 \in \mathscr{F}(O\times[t_0,t_0+\sigma]\times\Lambda, h_3, \omega)$, $F_\lambda^2 \in \mathscr{F}(O\times[t_0,t_0+\sigma]\times\Lambda, h_4, \omega)$, Let $F = F^1 + F^2$, $F_x = F_x^1 + F_x^2$, $F_\lambda = F_\lambda^1 + F_\lambda^2$ and $F_x, F_\lambda \in \mathscr{F}(O\times[t_0,t_0+\sigma]\times\Lambda, h, \omega)$, $h(t) = h_1(t) + h_2(t) + h_3(t) + h_4(t)$ $\omega(t) = t$.

By the assumptions, we have

$$(x(s,\lambda),\lambda) = (x^0(\lambda),\lambda) + \int_{t_0}^{s} D_t F(x(\tau,\lambda),t,\lambda), \lambda\in\Lambda, s\in[t_0,t_0+\sigma].$$

According to lemma 3, every solution x is a regulated and left-continuous function on $t\in[t_0,t_0+\sigma]$. If $\Delta\lambda\in R^l$ is such that $\|\Delta\lambda\| < \rho$, then

$$\|x(s,\lambda_0+\Delta\lambda) - x(s,\lambda_0)\| + \|\lambda_1 - \lambda_2\| \leq (\|x^0(\lambda_0+\Delta\lambda) - x^0(\lambda_0)\| + \|\lambda_1 - \lambda_2\|) + \left\|\int_{t_0}^{s} D_t J(\tau,t,\lambda)\right\|$$

where

$$J(\tau,t,\lambda) = F(x(\tau,\lambda_0+\Delta\lambda),t,\lambda) - F(x(\tau,\lambda_0),t,\lambda).$$

By(10), we obtain

$$\|J(\tau,t_1,\lambda_1) - J(\tau,t_2,\lambda_2)\|$$

$=\left\|F(x(\tau,\lambda_0+\Delta\lambda),t_1,\lambda_1)-F(x(\tau,\lambda_0),t_1,\lambda_2)-F(x(\tau,\lambda_0+\Delta\lambda),t_2,\lambda_1)+F(x(\tau,\lambda_0),t_2,\lambda_2)\right\|$
$\leq (\left\|x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0)\right\|+\left\|\lambda_1-\lambda_2\right\|)\left|h(t_1)-h(t_2)\right|$. and by using lemma 2, for every $s\in[t_0,t_0+\sigma]$, and $\lambda_1,\lambda_2\in\Lambda$. we have

$$\left\|x(s,\lambda_0+\Delta\lambda)-x(s,\lambda_0)\right\|+\left\|\lambda_1-\lambda_2\right\|$$
$$\leq (\left\|x^0(\lambda_0+\Delta\lambda)-x^0(\lambda_0)\right\|+\left\|\lambda_1-\lambda_2\right\|)+\int_{t_0}^{s}(\left\|x(\tau,\lambda_0+\Delta\lambda)-x(\tau,\lambda_0)\right\|+\left\|\lambda_1-\lambda_2\right\|)dh(\tau).$$

Consequently (using lemma 4)

$$\left\|x(s,\lambda_0+\Delta\lambda)-x(s,\lambda_0)\right\|+\left\|\lambda_1-\lambda_2\right\|\leq (\left\|x^0(\lambda_0+\Delta\lambda)-x^0(\lambda_0)\right\|+\left\|\lambda_1-\lambda_2\right\|)e^{h(t_0+\sigma)-h(t_0)}$$

So we can see that when $\Delta\lambda\to 0,\left\|\lambda_1-\lambda_2\right\|\to 0$, $(x(s,\lambda_0+\Delta\lambda),\lambda_1)$ uniformly for $(x(s,\lambda_0),\lambda_2)$.

Let $A_1(\tau,t,\lambda)=F_x(x(\tau,\lambda_0),t,\lambda)), A_2(\tau,t,\lambda)=F_\lambda(x(\tau,\lambda_0),t,\lambda))$, Because of $F_x,F_\lambda\in\mathscr{F}(O\times[t_0,t_0+\sigma]\times\Lambda,h,\omega)$ then $A_1(\tau,t,\lambda),A_2(\tau,t,\lambda)$ satisfies (10). By lemma 6 (19) has a unique solution $Z:O\times[t_0,t_0+\sigma]\times\Lambda\to R^{n\times l}$ Using lemma 5, the solution is regulated. Then there exists a constant M > 0, such that $\left\|Z(t)\right\|\leq M$, $t\in[t_0,t_0+\sigma]$.

For every $\Delta\lambda\in R^l$, such that $\left\|\Delta\lambda\right\|<\rho$, let

$$\varphi(r,\Delta\lambda)=\frac{(x(r,\lambda_0+\Delta\lambda),\lambda_1)-(x(r,\lambda_0),\lambda_2)-Z(r)\Delta\lambda}{\left\|\Delta\lambda\right\|}, r\in[t_0,t_0+\sigma],\lambda_1,\lambda_2\in\Lambda.$$

Next, we will prove that if $\Delta\lambda\to 0$, $\left\|\lambda_1-\lambda_2\right\|\to 0$, then $\varphi(r,\Delta\lambda)\to 0$ uniformly for $r\in[t_0,t_0+\sigma],\lambda\in\Lambda$

Let $\varepsilon>0$ be given, there exists a $\delta>0$ such that if $\Delta\lambda\in R^l$ and $\left\|\Delta\lambda\right\|<\rho$ then $\left\|(x(t,\lambda_0+\Delta\lambda),\lambda_1)-(x(t,\lambda_0),\lambda_2)\right\|<\varepsilon$ and

$\dfrac{\left\|(x^0(\lambda_0+\Delta\lambda),\lambda_1)-(x^0(\lambda_0),\lambda_2)-x^0_\lambda(\lambda_0)\Delta\lambda\right\|}{\left\|\Delta\lambda\right\|}<\varepsilon$ It is obvious that

$$\varphi(t_0,\Delta\lambda)=\frac{(x(t_0,\lambda_0+\Delta\lambda),\lambda_1)-(x(t_0,\lambda_0),\lambda_2)-Z(t_0)\Delta\lambda}{\left\|\Delta\lambda\right\|}$$
$$=\frac{(x^0(\lambda_0+\Delta\lambda),\lambda_1)-(x^0(\lambda_0),\lambda_2)-x^0_\lambda(\lambda_0)\Delta\lambda}{\left\|\Delta\lambda\right\|}$$

$$\varphi(r,\Delta\lambda)-\varphi(t_0,\Delta\lambda)=\frac{(x(r,\lambda_0+\Delta\lambda),\lambda_1)-(x(t_0,\lambda_0+\Delta\lambda),\lambda_1)}{\left\|\Delta\lambda\right\|}-\frac{(x(r,\lambda_0),\lambda_2)-(x(t_0,\lambda_0),\lambda_2)}{\left\|\Delta\lambda\right\|}$$

$-\dfrac{(Z(r)-Z(t_0))\Delta\lambda}{\|\Delta\lambda\|}=\int_{t_0}^{r} D_t W(\tau,t,\lambda)$ where

$$W(\tau,t,\lambda)=\frac{F(x(\tau,\lambda_0+\Delta\lambda),t,\lambda)-F(x(\tau,\lambda_0),t,\lambda)-[F_x(x(\tau,\lambda_0),t,\lambda_0)Z(\tau)+F_\lambda(x(\tau,\lambda_0),t,\lambda_0)]\Delta\lambda}{\|\Delta\lambda\|},$$

$$\begin{aligned}&W(\tau,t,\lambda_1)-W(\tau,s,\lambda_2)\\&=\frac{F(x(\tau,\lambda_0+\Delta\lambda),t,\lambda_1)-F(x(\tau,\lambda_0),t,\lambda_2)-[F_x(x(\tau,\lambda_0),t,\lambda_0)Z(\tau)+F_\lambda(x(\tau,\lambda_0),t,\lambda_0)]\Delta\lambda}{\|\Delta\lambda\|}\\&-\frac{F(x(\tau,\lambda_0+\Delta\lambda),s,\lambda_1)-F(x(\tau,\lambda_0),s,\lambda_2)-[F_x(x(\tau,\lambda_0),s,\lambda_0)Z(\tau)+F_\lambda(x(\tau,\lambda_0),s,\lambda_0)]\Delta\lambda}{\|\Delta\lambda\|}.\end{aligned}$$

Let

$F(1)=(F_x(x(\tau,\lambda_0),t,\lambda_0),F_\lambda(x(\tau,\lambda_0),t,\lambda_0)),$

$F(2)=(F_x(x(\tau,\lambda_0),s,\lambda_0),F_\lambda(x(\tau,\lambda_0),s,\lambda_0))$. Thus

$$\begin{aligned}&\|W(\tau,t,\lambda_1)-W(\tau,s,\lambda_2)\|\\&\le\frac{\|F(x(\tau,\lambda_0+\Delta\lambda),t,\lambda_1)-F(x(\tau,\lambda_0),t,\lambda_2)-F(1)((x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2))\|}{\|(x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2)\|}\\&\times\frac{\|(x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2)\|}{\|\Delta\lambda\|}\\&\le\frac{\|F(x(\tau,\lambda_0+\Delta\lambda),s,\lambda_1)-F(x(\tau,\lambda_0),s,\lambda_2)-F(2)((x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2))\|}{\|(x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2)\|}\\&\times\frac{\|(x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2)\|}{\|\Delta\lambda\|}\\&+\frac{\|F(1)-F(2)\|\|(x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2)-Z(\tau)\Delta\lambda\|}{\|\Delta\lambda\|},\end{aligned}$$

Because of the function $(x,\lambda)\to F(x,t,\lambda)$ is continuously differentiable on $O\times[t_0,t_0+\sigma]\times\Lambda$ and the definition of the $\varphi(r,\Delta\lambda)$, so for any given $\varepsilon>0$, $t,s\in[t_0,t_0+\sigma]$ we have

$$\begin{aligned}&\|W(\tau,t,\lambda_1)-W(\tau,s,\lambda_2)\|\\&\le 2\varepsilon\frac{\|(x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2)\|}{\|\Delta\lambda\|}+\|F(1)-F(2)\|\|\varphi(\tau,\Delta\lambda)\|\\&\le 2\varepsilon\Big(\frac{\|(x(\tau,\lambda_0+\Delta\lambda),\lambda_1)-(x(\tau,\lambda_0),\lambda_2)-Z(\tau)\Delta\lambda\|}{\|\Delta\lambda\|}+\frac{\|Z(\tau)\Delta\lambda\|}{\|\Delta\lambda\|}\Big)+\|F(1)-F(2)\|\|\varphi(\tau,\Delta\lambda)\|.\end{aligned}$$

and thus (using $F_x,F_\lambda\in\mathscr{F}(O\times[t_0,t_0+\sigma]\times\Lambda,h,\omega)$)

$$\leq 2\varepsilon(\|\varphi(\tau,\Delta\lambda)\|+M)+2|h(t_0+\sigma)-h(t_0)|\|\varphi(\tau,\Delta\lambda)\|.$$

$$\|W(\tau,t,\lambda_1)-W(\tau,s,\lambda_2)\|\leq 2\varepsilon(\|\varphi(\tau,\Delta\lambda)\|+M)+2|h(t)-h(s)|\|\varphi(\tau,\Delta\lambda)\|$$

Consequently,

$$\|\varphi(r,\Delta\lambda)\|\leq\|\varphi(r,\Delta\lambda)-\varphi(t_0,\Delta\lambda)\|+\|\varphi(t_0,\Delta\lambda)\|\leq\varepsilon+\left\|\int_{t_0}^{r}D_tW(\tau,t,\lambda)\right\|$$

$$\leq\varepsilon+\int_{t_0}^{r}(2\varepsilon\|\varphi(\tau,\Delta\lambda)\|+2M\varepsilon+2|h(t_0+\sigma)-h(t_0)|\|\varphi(\tau,\Delta\lambda)\|)d\tau$$

$$\leq\varepsilon(2M\sigma+1)+2(\varepsilon+(h(t_0+\sigma)-h(t_0)))\int_{t_0}^{r}\|\varphi(\tau,\Delta\lambda)\|d\tau.$$

Finally, Gronwall's inequality leads to the estimate

$$\|\varphi(r,\Delta\lambda)\|\leq\varepsilon(2M\sigma+1)e^{(\varepsilon+(h(t_0+\sigma)-h(t_0)))\sigma}$$

Since$\epsilon\to 0_+$,we have that if $\Delta\lambda\to 0$ and $\|\lambda_1-\lambda_2\|\to 0$, then $\varphi(r,\Delta\lambda)\to 0$ uniformly for $r\in[t_0,t_0+\sigma],\lambda\in\Lambda$.

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