

Numerical Solution of Singularly Perturbed Two Point Boundary Value Problems by Using Non-Polynomial Exponential Spline Functions

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Abstract. This paper presents the application of non-polynomial Exponential spline method for finding the numerical solution of singularly perturbed boundary value problems. Two numerical examples are considered to demonstrate the usefulness of the method and to show that the method converges with sufficient accuracy to the exact solutions.

Keywords: Non-polynomial Exponential spline method, singularly perturbation, truncation error, exact solution.

AMS Mathematics Subject Classification (2010): 41A15

1. Introduction

Singular perturbation problems containing a small perturbation parameter, arise very frequently in many branches of applied mathematics such as, fluid dynamics, quantum mechanics, chemical reactor theory, elasticity, aerodynamics, and the other domain of the great world of fluid motion [1-3]

A well known fact is that the solution of such problems has a multiscale character, i.e. there are thin transition layers where the solution varies very rapidly, while away from the layer the solution behaves regularly and varies slowly. Numerically, the presence of the perturbation parameter leads to difficulties when classical numerical techniques are used to solve such problems, this is due to the presence of the boundary layers in these problems.

We consider a second order singularly perturbed boundary problem [4-5]:

$$\varepsilon y'' + w(x)y' + u(x)y = r(x), x \in [a, b] \quad (1)$$

with the boundary conditions

$$\text{and } y(b) = \gamma_2 y(a) = \gamma_1$$

where ε is a small positive parameter $0 < \varepsilon < 1$, γ_1 and γ_2 are given constants, $w(x)$, $u(x)$ and $r(x)$ are assumed to be sufficiently continuously differentiable functions. The non-polynomial Exponential spline method [6-16] developed in this paper has lower computational cost and its only requires solving $n + 1$ linear or non-linear equations.

2. Derivation of the method

We divide the interval $[a, b]$ into $n + 1$ equal subintervals using the point

$$x_i = a + ih, \quad i = 0, 1, 2, \dots, n, n + 1,$$

With $a = x_0$, $b = x_{n+1}$ and $h = \frac{b-a}{n+1}$, where n arbitrary positive integer.

Let $y(x)$ be the exact solution and y_i be an approximation to $y(x_i)$ obtained by the non polynomial Exponential spline $E_i(x)$ passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}) , we do not only require that $E_i(x)$ satisfies interpolatory conditions at x_i and x_{i+1} but also the continuity of first derivative at the common nodes (x_i, y_i) are fulfilled. We write $E_i(x)$ in the form [7-8]:

$$E_i(x) = a_i e^{k(x-x_i)} + b_i e^{-k(x-x_i)} + c_i(x - x_i) + d_i \quad (2)$$

where a_i, b_i, c_i and d_i are constants and k is free parameter to be determined later.

A non-polynomial Exponential spline function $E(x)$ of class $C^2[a, b]$ interpolates $y(x)$ at the grid points x_i , $i = 0, 1, 2, \dots, n + 1$ depends on a parameter k , and reduces to ordinary spline $E(x)$ in $[a, b]$ as $k \rightarrow 0$.

To derive expression for the coefficient of equation (2) in term y_i, y_{i+1}, F_i and F_{i+1} , we first define:

$$E_i(x_i) = y_i, \quad E_i(x_{i+1}) = y_{i+1} \\ E_i''(x_i) = F_i, \quad E_i''(x_{i+1}) = F_{i+1}. \quad (3)$$

From algebraic manipulation, we get the following expression:

$$a_i = \frac{h^2(-F_i e^{-\theta} + F_{i+1})}{\theta^2(e^\theta - e^{-\theta})} \\ b_i = \frac{h^2(F_i e^\theta - F_{i+1})}{\theta^2(e^\theta - e^{-\theta})} \\ c_i = \frac{-h(F_{i+1} - k^2 y_{i+1} + k^2 y_i - F_i)}{\theta^2} \\ d_i = \frac{(\theta^2 y_i - h^2 F_i)}{\theta^2} \quad (4)$$

where $\theta = kh$ and $i = 0, 1, 2, \dots, n$.

We applying the first derivative at (x_i, y_i) , that is $E'_{i-1}(x_i) = E'_i(x_i)$, gives the following consistency relation for $i = 1, \dots, n$:

$$y_{i-1} - 2y_i + y_{i+1} = \left(\frac{(\theta e^{-\theta} - 2\theta e^\theta + e^{2\theta} + e^{-2\theta} - 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)} \right) F_{i-1} + \\ 2 \left(\frac{(\theta e^{2\theta} - \theta e^{-2\theta} - e^{2\theta} - e^{-2\theta} + 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)} \right) F_i + \\ \left(\frac{(\theta e^{-\theta} - 2\theta e^\theta + e^{2\theta} + e^{-2\theta} - 2)h^2}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)} \right) F_{i+1} \quad (5)$$

which can further be written as

$$\varepsilon(y_{i-1} - 2y_i + y_{i+1}) = h^2 \varepsilon[\lambda(F_{i-1} + F_{i+1}) + 2\varepsilon\eta F_i] \quad (6)$$

where

$$\lambda = \frac{(\theta e^{-\theta} - 2\theta e^\theta + e^{2\theta} + e^{-2\theta} - 2)}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}$$

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$$\eta = \frac{(\theta e^{2\theta} - \theta e^{-2\theta} - e^{2\theta} - e^{-2\theta} + 2)}{\theta^2(e^{2\theta} + e^{-2\theta} - 2)}$$

the local truncation errors,

$n = 1, 2, \dots, n - 1$, in equation (6) can be obtained as follows: t_i

First we re-write the equation.(6) in the form ,

$$\varepsilon(y_{i-1} - 2y_i + y_{i+1}) = h^2 \varepsilon \left[\lambda y_{i-1}^{(2)} + 2\varepsilon \eta y_i^{(2)} + \lambda y_{i+1}^{(2)} \right] \quad (7)$$

the terms $y_{i-1}^{(2)}$, $y_i^{(2)}$ and $y_{i+1}^{(2)}$ in equation (7) are expanded around the point x_i using Taylor series and the expressions for t_i , $n = 1, 2, \dots, n - 1$. can be obtained,

$$t_i = h^2(1 - 2\lambda - 2\eta)y_i^{(2)} + \frac{h^2}{12}(1 - 12\lambda)y_i^{(4)} + \frac{h^6}{360}(1 - 30\lambda)y_i^{(6)} + O(h^8) \quad (8)$$

Now the (8) gives rise to the class of methods of different orders as follow:

Second order method:

For any choice of arbitrary λ and η with $\lambda = \frac{1}{6}$, $\eta = \frac{1}{3}$ and $\lambda + \eta = \frac{1}{2}$, then local truncation error is

$$t_i = \frac{-h^4}{12} y_i^{(4)} + O(h^6), i = 1, 2, \dots \quad (9)$$

Fourth order method:

If $\lambda = \frac{1}{12}$ and $\eta = \frac{5}{12}$, $T_i = O(h^6)$ then the resulting method is fourth order method.

Then the local truncation error is

$$t_i = \frac{-h^6}{240} y_i^{(6)} + O(h^8), i = 1, 2, \dots n \quad (10)$$

3. Numerical scheme

At the grid points x , Eq. (1) may be discretized by

$$\varepsilon F_i + w(x)y' + u(x)y = r(x) \quad (11)$$

Solving Eq (11) for F_i , we get

$$\varepsilon F_i = -w(x)y' - u(x)y + r(x) \quad (10)$$

and approximate first derivative by using finite-difference.

The following approximation for the first-order derivative of y in Eq. (12) can be used

$$y'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}$$

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}$$

So Eq. (12) becomes

$$\varepsilon F_{i-1} = -\frac{w(x_{i-1})(-y_{i+1} + 4y_i - 3y_{i-1})}{2h} - u(x_{i-1})y_{i-1} + r(x_{i-1})$$

$$\varepsilon F_i = -\frac{w(x_i)(y_{i+1} - y_{i-1})}{2h} - u(x_i)y_i + r(x_i)$$

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$$\varepsilon F_{i+1} = -\frac{w(x_{i+1})(3y_{i+1}-4y_i+y_{i-1})}{2h} - u(x_{i+1})y_{i+1} + r(x_{i+1}) \quad (13)$$

Substituting Eq. (13) in Eq. (5), we get the following equation:

$$\begin{aligned} & \left[-\varepsilon + h\lambda \left(\frac{3w(x_{i-1})}{2} - \frac{w(x_{i+1})}{2} \right) + h\beta w(x_i) - h^2\lambda u(x_{i-1}) \right] y_{i-1} + [2\varepsilon + \\ & 2h\lambda(-w(x_{i-1}) + w(x_{i+1})) - 2\eta h^2\lambda u(x_i)] y_i + \left[-\varepsilon + h\lambda \left(\frac{w(x_{i-1})}{2} - \frac{3w(x_{i+1})}{2} \right) - \right. \\ & \left. h\eta w(x_i) - h^2\lambda u(x_{i+1}) \right] y_{i+1} = -\lambda h^2 r(x_{i-1}) - 2\eta h^2 r(x_i) - \\ & \lambda h^2 r(x_{i+1}) \end{aligned} \quad (14)$$

4. Stability analysis

The tri-diagonal linear system (14) can be written in the following matrix form,

$$AY + h^2CR = G \quad (15)$$

where $A = N + hBQ - h^2Bu$ and A is tri-diagonal and diagonally matrix of order $n - 1$

Here $N = (L_{ij})$ is a tri-diagonal matrix defined by are tri-diagonal matrices defined by

$$L_{ij} = \begin{cases} 2\varepsilon & i = j = 1, 2, \dots, n - 1 \\ -\varepsilon & |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $BQ = \psi_{ij}$, $Bq = \Phi_{ij}$ are tri-diagonal matrices defined by

$$\psi_{ij} = \begin{cases} 2\lambda(-w(x_0) + w(x_2)), & i = j = 1 \\ \lambda \left(\frac{3w(x_{i-1})}{2} - \frac{w(x_{i+1})}{2} \right) - h\eta w(x_i), & i > j \\ 2\lambda(-w(x_{i-1}) + w(x_{i+1})) & i = j \\ \lambda \left(\frac{w(x_{i-1})}{2} - \frac{3w(x_{i+1})}{2} \right) - h\eta w(x_i), & i < j \\ 2\lambda(-w(x_{n-2}) + w(x_n)) & i = j = n - 1 \end{cases}$$

And

$$\Phi_{ij} = \begin{cases} 2\eta u(x_i) & i = j = 1, 2, \dots, n - 1 \\ \lambda u(x_{i-1}) & i > j \\ \lambda u(x_{i+1}) & i < j \end{cases}$$

And

$$Y = (y_1, y_2, \dots, y_{n-2}, y_{n-1})^T R = (r(x_1), r(x_2), \dots, r(x_{n-2}), r(x_{n-1}))^T$$

The tri-diagonal matrix C is defined by

$$C = \begin{bmatrix} 2\eta & \lambda & 0 & \dots & \dots & 0 \\ \lambda & 2\eta & \lambda & 0 & \dots & 0 \\ 0 & \lambda & 2\eta & \lambda & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \lambda & 2\eta & \lambda \\ 0 & 0 & \dots & \dots & \lambda & 2\eta \end{bmatrix}$$

where $G = (g_1, 0, 0, 0, \dots, 0, g_{n-1})^T$

We assume that

$$Y = (y_1, y_2, \dots, y_{n-2}, y_{n-1})^T$$

Be the exact solution of the given boundary value problem (1) at nodal point x_i , for $i = 0, 1, 2, \dots, n - 1$ and then we have

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$$A\bar{Y} + h^2DR = T(h) + G \quad (16)$$

If we subtract equation (12) from equation (13) we get the following

$$A(\bar{Y} - Y) = AE = T(h) \quad (17)$$

5. Convergence analysis

Our main purpose now is to derive a bound $\|E\|$. We now turn back to the error equation in (13) and rewrite it in the form

$$E = A^{-1}T = [N + hBQ - h^2Bu]^{-1}T = [I + N^{-1}(hBQ - h^2Bu)]^{-1}N^{-1}T$$

$$\|E\|_\infty \leq \|[I + N^{-1}(hBQ - h^2Bu)]^{-1}\|_\infty \|N^{-1}\|_\infty \|T\|_\infty \quad (18)$$

In order to derive the bound on $\|E\|_\infty$ the following two lemmas are needed.

Lemma 1. [17] If G is a square matrix of order n and $\|G\| < 1$, then the $(I + G)^{-1}$ exists and $\|(I + G)^{-1}\| \leq (1 - \|G\|)^{-1}$

Lemma 2: The matrix $(N + hBQ - h^2Bu)$ is nonsingular $\|w\|_\infty < \frac{8h\varepsilon}{(a-b)^2(8\lambda+2\eta)}$ and $\|u\|_\infty < \frac{8(1-\varepsilon)}{(a-b)^2}$ where. $0 < \varepsilon < 1$

Proof:

Since, $A = (N + hBQ - h^2Bu) = (I + N^{-1}(hBQ - h^2Bu))N$ and the matrix N is nonsingular, so to prove A nonsingular it is sufficient to show $(I + N^{-1}(hBQ - h^2Bu))$ nonsingular.

Since

$$\|N^{-1}(hBQ - h^2Bu)\|_\infty \leq \|N^{-1}\|_\infty (\|hBQ - h^2Bu\|_\infty) \leq \|N^{-1}\|_\infty (\|hBQ\|_\infty + \|h^2Bu\|_\infty) \quad (19)$$

Moreover,

$$[17]\|N^{-1}\|_\infty < \frac{(a-b)^2}{8h^2}$$

$$\text{and } \|h^2Bu\|_\infty \leq h^2\|u\|_\infty \text{ and } \|hBQ\|_\infty \leq h(8\lambda + 2\eta)\|w\|_\infty$$

$$\text{and } \|u\|_\infty = \max_{a \leq x \leq b} |u(x_i)| \quad \|w\|_\infty = \max_{a \leq x \leq b} |w(x_i)|$$

There for, substituting $\|N^{-1}\|_\infty$, $\|hBQ\|_\infty$, $\|h^2Bu\|_\infty$, in equation (17) we get,

$$\|N^{-1}(hBQ - h^2Bu)\|_\infty \leq \frac{(b-a)^2}{8h^2}(8\lambda + 2\eta)\|w\|_\infty + \frac{(b-a)^2}{8}\|u\|_\infty \quad (20)$$

Since ,

$$\begin{cases} \|w\|_\infty < \frac{8h\varepsilon}{(a-b)^2(8\lambda+2\eta)} \\ \|u\|_\infty < \frac{8(1-\varepsilon)}{(a-b)^2} \end{cases} \quad (21)$$

There for equations (20) and (21) leads to $\|N^{-1}(hBQ - h^2Bu)\|_\infty \leq 1$. From Lemma (1) it show that the matrix A is nonsingular. Since $\|N^{-1}(hBQ - h^2Bu)\|_\infty \leq 1$ so using Lemma (1) and equation (16) follow that

$$\|E\|_\infty \leq \frac{\|N^{-1}\|_\infty \|T\|_\infty}{1 - \|N^{-1}\|_\infty \|hBQ - h^2Bu\|_\infty}$$

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6. Numerical results

We solve two singular perturbed problems using different values of h and ε . The numerical solutions are computed and compared with the exact solutions at grade points. All calculations are implemented by Maple 13.

Example 1. Consider the following equation with variable coefficients

$$\varepsilon y'' + xy' - y = -(1 + \varepsilon\pi^2) \cos(\pi x) - (\pi x) \sin \pi x$$

$$y(-1) = -1, y(1) = 1$$

The exact solution is given by:

$$y(x) = \cos(\pi x) + x + \frac{x \operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) + \sqrt{\frac{2\varepsilon}{\pi}} e^{-\left(\frac{x^2}{2\varepsilon}\right)} }{x \operatorname{erf}\left(\frac{1}{\sqrt{2\varepsilon}}\right) + \sqrt{\frac{2\varepsilon}{\pi}} e^{-\left(\frac{1}{2\varepsilon}\right)}}$$

The numerical result of the example1 are presented in table 1 and figure 1 for different values subinterval N and $\varepsilon = 1/64$.

Example 2. Consider the following equation with variable coefficients

$$\varepsilon y'' - \frac{1}{x} y' - (1 + x^2)y = x$$

The exact solution is $y(x) = e^{(x^2)}$. The maximum absolute errors are tabulated in Table 2 for different values ε and h .

Table 1: Numerical solution of Example 1 at different value of subintervals

x	Numerical Sol.				Exact Sol.
	N=16	N=32	N=64	N=128	
-7/8	-0.9190258015	-0.9219199438	-0.9237709378	-0.9240415171	-.9243033330
-6/8	-0.6984560039	-0.7035617854	-0.7068294729	-0.7073106180	-.7077785740
-5/8	-0.3722043319	-0.3783607642	-0.3822747958	-0.3828522232	-.3834150957
-4/8	-0.009537748283	-0.0038551921	-0.0003641383	-0.0001416495	-0.000141523
-3/8	0.3881155740	0.3845924696	0.3827696718	0.3825442896	0.3823396261
-2/8	0.7079360848	0.7077704103	0.7085372990	0.7087543527	.7090026623
-1/8	0.9384379783	0.9408096810	0.9435055183	0.9440534564	0.9446355666
0	1.092840722	1.095498023	1.098469883	1.099075370	1.099715459
1/8	1.188481264	1.190838603	1.193508694	1.194053107	1.194635567
2/8	1.204935832	1.206883138	1.208472633	1.208735746	1.209002662

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3/8	1.131158356	1.132521790	1.132618238	1.132502994	1.132339626
4/8	0.9998408258	1.000964472	1.000152356	0.9998016554	.9993681064
5/8	0.8674062236	0.8685359797	0.8674978662	0.8670873438	0.8665849039
6/8	0.7927591290	0.7938125624	0.7929781403	0.7926384601	.7922214264
7/8	0.8258581569	0.8265503836	0.8261169714	0.8259288994	0.8256966672

Table 2: Numerical solution of Example 2 at different value of subintervals

x	Numerical Sol.				Exact Sol.
	N=16	N=32	N=64	N=128	
1	1.009050167	1.010040167	1.010050067	1.010050166	1.010050167
2	1.039810774	1.040800774	1.040810674	1.040810773	1.040810774
3	1.093174284	1.094164284	1.0941741839	1.094174283	1.094174284
4	1.172510871	1.173500871	1.173510771	1.173510869	1.173510871
5	1.283025417	1.284015417	1.284025317	1.284025416	1.284025417
6	1.432329415	1.433319415	1.4333293149	1.433329414	1.433329415
7	1.631316220	1.632306220	1.632316120	1.632316219	1.632316220
8	1.895480879	1.896470879	1.896480779	1.896480878	1.896480879
9	2.246907987	2.247897987	2.247907887	2.247907986	2.247907987
10	2.7172818289	2.718271827	2.718281728	2.718281827	2.718281828

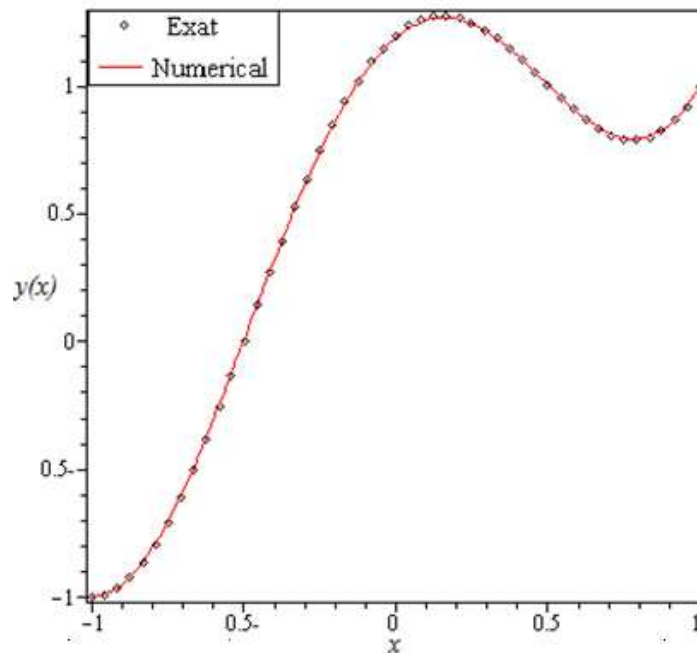


Figure 1: Comparison of exact and numerical solutions of example 1 for $N = \varepsilon = 64$.

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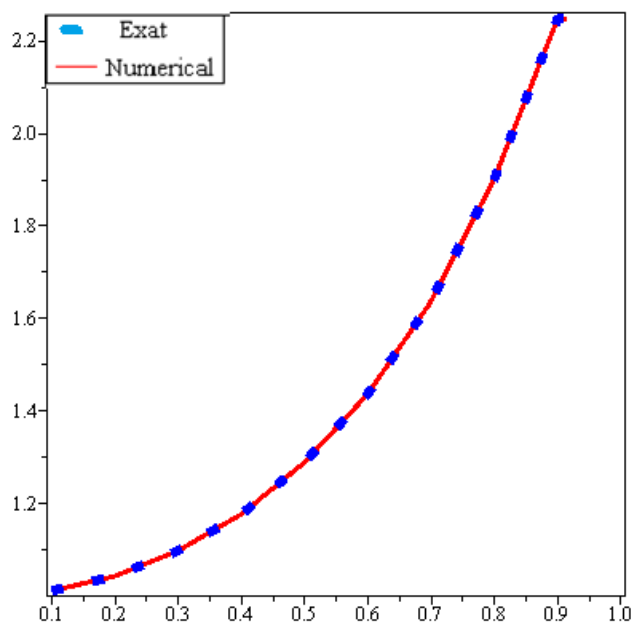


Figure 2: Comparison of exact and numerical solutions of example 2 for $N = \varepsilon = 10$.

7. conclusion

in this paper, a numerical technique for singularly perturbed boundary value problems using non-polynomial exponential spline functions is derived. Simplicity of the adaptation of non-polynomial exponential spline and obtaining acceptable solutions can be noted as advantages of given numerical methods. The method is tested on two problems and the results obtained are very encouraging. The method is simple and easy to apply.

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