Journal of Mathematics and Informatics
Vol. 21, 2021, 43-52
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 14 November 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v21a04199

Journal of
Mathematics and Informatics

Quadratically Optimal Bi-Matrix Games

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Received 10 October 2021; accepted 12 November 2021

Abstract. In this paper, we introduce the class of quadratically optimal (bi-matrix) games, which are bi-matrix games whose set of equilibrium points contain all pairs of probability vectors which maximize the expected pay-off of some pay-off matrix. We call the equilibrium points obtained in this way, quadratically optimal equilibrium points. We prove the existence of quadratically optimal equilibrium points of identical bi-matrix games, i.e. bi-matrix games for which the two pay-off matrices are equal, from which it easily follows that weakly potential bi-matrix games (a generalization of potential bi-matrix games) are quadratically optimal. We also show that those weakly potential square bi-matrix games which have potential matrices that are two-way matrices are quadratically and symmetrically solvable games (i.e. there exists a square pay-off matrix whose expected pay-off maximizing probability vectors subject to the requirement that the two probability vectors (row probability vector and column probability vector) being equal) are equilibrium points of the bi-matrix game. We show by means of an example of a 2×2 identical symmetric potential bi-matrix game that for every potential matrix of the game, the set of pairs of probability distributions that maximizes the expected pay-off of the potential matrix is a strict subset of the set of equilibrium points of the potential game.

Keywords: bi-matrix games, equilibrium points, weakly potential, quadratically optimal, expected pay-off maximization.

AMS Subject Classification (2010): 90C20, 91A05, 91A10

1. Introduction

In this paper we introduce a class of bi-matrix games called quadratically optimal (bi-matrix) games. Such bi-matrix games are uniquely defined by the property that, their set of equilibrium points contains all pairs of probability vectors which maximize the expected pay-off for some pay-off matrix of the same dimension as those of the bi-matrix game under consideration. Equilibrium points of bi-matrix games, obtained in this manner, i.e.

by maximizing the expected of a pay-off matrix, are called quadratically optimal equilibrium points. Comprehensive discussions on bi-matrix games for our purposes can be found in Majumdar [3], Stahl [8]. Further and more advanced results are available in Chandrasekaran (undated).

In what follows, we prove the existence of quadratically optimal equilibrium points of identical bi-matrix games, i.e. bi-matrix games for which the two pay-off matrices are equal, from which it easily follows that weakly potential bi-matrix games (a generalization of potential bi-matrix games) are quadratically optimal. An alternative way of viewing quadratically optimal equilibrium points is that they are those equilibrium points which maximize the expected pay-off of the pay-off matrix of an identical bi-matrix game. We also show that those weakly potential square bi-matrix games which have potential matrices that are "two-way matrices" are quadratically and symmetrically solvable games, i.e. there exists a square pay-off matrix whose expected pay-off maximizing probability vectors subject to the two probability vectors (row probability vector and column probability vector) being equal, are equilibrium points of the bi-matrix game. We show by means of an example of a 2×2 identical symmetric potential bi-matrix game that for every potential matrix of the game, (a) the set of pairs of probability distributions that maximizes the expected pay-off of the potential matrix is a strict subset of the set of equilibrium points of the potential game, as well as the result (b) that the set of pairs of probability distributions that maximizes the expected pay-off of the potential matrix "subject to the requirement that the two probability distributions are equal", is a strict subset of the set of symmetric equilibrium points of the potential game.

2. The model

Let A and B be two m×n (real-valued) matrices for some positive integers m and n. The pair (A,B) is referred to as an **(m×n)-bi-matrix game**. A and B represent the **pay-off matrices** of the **row-player** (the player who chooses a row) and the **column player** (the player who chooses a column) respectively.

Let

$$\Delta^{m-1} = \{x \in R_+^m \mid \textstyle\sum_{i=1}^m x_i = 1\}$$

be the set of all possible **probability vectors/distributions over rows**, and

$$\Delta^{n-1} = \{ y \in \mathbb{R}_+^n \mid \sum_{j=1}^n y_j = 1\}$$

be the set of all **possible probability vectors/distributions over columns**.

Given an (m×n)-bi-matrix game (A,B) and a pair of probability distributions (x,y), $x^T Ay$ denotes the **expected pay-off of the row player** and $x^T By$ the **expected pay-off of the column player**.

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As in Mangasarian and Stone (1964) a pair $(x^*, y^*) \in \Delta^{m-1} \times \Delta^{n-1}$ is said to be an **equilibrium point** of the $(m\times n)$- bi-matrix game (A,B) if and only if $x^{*T}Ay^* \geq x^TAy^*$ for all $x\in\Delta^{m-1}$ and $x^{*T}By^* \geq x^{*T}By$ for all $y\in\Delta^{n-1}$.

An easier way to define an equilibrium point is the following:
A pair of vectors $(x^*, y^*) \in \Delta^{m-1} \times \in \Delta^{n-1}$ where x^* specifies the probabilities with each row is chosen, y^* specifies the probabilities with which each column is chosen and the pair (x^*,y^*) satisfies the following properties:

(i) a row of the matrix A is chosen with positive probability only if the expected pay-off from that row of the matrix A is (one of) the highest;

(ii) a column of the matrix B is chosen with positive probability only if the expected pay-off from that column of the matrix B is (one of) the highest.

Let c and d be real numbers such that if C is the $m\times n$ with all entries equal to c and D is the $m\times n$ with all entries equal to d, then for all $x\in\Delta^{m-1}$ and $y\in\Delta^{n-1}$, $x^TCy = c$ and $x^TDy = d$. Further if s and t are positive real numbers then for all $x\in\Delta^{m-1}$ and $y\in\Delta^{n-1}$, $x^T(sA)y = sx^TAy$ and $x^T(tB)y = tx^TBy$. Thus (x^*, y^*) is an equilibrium point of the $(m\times n)$-bi-matrix game (A,B) if and only if it is an equilibrium point of the $(m\times n)$-bi-matrix game (sA+C, tB+D) where s, t are positive real numbers and C and D are $m\times n$ matrices with all entries of C being equal to a real number c and all entries of D being a real number d.

A $(m\times n)$-bi-matrix game (A,B) is said to be a **quadratically optimal (bi-matrix) game** if there exists an $m\times n$ matrix P such that <u>all (optimal) solutions</u> of the bilinear programming problem:

Maximize x^TPy
s.t. $x\in\Delta^{m-1}$, $y\in\Delta^{n-1}$,
are equilibrium points of (A,B).

In other words if (x^*,y^*) solves
Maximize x^TPy
s.t. $x\in\Delta^{m-1}$, $y\in\Delta^{n-1}$,
then (x^*,y^*) is an equilibrium point of (A,B).

Any solution of the above bilinear programming problem, (x^*, y^*) is said to be a **quadratically optimal equilibrium point** of (A,B).

<u>Note that</u> the above problem is equivalent to the following quadratic programming problem:

Maximize z^TFz
s.t. $z = \binom{x}{y} \in \mathbb{R}_+^{m+n}$, $x\in\Delta^{m-1}$, $y\in\Delta^{n-1}$,

where F is the (m+n) × (m+n) matrix $\begin{bmatrix} 0 & P \\ P^T & 0 \end{bmatrix}$. Clearly $F = F^T$.

A bi-matrix game (A,-A) for some m×n matrix A is said to be an **(m×n)-zero-sum game**.

For an (m×n)-bi-matrix game (A,B), $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, let A_i, B_i denote the i^{th} rows of A and B respectively and A^j, B^j denote the j^{th} columns of A and B respectively.

An (m×n)-bi-matrix game (A,B) with m = n is said to be a **square bi-matrix game (of size m)**. A square bi-matrix game (A,A^T) is said to be a **symmetric bi-matrix game (of size m)**.

One of the earliest known research on symmetric bi-matrix games is due to Griesmer, Hoffman and Robinson (1963) [2].

It is easy to verify that if (x^*, y^*) is an equilibrium point of a symmetric bi-matrix game (A,A^T) then so is (y^*, x^*).

An equilibrium point (x^*,y^*) of a <u>square</u> bi-matrix game (A,B) is said to be a **symmetric equilibrium point** if $x^* = y^*$.

A square bi-matrix game (A,B) is said to be a **quadratically and symmetrically solvable game** if there exists a square matrix P of the same size as A and B, such that if x^* solves the following quadratic programming problem:

Maximize x^TPx
s.t. $x \in \Delta^{m-1}$,
then (x^*,x^*) is an equilibrium point of (A,B).

Any solution to the above problem is called a **quadratically optimal symmetric equilibrium point** of (A,B).

In the sequel we will be requiring the following concept.

For a natural number m, let $\mathcal{A}(m)$= {A| A is a square matrix of size m satisfying the following property for all $x,y,z \in \Delta^{m-1}$: $x^TA(y-z) > 0$ <u>if and only if</u> $(y-z)^TAx > 0$}.

A matrix in $\mathcal{A}(m)$ is said to be a **two-way matrix of size m**.

Following Monderer and Shapley [7] (1996), we say that an (m×n)-bi-matrix game (A,B) is a **(m×n)- potential bi-matrix game** if there exits an (m×n)-matrix P such that for all $x,z \in \Delta^{m-1}$ and $y,w \in \Delta^{n-1}$: (i) $(x-z)^TAy > 0$ <u>if and only if</u> $(x-z)^TPy > 0$; (ii) $x^TB(y-w) > 0$ <u>if and only if</u> $(x-z)^TPy$. Such a matrix P is said to be a **potential matrix** of the bi-matrix game (A,B).

However, one of our desired results (concerning existence of equilibrium point) holds for a more general class of bi-matrix games.

An (m×n)-bi-matrix game (A,B) is a **(m×n)- weakly potential bi-matrix game** if there exits an (m×n)-matrix P such that for all $x,z \in \Delta^{m-1}$ and $y,w \in \Delta^{n-1}$: (i) $(x-z)^TAy > 0$

implies $(x-z)^T Py > 0$; (ii) $x^T B(y-w) > 0$ implies $(x-z)^T Py$. Such a matrix P is said to be a **potential matrix** of the bi-matrix game (A,B).

A $(m\times n)$-bi-matrix game (A,A) for some $m\times n$ matrix A is said to be an **identical bi-matrix game**.

Note 1: Identical bi-matrix games are weakly potential bi-matrix games.

Note 2: It is easy to verify that if A is a symmetric (i.e. $A = A^T$ and hence square) non-zero matrix, then the two-person zero sum game (A,-A) is not a weakly potential bi-matrix game, for if there exists a square matrix P of the same size as A (say m) such that for all $x,y,z\in \Delta^{m-1}$: (a) $(x-z)^T Ay > 0$ implies $(x-z)^T Py > 0$; (b) $-y^T A(x-z) > 0$ implies $y^T P(x-z) > 0$, then since $A = A^T$ we have from (b) that $- y^T A^T(x-z) > 0$ implies $y^T P(x-z) > 0$, so that $-(x-z)^T Ay > 0$ implies $(x-z)^T Py > 0$. But this is not possible unless A is the zero-matrix, contrary to our assumption about A.

In the next section we have our main results concerning existence of equilibrium points.

3. Main results for weakly potential and quadratically optimal bi-matrix games

The next proposition is of considerable interest in what follows. It is about the existence of quadratically optimal equilibrium points for identical matrix games.

Proposition 1: Suppose A is a $m\times n$ matrix. Then the identical bi-matrix game (A,A) has a quadratically optimal equilibrium point, i.e. (A,A) is a quadratically optimal game.

Proof: Consider the bilinear programming problem denoted BLP(1):

Maximize $x^T Ay$

s.t. $x\in \Delta^{m-1}$, $y\in \Delta^{n-1}$.

It is easy to see that the above problem has an optimal solution.

Let x^*, y^* be a solution.

Then (x^*,y^*) is the desired quadratically optimal equilibrium point. □

Note 3: The category of bi-matrix games discussed in Proposition 1 includes "coordination games" of the form (A,A) where A is a 2×2 matrix of the form $\begin{bmatrix} a & c \\ d & b \end{bmatrix}$ or $\begin{bmatrix} c & a \\ b & d \end{bmatrix}$ where a and b are any two real numbers satisfying $a \neq b$ and c and d are real numbers with min{a,b} > max {c,d}.

An immediate consequence of Proposition 1 is the following interesting result.

Theorem 1: Every weakly potential bi-matrix game has a quadratically optimal equilibrium point, i.e. every weakly potential bi-matrix game is a quadratically optimal game.

The following example shows that there are quadratically optimal games which are not weakly potential bi-matrix games, thus implying that quadratically optimal games are a non-trivial extension of the class of weakly potential bi-matrix games.

Example 1: Consider the two person zero-sum game (A,-A) where A = $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Since A = A^T, by Note 1 (in section 2) it is not a weakly potential bi-matrix game. However, it has a unique equilibrium point ((1,0), (0,1)) which is also the unique (optimal) solution of

Maximize x^TPy

s.t. $x\in\Delta^1$, $y\in\Delta^1$, where

$P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Note 4: It is easily verified that "matching pennies", i.e. the two-person zero sum game (A, -A) with A = $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is not a quadratically optimal game. It is well known that the unique equilibrium point of this game is $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$. Towards a contradiction suppose there exists a square matrix P of size 2 such that the unique optimal solution of the following bilinear programming problem is $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$:

Maximize x^TPy

s.t. $x\in\Delta^1$, $y\in\Delta^1$, where

$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$.

Since $x^TPy = (p_{11} + p_{22} - p_{12} - p_{21}) x_1y_1 + (p_{12}-p_{22})x_1 + (p_{21}-p_{22})y_1$, for any $y\in\Delta^1$ the corresponding $\{x\in\Delta^1 | x^TPy \geq z^TPy$ for all $z\in\Delta^1\}$ must include either (1,0) or (0,1).
Similarly, for any $x\in\Delta^1$ the corresponding $\{y\in\Delta^1 | x^TPy \geq x^TPw$ for all $w\in\Delta^1\}$ must include either (1,0) or (0,1).

Thus the set of optimal solutions of the above bilinear programming must at least one point from $\{(1,0), (0,1)\}\times\{(1,0), (0,1)\}$, contrary to the requirement that $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is the unique solution of the above bilinear programming problem.
Thus, "matching pennies" is not a quadratically optimal game. Hence it is not a weakly potential bi-matrix game either.

The following example shows that there are potential (identical symmetric) bi-matrix games with <u>no potential matrix</u> P being such that expected pay-off maximizing pairs of probability vectors for P, i.e. the set of optimal solutions of the following bilinear programming problem denoted BLP(2):
Maximize x^TPy
s.t. $x\in\Delta^{m-1}$, $y\in\Delta^{n-1}$,
coincides with the set of equilibrium points of the given weakly potential game.

Example 2: Let A = $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. We know that (A,A) has two equilibrium points, namely ((1,0), (1,0)) and ((0,1),(0,1)). A itself a potential matrix for (A,A) which is thus a potential game. Let P be a potential matrix with the entry in the i^{th} row and j^{th} column being p_{ij}. Suppose both ((1,0), (1,0)) and ((0,1),(0,1)) are optimal solutions of BLP(2). Then it must be the case that $p_{11} = p_{22}$. By theorem 1, BLP(2) has no other optimal solution (because if it did, then by the definition of quadratic optimality, it would be an equilibrium point of (A,A)).

Let x,z, y∈[0,1] where x and z are probabilities with which the first row is chosen and y is the probability with which the first column is chosen.

Then (x-z, z-x)A$\binom{y}{1-y}$ = 2(x-z)y –(x-z)(1-y) = 3(x-z)y –(x-z) = (x-z)[3y -1].

Also (x-z, z-x)P$\binom{y}{1-y}$ = $(p_{11}- p_{21})$(x-z)y + $(p_{12}-p_{22})$(x-z)(1-y) = (x-z)[$(p_{11}-p_{21})$y + $(p_{11}-p_{12})$y – $(p_{11}-p_{12})$].

(x-z)[3y -1] > 0 implies (x-z)[$(p_{11}-p_{21})$y + $(p_{11}-p_{12})$y – $(p_{11}-p_{12})$] only if

$$\frac{1}{3} \geq \frac{p_{11}-p_{12}}{2p_{11}-p_{12}-p_{21}} \geq \frac{1}{3}\text{, i.e. } \frac{p_{11}-p_{12}}{2p_{11}-p_{12}-p_{21}} = \frac{1}{3}.$$

Similarly, if we let x,y, w∈[0,1] where x is the probability with which the first row is chosen and y and w are the probabilities with which the first column is chosen, then

(x, 1-x)A$\binom{y-w}{w-y}$ = (y-w)[3y -1], and

(x, 1-x)P$\binom{y-w}{w-y}$ = $(p_{11}- p_{12})$(y-w)x + $(p_{21}-p_{11})$(y-w)(1-x) = (y-w)[$(2p_{11}-p_{12}-p_{21})$x –$(p_{11}-p_{21})$].

Thus, (y-w)[3y -1] > 0 implies (y-w)[$(2p_{11}-p_{12}-p_{21})$x –$(p_{11}-p_{21})$] > 0 only if

$$\frac{p_{11}-p_{21}}{2p_{11}-p_{12}-p_{21}} = \frac{1}{3}.$$

$\frac{p_{11}-p_{12}}{2p_{11}-p_{12}-p_{21}} = \frac{1}{3} = \frac{p_{11}-p_{21}}{2p_{11}-p_{12}-p_{21}}$ implies $p_{11} – p_{12} = p_{11} – p_{21}$, whence $p_{12} = p_{21}$.

But then, $\frac{p_{11}-p_{12}}{2p_{11}-p_{12}-p_{21}} = \frac{1}{2} = \frac{p_{11}-p_{21}}{2p_{11}-p_{12}-p_{21}}$, contradicting $\frac{p_{11}-p_{12}}{2p_{11}-p_{12}-p_{21}} = \frac{1}{3} = \frac{p_{11}-p_{21}}{2p_{11}-p_{12}-p_{21}}$.

Thus there does not exist any potential matrix P for (A,A) such that the set of solutions of BLP(2), coincides with the set of equilibrium points of (A,A).

4. Quadratically optimal and symmetrically solvable bi-matrix games

In this section we will be concerned with quadratically optimal and symmetrically solvable bi-matrix games.

The following is a result about the existence of quadratically optimal symmetric equilibria of a weakly potential square bi-matrix game whose potential matrix is a two-way matrix.

Theorem 2: Suppose (A,B) is a weakly potential square bi-matrix game (of size m) whose potential matrix P is a two-way matrix. Then it is a quadratically optimal and symmetrically solvable game.

Proof: Note that for any square matrix C all whose entries are a constant real number c and $x,y,z \in \Delta^{m-1}$, $x^TC(y-z) = (y-z)^TCx = 0$. Hence, without suppose $p_{ij} > 0$ for all $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$.

Let QP(2) be the quadratic programming problem defined as follows:

Maximize x^TPx

s.t. $x \in \Delta^{m-1}$.

It is well known that QP(2) has a solution. Suppose x^*solves QP(2)

Towards a contradiction suppose there exists $x \in \Delta^{m-1}$, such that $x^TPx^* > x^{*T}Px^*$.

Thus, $(x-x^*)^TPx^* > 0$.

Since $P \in \mathcal{A}(m)$, it must be the case that $x^{*T}P(x-x^*) > 0$.

Consider the point $x^*+ \theta(x-x^*)$ for $\theta \in (0,1)$.

Thus $[x^*+ \theta(x-x^*)]^TP[x^*+ \theta(x-x^*)] = x^{*T}Px^* + \theta[\theta(x-x^*)^TP(x-x^*) + (x-x^*)^TPx^* + x^{*T}P(x-x^*)]$.

Since $(x-x^*)^TPx^* > 0$ and $x^{*T}P(x-x^*) > 0$, we can choose $\theta > 0$, θ sufficiently small so that $\theta(x-x^*)^TP(x-x^*) + (x-x^*)^TPx^* + x^{*T}P(x-x^*) > 0$.

Thus $\theta[\theta(x-x^*)^TP(x-x^*) + (x-x^*)^TPx^* + x^{*T}P(x-x^*)] > 0$, so that we have $[x^*+ \theta(x-x^*)]^TP[x^*+ \theta(x-x^*)] > x^{*T}Ax^*$, contradicting the optimality of x^* for QP(1).

Hence it must be the case that $x^{*T}Px^* \geq x^TPx^*$ for all $x \in \Delta^{m-1}$.

Since P is a weakly potential matrix we get $x^{*T}Px^* \geq x^{*T}Px$ for all $x \in \Delta^{m-1}$.

Since P is a weakly potential matrix for the bi-matrix game (A,B), it follows that (x^*, x^*) is an equilibrium point for (A,B). □

An identical bi-matrix game (A,A) with $A = A^T$ is said to be an **identical symmetric bi-matrix game**.

If (A,A^T) is an identical symmetric bi-matrix game then (i) (A,A^T) is a potential game with A being a potential matrix of the game; (ii) $A \in \mathcal{A}(m)$, i.e. A is a "two-way" matrix.

Thus we have the following corollary of Theorem 2.

Corollary of Theorem 2: Suppose A is a square matrix satisfying $A = A^T$. Then the identical symmetric bi-matrix game (A,A^T) is a <u>quadratically and symmetrically solvable game</u>.

Note 5: The following example of an identical symmetric bi-matrix game with $A = A^T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ suggested to me by Professor Mallikarjuna Rao shows that optimal solutions of QP(1) may be a strict subset of the set of symmetric equilibrium points of a symmetric bi-matrix game.

In this game there are two symmetric equilibrium points x with $x_1 = 1$ and y with $y_1 = 0$. However the unique optimal solution of QP(1) is x.

Example 3: For (A,A^T) with $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, the two quadratically optimal symmetric equilibrium points are the two solutions of:
Maximize x^TPx
s.t. $x \in \Delta^1$,
where $P = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some strictly positive real number 'a'.
However, as noted in example 2, such a P <u>cannot</u> be a potential matrix for (A,A).

Note 6: We saw in note 4 that "matching pennies" (i.e. the two-person zero sum game (A,-A) with $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$) is not quadratically optimal and hence not a weakly potential bi-matrix game. However, it is a quadratically and symmetrically solvable game since $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is the unique optimal solution of
Maximize x^TPx
s.t. $x \in \Delta^1$, where $P = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, i.e. the negative of the identity matrix of size 2.

Note 7: McLennan and Tourkey [6] (2004, 2010) consider "imitation games" which are square bi-matrix games of the form (A,I_m) where A is an $m \times m$ matrix and I_m is an identity matrix. In this context they are concerned with the set of symmetric equilibrium points of the symmetric game (A,A^T). One of the statements of Lemma 1 in McLennan and Tourkey (2010) [5] is an equivalent version of the definition of symmetric equilibrium points of such games and Proposition 1 in the same paper is valid only for the situation where A is a symmetric matrix, i.e. $A = A^T$. Note that a proof of the existence of a symmetric equilibrium of any symmetric bi-matrix game without using any fixed-point theorem argument is available in McLennan and Tourkey [5].

Note 8: Professor Andrew Mclennan [5] suggested that I take a look at his that paper that appeared in American Political Science Review- which however in the context of the latest version is totally irrelevant, except that what we refer to as an identical symmetric game, is called a "game of common interest" in his APSR paper.

Acknowledgments. This paper is a revised version of an earlier one entitled "Equilibrium points of bi-matrix games using bilinear and linear programming". I owe Professor R. Mallikarjuna Rao deep gratitude for suggesting Mangasarian and Stone (1964) to me and for an enormous amount of help and feedback on earlier versions of this paper. In fact, Professor Rao is the only learned and respected colleague who spotted logical errors in the proof of earlier versions of this paper, and my vocabulary of English language is not enough to express my gratitude to him that I wish to do, for such help.

I would also like to thank Dr. Subhadip Chakraborty and Professor A. Tripathy (Arabinda-da: my departmental and institutional colleague at two jobs!) for comments, corrections and for pointing out major gaps in this and related work and Professor Ratul Lahkar for raising clarificatory questions about the applications of the theory discussed in this paper. It is a matter of great pleasure for me to gratefully acknowledge comments, advice about being cautious and suggesting a re-look at the Lemke-Howson algorithm- all from Professor Andrew Mclennan. Finally, I would like to thank an anonymous reviewer and the editor of this journal for their invaluable comments.

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