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Pathway Fractional Integral Operator on Some τ-extensions of Lauricella Functions Of Several Variables

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Abstract. In this paper, we establish new results as images of τ-extensions of Lauricella functions of several variables using the pathway fractional integral operator. The pathway fractional integral operator's motivation is a switching mechanism that transforms one fractional form to another with important applications. Some interesting special cases of our main findings are also highlighted.

Keywords: Lauricella functions; Pathway fractional integral operator; Riemann-Liouville fractional integral operator; Laplace transform

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1. Introduction and preliminaries

During the last few years, fractional calculus has emerged as an important tool for modeling analysis, with applications in a variety of fields such as material science, mechanics, power, economics, control theory, science and technology [13, 17, 31, 37]. Furthermore, a number of researchers have investigated a wide range of fractional calculus operators in terms of properties, implementation methods, and complex modifications.

Recently, Nair [28] introduced and defined the pathway fractional integral operator as follows

If $f(x) \in L(a,b)$, $\eta \in C, R(\eta) > 0$, $a > 0$ with $\xi < 1$ (pathway parameter), then

$$\left(P_{0+}^{(\eta,\xi)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\xi)}\right]} \left[1 - \frac{a(1-\xi)t}{x}\right]^{\frac{\eta}{1-\xi}} f(t)\, dt. \tag{1}$$

If $\xi = 1$ in equation (1), then $\left[1 - \frac{a(1-\xi)t}{x}\right]^{\frac{\eta}{1-\xi}} \to e^{-\frac{a\eta}{x}t}$ and pathway fractional integral operator switches to Laplace integral transform (LIT) [23] with parameter $\frac{a\eta}{x}$ as

$$\left(P_{0+}^{(\eta,1)} f\right)(x) = x^{\eta} \int_0^{\infty} e^{-\frac{a\eta}{x}t} f(t)\, dt = x^{\eta} L_f\left(\frac{a\eta}{x}\right) \tag{2}$$

If $\xi = 0$, $a = 1$ and replacing η by $\eta - 1$ in equation (1), then pathway fractional integral operator switches to the Left-sided Riemann-Liouville (R-L) fractional integral operator $\left(I_{0+}^{\eta}\right)$ [23]

$$\left(P_{0+}^{(\eta-1,0)} f\right)(x) = \int_0^{x} (x-t)^{\eta-1} f(t)\, dt = \Gamma(\eta)\left(I_{0+}^{\eta}\right)(x) \tag{3}$$

Many authors have contributed works on the Pathway fractional integral operator, which is associated with a variety of special functions including the Aleph function and generalized polynomials [20, 34], the H-function, the M-series [8, 9, 24], the Mittag-Leffler type function [27], the generalized k-Mittag-Leffler function [30], the new generalized Mittag-Leffler function [33], the Struve function [29], the composition of two functions [3, 15], the Mainardi function [5], the product of two Aleph functions [6], the Gimel function [1], the composition of Hurwitz-Lerch zeta function [21], the weighted Chebyshev function [26], the incomplete H-functions [4] and the modified multivariable H-function [16].

In 1893, Lauricella [25] has defined the Lauricella functions $F_A^{(r)}, F_B^{(r)}$ and $F_D^{(r)}$ in r (real or complex) variables. The integral representation of Lauricella functions was defined by Tuan, V.K. et al. [35]. Lauricella functions have been transformed as summation formulas [32], decomposition formulas [18], extension formulas [1], reduction formulas [11], a cubic transformation formula [14], an expression over a finite field [10], incomplete Lauricella functions [12], incomplete Lauricella matrix functions [36], and τ-extension of Lauricella functions of several variables [22] with applications on heat equations [19].

Motivated by the above-mentioned works, we apply Pathway fractional integral operator on the τ-extension Lauricella functions $F_A^{(n)}, F_B^{(n)}$ and $F_D^{(n)}$ of n variables [22] in the present investigation. These functions are expressed in terms of additional parameters $\tau_1, \tau_2, \dots, \tau_n$, which are defined as follows

Let $\alpha, \alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n \in C$ and $\gamma, \gamma_1, \gamma_2, \dots, \gamma_n \in C \setminus Z_0^-$, where $Z_0^- = Z^- \bigcup \{0\}$, then

$$F_A^{(n),\tau_1,\dots,\tau_n}\left[\alpha, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n\right]$$
$$= \frac{\Gamma(\gamma_1)\dots\Gamma(\gamma_n)}{\Gamma(\beta_1)\dots\Gamma(\beta_n)} \sum_{m_1,\dots,m_n=0}^{\infty} \left(\frac{(\alpha)_{m_1+\dots+m_n}\Gamma(\beta_1+\tau_1 m_1)\dots\Gamma(\beta_n+\tau_n m_n)}{\Gamma(\gamma_1+\tau_1 m_1)\dots\Gamma(\gamma_n+\tau_n m_n)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}\right), \tag{4}$$
$$\left(\tau_1, \tau_2, \dots, \tau_n > 0\,; |x_1| + |x_2| + \dots + |x_n| < 1\right)$$

$$F_B^{(n),\tau_1,\dots,\tau_n}\left[\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n\right]$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\beta_1)\dots\Gamma(\beta_n)}\sum_{m_1,\dots,m_n=0}^{\infty}\left(\frac{(\alpha)_{m_1}\dots(\alpha)_{m_n}\Gamma(\beta_1+\tau_1 m_1)\dots\Gamma(\beta_n+\tau_n m_n)}{\Gamma(\gamma+\tau_1 m_1+\dots+\tau_n m_n)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}\right), \tag{5}$$
$$\left(\tau_1,\tau_2,\dots,\tau_n>0\ ;\ \max\{|x_1|,|x_2|,\dots,|x_n|\}<1\right)$$

and

$$F_D^{(n),\tau_1,\dots,\tau_n}\left[\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n\right]$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{m_1,\dots,m_n=0}^{\infty}\left(\frac{\Gamma(\alpha+\tau_1 m_1+\dots+\tau_n m_n)(\beta_1)_{m_1}\dots(\beta_1)_{m_n}}{\Gamma(\gamma+\tau_1 m_1+\dots+\tau_n m_n)}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}\right), \tag{6}$$
$$\left(\tau_1,\tau_2,\dots,\tau_n>0\ ;\ \max\{|x_1|,|x_2|,\dots,|x_n|\}<1\right).$$

2. Main results

In this section, we derive the images for some τ-extensions of Lauricella functions of several variables under the pathway fractional integral operator $p_{0+}^{(\eta,\xi)}$.

Theorem 1. For $\eta,\rho,\alpha,\beta_j\in C,\gamma_j\in C\setminus Z_0^-,R(\eta)>0,R(\rho)>0,R(\alpha)>0,a>0,$ $R(\beta_j)\geq 0,\ b_j\in R\ \ (j=1,2,\dots,n\ ;n\in N),\ R\left(1+\frac{\eta}{1-\xi}\right)>\max\{0,-R(\rho)\};$ $\tau_1,\tau_2,\dots,\tau_n>0;\ |b_1t^{\sigma_1}|+|b_2t^{\sigma_2}|+\dots+|b_nt^{\sigma_n}|<1$ and $\xi<1$, then following results hold true

$$P_{0+}^{(\eta,\xi)}\left[t^{\rho-1}F_A^{(n),\tau_1,\dots,\tau_n}\left(\alpha,\beta_1,\dots,\beta_n;\gamma_1,\dots,\gamma_n;b_1t^{\sigma_1},\dots,b_nt^{\sigma_n}\right)\right]$$
$$=\frac{x^{\eta+\rho}\,\mathrm{B}\left(\rho,1+\frac{\eta}{1-\xi}\right)}{\left[a(1-\xi)\right]^{\rho}}\sum_{m_1,\dots,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\dots+\sigma_nm_n}(\alpha)_{m_1+\dots+m_n}(\beta_1)_{\tau_1m_1}\dots(\beta_n)_{\tau_nm_n}}{\left(1+\rho+\frac{\eta}{1-\xi}\right)_{\sigma_1m_1+\dots+\sigma_nm_n}(\gamma_1)_{\tau_1m_1}\dots(\gamma_n)_{\tau_nm_n}}\right.$$
$$\left.\times\left(\frac{b_1x^{\sigma_1}}{\left[a(1-\xi)\right]^{\sigma_1}}\right)^{m_1}\frac{1}{m_1!}\dots\left(\frac{b_nx^{\sigma_n}}{\left[a(1-\xi)\right]^{\sigma_n}}\right)^{m_n}\frac{1}{m_n!}\right\}. \tag{7}$$

Proof: Using equations (1) and (4), in the LHS of equation (7), we have

$$P_{0+}^{(\eta,\xi)}\left[t^{\rho-1}F_A^{(n),\tau_1,\ldots,\tau_n}\left(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;b_1t^{\sigma_1},\ldots,b_nt^{\sigma_n}\right)\right]$$

$$=x^{\eta}\int_0^{\left[\frac{x}{a(1-\xi)}\right]}\left\{\left[1-\frac{a(1-\xi)t}{x}\right]^{\frac{\eta}{1-\xi}}t^{\rho-1}\frac{\Gamma(\gamma_1)\ldots\Gamma(\gamma_n)}{\Gamma(\beta_1)\ldots\Gamma(\beta_n)}\right.$$

$$\left.\times\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha)_{m_1+\ldots+m_n}\Gamma(\beta_1+\tau_1m_1)\ldots\Gamma(\beta_n+\tau_nm_n)}{\Gamma(\gamma_1+\tau_1m_1)\ldots\Gamma(\gamma_n+\tau_nm_n)}\frac{\left(b_1t^{\sigma_1}\right)^{m_1}}{m_1!}\ldots\frac{\left(b_nt^{\sigma_n}\right)^{m_n}}{m_n!}\right\}dt.$$

By changing the order of integration and summation, and after some simplifications using the beta function formula, we get the desired result (7).

Special cases

Corollary 1.1. Putting $\xi=1$ in equation (7), we obtain the following result

$$P_{0+}^{(\eta,1)}\left[t^{\rho-1}F_A^{(n),\tau_1,\ldots,\tau_n}\left(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;b_1t^{\sigma_1},\ldots,b_nt^{\sigma_n}\right)\right]$$

$$=\frac{\Gamma(\rho)x^{\eta+\rho}}{(a\eta)^{\rho}}\sum_{m_1,\ldots,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\ldots+\sigma_nm_n}(\alpha)_{m_1+\ldots+m_n}(\beta_1)_{\tau_1m_1}\cdots(\beta_n)_{\tau_nm_n}}{(\gamma_1)_{\tau_1m_1}\cdots(\gamma_n)_{\tau_nm_n}}\right.$$

$$\left.\times\left(\frac{b_1x^{\sigma_1}}{(a\eta)^{\sigma_1}}\right)^{m_1}\frac{1}{m_1!}\cdots\left(\frac{b_nx^{\sigma_n}}{(a\eta)^{\sigma_n}}\right)^{m_n}\frac{1}{m_n!}\right\}. \quad (8)$$

This is the same as the Laplace transform of the given function.

Corollary 1.2. Taking $\xi=0,\ a=1$ and replacing η by $\eta-1$ in equation (7), we obtain the following result

$$P_{0+}^{(\eta-1,0)}\left[t^{\rho-1}F_A^{(n),\tau_1,\ldots,\tau_n}\left(\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;b_1t^{\sigma_1},\ldots,b_nt^{\sigma_n}\right)\right]=\frac{\Gamma(\eta)\Gamma(\rho)x^{\eta+\rho-1}}{\Gamma(\eta+\rho)}$$

$$\times\sum_{m_1,\ldots,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\ldots+\sigma_nm_n}(\alpha)_{m_1+\ldots+m_n}(\beta_1)_{\tau_1m_1}\cdots(\beta_n)_{\tau_nm_n}}{(\eta+\rho)_{\sigma_1m_1+\ldots+\sigma_nm_n}(\gamma_1)_{\tau_1m_1}\cdots(\gamma_n)_{\tau_nm_n}}\frac{\left(b_1x^{\sigma_1}\right)^{m_1}}{m_1!}\cdots\frac{\left(b_nx^{\sigma_n}\right)^{m_n}}{m_n!}\right\}. \quad (9)$$

This is the same as the left-sided Riemann-Liouville fractional integral of the given function.

Theorem 2. For $\eta,\rho,\ \alpha_j,\ \beta_j\in C,\gamma\in C\setminus Z_0^-,R(\eta)>0,R(\rho)>0,a>0,R(\alpha_j)\geq 0,$

$R(\beta_j)\geq 0, b_j\in R, \left(j=1,2,...,n\,;n\in N\right), R\left(1+\frac{\eta}{1-\xi}\right)>\max\left\{0,-R(\rho)\right\};\tau_1,\,\tau_2,...,\tau_n>0$

; $\max\left\{\left|b_1t^{\sigma_1}\right|,\left|b_2t^{\sigma_2}\right|,\,...\,,\left|b_nt^{\sigma_n}\right|\right\}<1$ and $\xi<1$, then following results hold true

$$P_{0+}^{(\eta,\xi)}\left[t^{\rho-1}F_B^{(n),\tau_1,...,\tau_n}\left(\alpha_1,...,\alpha_n,\beta_1,...,\beta_n;\gamma;b_1t^{\sigma_1},...,b_nt^{\sigma_n}\right)\right]$$

$$=\frac{x^{\eta+\rho}\,\mathrm{B}\left(\rho,1+\frac{\eta}{1-\xi}\right)}{\left[a\left(1-\xi\right)\right]^{\rho}}\sum_{m_1,...,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\,...\,+\sigma_nm_n}(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{\tau_1m_1}\cdots(\beta_n)_{\tau_nm_n}}{\left(1+\rho+\frac{\eta}{1-\xi}\right)_{\sigma_1m_1+...+\sigma_nm_n}(\gamma)_{\tau_1m_1+\,...\,+\tau_nm_n}}\right.$$

$$\left.\times\left(\frac{b_1x^{\sigma_1}}{\left[a\left(1-\xi\right)\right]^{\sigma_1}}\right)^{m_1}\frac{1}{m_1!}\cdots\left(\frac{b_nx^{\sigma_n}}{\left[a\left(1-\xi\right)\right]^{\sigma_n}}\right)^{m_n}\frac{1}{m_n!}\right\}. \tag{10}$$

Proof: Using equations (1) and (5), in the LHS of equation (10), we have

$$P_{0+}^{(\eta,\xi)}\left[t^{\rho-1}F_B^{(n),\tau_1,...,\tau_n}\left(\alpha_1,...,\alpha_n,\beta_1,...,\beta_n;\gamma;b_1t^{\sigma_1},...,b_nt^{\sigma_n}\right)\right]$$

$$=x^{\eta}\int_0^{\left[\frac{x}{a(1-\xi)}\right]}\left\{\left[1-\frac{a(1-\xi)t}{x}\right]^{\frac{\eta}{1-\xi}}t^{\rho-1}\frac{\Gamma(\gamma)}{\Gamma(\beta_1)\ldots\Gamma(\beta_n)}\right.$$

$$\left.\times\sum_{m_1,...,m_n=0}^{\infty}\frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}\Gamma(\beta_1+\tau_1m_1)\ldots\Gamma(\beta_n+\tau_nm_n)}{\Gamma(\gamma+\tau_1m_1+\ldots+\tau_nm_n)}\frac{\left(b_1t^{\sigma_1}\right)^{m_1}}{m_1!}\cdots\frac{\left(b_nt^{\sigma_n}\right)^{m_n}}{m_n!}\right\}dt.$$

By changing the order of integration and summation, and after some simplifications using the beta function formula, we get the desired result (10).

Special cases

Corollary 2.1. Putting $\xi=1$ in equation (10), we obtain the following result

$$P_{0+}^{(\eta,1)}\left[t^{\rho-1}F_B^{(n),\tau_1,...,\tau_n}\left(\alpha_1,...,\alpha_n,\beta_1,...,\beta_n;\gamma;b_1t^{\sigma_1},...,b_nt^{\sigma_n}\right)\right]$$

$$=\frac{\Gamma(\rho)\,x^{\eta+\rho}}{(a\eta)^{\rho}}\sum_{m_1,...,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\,...\,+\sigma_nm_n}(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{\tau_1m_1}\cdots(\beta_n)_{\tau_nm_n}}{(\gamma)_{\tau_1m_1+\,...\,+\tau_nm_n}}\right.$$

$$\left.\times\left(\frac{b_1x^{\sigma_1}}{(a\eta)^{\sigma_1}}\right)^{m_1}\frac{1}{m_1!}\cdots\left(\frac{b_nx^{\sigma_n}}{(a\eta)^{\sigma_n}}\right)^{m_n}\frac{1}{m_n!}\right\}. \tag{11}$$

This is the same as the Laplace transform of the given function.

Corollary 2.2. Taking $\xi = 0$, $a = 1$ and replacing η by $\eta - 1$ in equation (10), we obtain the following result

$$P_{0+}^{(\eta-1,0)}\left[t^{\rho-1}F_B^{(n),\tau_1,\dots,\tau_n}\left(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n;\gamma;b_1t^{\sigma_1},\dots,b_nt^{\sigma_n}\right)\right]$$

$$=\frac{\Gamma(\eta)\Gamma(\rho)x^{\eta+\rho-1}}{\Gamma(\eta+\rho)}\sum_{m_1,\dots,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\dots+\sigma_nm_n}(\alpha)_{m_1}\cdots(\alpha)_{m_n}(\beta_1)_{\tau_1m_1}\cdots(\beta_n)_{\tau_nm_n}}{(\eta+\rho)_{\sigma_1m_1+\dots+\sigma_nm_n}(\gamma)_{\tau_1m_1+\dots+\tau_nm_n}}\right.$$

$$\left.\times\frac{\left(b_1x^{\sigma_1}\right)^{m_1}}{m_1!}\cdots\frac{\left(b_nx^{\sigma_n}\right)^{m_n}}{m_n!}\right\}. \tag{12}$$

This is the same as the left-sided Riemann-Liouville fractional integral of the given function.

Theorem 3. For $\eta, \rho, \alpha, \beta_j \in C$, $\gamma \in C\setminus Z_0^-$, $R(\eta)>0$, $R(\rho)>0$, $R(\alpha)>0$, $a>0$, $R(\beta_j)\geq 0,, b_j \in R\,(j=1,2,\dots,n;n\in N), R\left(1+\frac{\eta}{1-\xi}\right)>\max\{0,-R(\rho)\}$; $\tau_1,\tau_2,\dots,\tau_n>0$; $\max\left\{\left|b_1t^{\sigma_1}\right|,\left|b_2t^{\sigma_2}\right|,\dots,\left|b_nt^{\sigma_n}\right|\right\}<1$ and $\xi<1$, then following results hold true

$$P_{0+}^{(\eta,\xi)}\left[t^{\rho-1}F_D^{(n),\tau_1,\dots,\tau_n}\left(\alpha,\beta_1,\dots,\beta_n;\gamma;b_1t^{\sigma_1},\dots,b_nt^{\sigma_n}\right)\right]$$

$$=\frac{x^{\eta+\rho}\,\mathrm{B}\left(\rho,1+\frac{\eta}{1-\xi}\right)}{\left[a(1-\xi)\right]^{\rho}}\sum_{m_1,\dots,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\dots+\sigma_nm_n}(\alpha)_{\tau_1m_1+\dots+\tau_nm_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{\left(1+\rho+\frac{\eta}{1-\xi}\right)_{\sigma_1m_1+\dots+\sigma_nm_n}(\gamma)_{\tau_1m_1+\dots+\tau_nm_n}}\right.$$

$$\left.\times\left(\frac{b_1x^{\sigma_1}}{\left[a(1-\xi)\right]^{\sigma_1}}\right)^{m_1}\frac{1}{m_1!}\cdots\left(\frac{b_nx^{\sigma_n}}{\left[a(1-\xi)\right]^{\sigma_n}}\right)^{m_n}\frac{1}{m_n!}\right\}. \tag{13}$$

Proof: Using equations (1) and (6), in the LHS of equation (13), we have

$$P_{0+}^{(\eta,\xi)}\left[t^{\rho-1}F_D^{(n),\tau_1,\dots,\tau_n}\left(\alpha,\beta_1,\dots,\beta_n;\gamma;b_1t^{\sigma_1},\dots,b_nt^{\sigma_n}\right)\right]$$

$$=x^{\eta}\int_0^{\left[\frac{x}{a(1-\xi)}\right]}\left\{\left[1-\frac{a(1-\xi)t}{x}\right]^{\frac{\eta}{1-\xi}}t^{\rho-1}\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\right.$$

$$\left.\times\sum_{m_1,\dots,m_n=0}^{\infty}\frac{\Gamma(\alpha+\tau_1m_1+\dots+\tau_nm_n)(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{\Gamma(\gamma+\tau_1m_1+\dots+\tau_nm_n)}\frac{\left(b_1t^{\sigma_1}\right)^{m_1}}{m_1!}\cdots\frac{\left(b_nt^{\sigma_n}\right)^{m_n}}{m_n!}\right\}dt.$$

By changing the order of integration and summation, and after some simplifications using the beta function formula, we get the desired result (13).

Special Cases

Corollary 3.1. Putting $\xi=1$ in equation (13), we obtain the following result

$$P_{0+}^{(\eta,1)}\left[t^{\rho-1}F_D^{(n),\tau_1,\ldots,\tau_n}\left(\alpha,\beta_1,\ldots,\beta_n;\gamma;b_1t^{\sigma_1},\ldots,b_nt^{\sigma_n}\right)\right]$$
$$=\frac{\Gamma(\rho)\,x^{\eta+\rho}}{(a\eta)^{\rho}}\sum_{m_1,\ldots,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\ldots+\sigma_nm_n}(\alpha)_{\tau_1m_1+\ldots+\tau_nm_n}(\beta_1)_{\tau_1m_1}\cdots(\beta_n)_{\tau_nm_n}}{(\gamma)_{\tau_1m_1+\ldots+\tau_nm_n}}\right.$$
$$\left.\times\left(\frac{b_1x^{\sigma_1}}{(a\eta)^{\sigma_1}}\right)^{m_1}\frac{1}{m_1!}\cdots\left(\frac{b_nx^{\sigma_n}}{(a\eta)^{\sigma_n}}\right)^{m_n}\frac{1}{m_n!}\right\}. \qquad (14)$$

Corollary 3.2. Taking $\xi=0,\ a=1$ and replacing η by $\eta-1$ in equation (13), we obtain the following result

$$P_{0+}^{(\eta-1,0)}\left[t^{\rho-1}F_D^{(n),\tau_1,\ldots,\tau_n}\left(\alpha,\beta_1,\ldots,\beta_n;\gamma;b_1t^{\sigma_1},\ldots,b_nt^{\sigma_n}\right)\right]$$
$$=\frac{\Gamma(\eta)\Gamma(\rho)\,x^{\eta+\rho-1}}{\Gamma(\eta+\rho)}\sum_{m_1,\ldots,m_n=0}^{\infty}\left\{\frac{(\rho)_{\sigma_1m_1+\ldots+\sigma_nm_n}(\alpha)_{\tau_1m_1+\ldots+\tau_nm_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\eta+\rho)_{\sigma_1m_1+\ldots+\sigma_nm_n}(\gamma)_{\tau_1m_1+\ldots+\tau_nm_n}}\right. \qquad (15)$$
$$\left.\times\frac{\left(b_1x^{\sigma_1}\right)^{m_1}}{m_1!}\cdots\frac{\left(b_nx^{\sigma_n}\right)^{m_n}}{m_n!}\right\}.$$

This is the same as the left-sided Riemann-Liouville fractional integral of the given function.

5. Concluding remark

In this study, we used the pathway fractional integral operator $p_{0+}^{(\eta,\xi)}$ on some τ-extensions of Lauricella functions with several variables. The results obtained here are quite general in nature, and they also have interesting special cases. Furthermore, we can establish some more properties of τ-extensions of Lauricella functions with several variables.

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