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A Study of *k*-generalized Mittag-Leffler Type Function with Four Parameters

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Abstract. This study introduces a new generalization of the Mittag-Leffler function, the k-generalized Mittag-Leffler type function ${}_{\gamma,\delta}E_{k,\alpha,\beta}(z)$, and investigates its interesting and important basic properties, such as useful relationships between Mittag-Leffler functions, recurrence relations, differential formulas, integral representations, and images of this function under the Euler (Beta), Laplace, and Whittaker transforms. Some interesting special cases of the main finding are also addressed and demonstrated to be related to some well-known ones.

Keywords: Mittag-Leffler functions, *k*-Pochhammer symbol, *k*-Gamma function, the k-Beta function, Euler (Beta) transform, Laplace transform, Whittaker transform

AMS Mathematics Subject Classification (2010): 33E12, 44A10, 65R10

1. Introduction

The Mittag-Leffler function has grown in popularity and importance in recent years as a result of its applications to the solution of fractional-order differential, integral, integro-differential, and difference equations encountered in a variety of problems in applied sciences such as physics, chemistry, biology, and engineering. Furthermore, various extensions of this special function are significant in the analysis of fluid flow, electric networks, and statistical distribution theory. For various mathematical properties of Mittag-Leffler functions and their applications, the works of Agarwal *et al.* [1], Goreno *et al.* [5], Kilbas *et al.* [6], Haubold *et al.* [7], and Paneva-Konovska and Kiryakova [10] are worth mentioning.

The Mittag-Leffler $E_\alpha(z)$ was introduced in the nineteenth century by renowned mathematician Mittag-Leffler [9] as follow

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\alpha\in C,\operatorname{Re}(\alpha)>0. \tag{1}$$

In 1905, Wiman [18] generalized the function in (1) and defined it as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\alpha,\beta\in C,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0. \quad (2)$$

In 1971, another generalization of (2) appeared, defined by Prabhakar [11] as

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)\, n!},\alpha,\beta,\gamma\in C, \quad (3)$$
$$\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0.$$

In 2007, Shukla and Prajapati [12] developed and defined the generalization of (3)

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)\, n!},\alpha,\beta,\gamma\in C, \quad (4)$$
$$\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0 \text{, and } q\in(0,1)\cup N.$$

In 2009, Salim [14] defined the generalized form of (3) as

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)\,(\delta)_n},\ \alpha,\beta,\gamma,\delta\in C, \quad (5)$$
$$\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,\mathrm{Re}(\delta)>0.$$

In 2012, Salim and Ahmad [15] generalizes (5) and defined as

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)\,(\delta)_{pn}},\ \alpha,\beta,\gamma,\delta\in C \quad (6)$$
$$;\ \min\{\mathrm{Re}(\alpha),\ \mathrm{Re}(\beta),\ \mathrm{Re}(\gamma)\ \text{and}\ \mathrm{Re}(\delta)\}>0.$$

A further generalization of (6) is given by Khan and Ahmad [8] as follows

$$E_{\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\upsilon)_{\sigma n}(\delta)_{pn}},\alpha,\beta,\gamma,\delta,\mu,\upsilon,\rho,\sigma\in C$$
$$;\ p,q>0,\ q\le\mathrm{Re}(\alpha)+p \text{ with } \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta), \quad (7)$$
$$\mathrm{Re}(\gamma),\mathrm{Re}(\delta),\mathrm{Re}(\mu),\mathrm{Re}(\upsilon),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0.$$

Recently, Garg *et al.* [4] introduced and defined a new generalization of the Mittag-Leffler type function in order to avoid increasing the number of variables and parameters in the following manner

$${}_{\gamma,\delta}E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n}}{\Gamma(\alpha n+\beta)}z^n, \quad (8)$$

where $\alpha,\beta,\gamma,\delta\in C,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,\mathrm{Re}(\delta)>0,\ \mathrm{Re}(\alpha)>\mathrm{Re}(\delta)$.

The main goal of this study is to investigate a novel generalization of the aforementioned Mittag-Leffler type function and evaluate some of its basic properties.

2. Basic definitions and results

In this section, we have given several important definitions and well-known results, which are used further in this paper.

(a) The k-Pochhammer symbol $(x)_{n,k}$ was defined by Diaz and Pariguan [2] as

$$(x)_{n,k} = x(x+k)(x+2k),...,[x+(n-1)k], \tag{9}$$

where $x \in C$, $k \in R$ and $n \in N$.

(b) The k-Gamma function $\Gamma_k(x)$ was defined by Diaz and Pariguan [2] as

$$\Gamma_k(x) = \int_0^\infty \exp\left(-\frac{t^k}{k}\right) t^{x-1} dt, \tag{10}$$

where $x \in C$, $k \in R$, $\text{Re}(x) > 0$ with $\Gamma_k(x+k) = x\Gamma_k(x)$.

(c) The k-Beta function $B_k(x,y)$ was defined by Diaz and Pariguan [2] as

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt, \tag{11}$$

where $x \in C$, $k > 0$, $k \in R$, $\text{Re}(x) > 0, \text{Re}(y) > 0$ with

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}.$$

(d) The well-Known Gamma function $\Gamma(n)$ is defined as

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \quad \text{where } \text{Re}(n) > 0. \tag{12}$$

(e) The well-Known Beta function $B(m,n)$ is defined as [13]

$$B(m,n) = \int_0^1 t^{n-1}(1-t)^{m-1} dt = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \tag{13}$$

where $\text{Re}(m) > 0$, $\text{Re}(n) > 0$.

(f) The Fox-Wright function ${}_p\Psi_q(z)$ is defined as [16]

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),...,(a_p,A_p) \\ \\ (b_1,B_1),...,(b_q,B_q)\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i + A_i n)}{\prod_{j=1}^{q}\Gamma(b_j + B_j n)} \frac{z^n}{n!}, \tag{14}$$

where z, a_i, b_j, A_i, $B_j \in C$, $\text{Re}(a_i) > 0$, $\text{Re}(A_i) > 0$, $i = 1,..., p$, $\text{Re}(b_j) > 0$, $\text{Re}(B_j) > 0$, $j = 1,..., q$, and $1+\mathrm{Re}\left(\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i\right) \geq 0$.

(g) The Euler (Beta) transform of the function $f(z)$ is defined as [13]

$$B\{f(z);a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)dz. \tag{15}$$

(h) The Laplace transform of the function $f(z)$ is defined as [13]

$$L\{f(z);s\} = \int_0^\infty e^{-sz} f(z)dz, \tag{16}$$

where s is real or complex number.

(i) The Whittaker transform is defined as [17]

$$\int_0^\infty e^{-\frac{t}{2}} t^{\nu-1} W_{\lambda,\mu}(t)dt = \frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}-\mu+\nu\right)}{\Gamma(1-\lambda+\nu)}, \tag{17}$$

where $\mathrm{Re}(\mu+\nu) > -\frac{1}{2}$

and $W_{\lambda,\mu}(t)$ is the Whittaker confluent hypergeometric function.

The following well-known identities are necessary for our sequel (cf. [3]).

Proposition I. Let $\gamma \in C$ and $k, s \in R$, then the following identity holds

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1} \Gamma_k\left(\frac{k\gamma}{s}\right), \tag{18}$$

particularly, $$\Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1} \Gamma\left(\frac{\gamma}{k}\right). \tag{19}$$

Proposition II. Let $\gamma \in C$, $k, s \in R$ and $n \in N$, then the following identity holds

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq} \left(\frac{k\gamma}{s}\right)_{nq,k}, \tag{20}$$

particularly, $$(\gamma)_{nq,k} = (k)^{nq} \left(\frac{\gamma}{k}\right)_{nq}. \tag{21}$$

3. A new generalized Mittag-Leffler type function

In this section, we have introduced and studied a new generalized Mittag-Leffler type function, called the *k*-generalized Mittag-Leffler function

$${}_{\gamma,\delta}E_{k,\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{(\gamma)_{\delta n,k}}{\Gamma_k(\alpha n+\beta)} z^n, \tag{22}$$

where $k \in R, \alpha, \beta, \gamma, \delta \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0, \mathrm{Re}\,(\alpha) > \mathrm{Re}\,(\delta)$ and $(\gamma)_{\delta n,k}$ is the k-Pochhammer symbol and $\Gamma_k(\alpha n+\beta)$ is the k-Gamma function.

Special cases of $_{\gamma,\delta}E_{k,\alpha,\beta}(z)$

(i) Putting $k=1$, equation (22) reduces to Mittag-Leffler function given by Garg *et al.* [4]

$$_{\gamma,\delta}E_{1,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,1}}{\Gamma_1(\alpha n+\beta)}z^n = {}_{\gamma,\delta}E_{\alpha,\beta}(z). \tag{23}$$

(ii) Assigning $k=1, \delta=1,$ equation (22) reduces to another new generalized Mittag-Leffler type function

$$_{\gamma,1}E_{1,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{1n,1}}{\Gamma_1(\alpha n+\beta)}z^n = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}z^n = {}_{\gamma}E_{\alpha,\beta}(z). \tag{24}$$

(iii) Putting $k=1, \delta=0$, equation (22) reduces to the Mittag-Leffler function defined by Wiman [18],

$$_{\gamma,0}E_{1,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{0,1}}{\Gamma_1(\alpha n+\beta)}z^n = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}z^n = E_{\alpha,\beta}(z). \tag{25}$$

(iv) Finally, the Mittag-Leffler function of Gosta Mittag-Leffler [9] is obtained from equation (22) for $k=1, \delta=0, \beta=1$

$$_{\gamma,0}E_{1,\alpha,1}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{0,1}}{\Gamma_1(\alpha n+1)}z^n = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)} = E_{\alpha}(z). \tag{26}$$

4. Main results

In this section, we have derived some new basic properties of $_{\gamma,\delta}E_{k,\alpha,\beta}(z)$.

Theorem 1. The functional relationship between k-generalized Mittag-Leffler function $_{\gamma,\delta}E_{k,\alpha,\beta}(z)$ and generalized Mittag-Leffler function $_{\gamma,\delta}E_{\alpha,\beta}(z)$ is expressed as

$$_{\gamma,\delta}E_{k,\alpha,\beta}(z)=\left((k)^{1-\frac{\beta}{k}}\right)\,_{\frac{\gamma}{k},\delta}E_{\frac{\alpha}{k},\frac{\beta}{k}}\left(k^{\delta-\frac{\alpha}{k}}z\right). \tag{27}$$

Proof. By using definition (22), we have

$$_{\gamma,\delta}E_{k,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}}{\Gamma_k(\alpha n+\beta)}z^n.$$

Now, making use of the equations (19) and (21), we have

$$_{\gamma,\delta}E_{k,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(k)^{\delta n}\left(\frac{\gamma}{k}\right)_{\delta n}}{(k)^{\frac{\alpha n}{k}+\frac{\beta}{k}-1}\,\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}z^n$$

$$_{\gamma,\delta}E_{k,\alpha,\beta}(z)=\left((k)^{1-\frac{\beta}{k}}\right)_{\frac{\gamma}{k},\delta}E_{\frac{\alpha}{k},\frac{\beta}{k}}\left(k^{\delta-\frac{\alpha}{k}}z\right).$$

This completes the proof of the Theorem 1.

Recurrence relation

Theorem 2. Let $k\in R,\ \alpha,\ \beta,\ \gamma,\ \delta\in C,$

and (i) $\operatorname{Re}(\alpha)>0,\operatorname{Re}(\beta)>0,\operatorname{Re}(\gamma)>0,\operatorname{Re}(\delta)>0,$

(ii) $\operatorname{Re}(\alpha)>\operatorname{Re}(\delta),$

then

$$_{\gamma,\delta}E_{k,\alpha,\beta}(z)=\beta\,_{\gamma,\delta}E_{k,\alpha,\beta+k}(z)+\alpha z\frac{d}{dz}\,_{\gamma,\delta}E_{k,\alpha,\beta+k}(z). \tag{28}$$

Proof: Expressing the series form of $_{\gamma,\delta}E_{k,\alpha,\beta}(z)$ on the right-hand side of equation (28) in view of equation (22), we have

$$\beta\,_{\gamma,\delta}E_{k,\alpha,\beta+k}(z)+\alpha z\frac{d}{dz}\,_{\gamma,\delta}E_{k,\alpha,\beta+k}(z)$$

$$=\beta\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\,z^n}{\Gamma_k(\alpha n+\beta+k)}+\alpha z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\,z^n}{\Gamma_k(\alpha n+\beta+k)}$$

$$=\beta\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\,z^n}{\Gamma_k(\alpha n+\beta+k)}+\alpha n\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\,z^n}{\Gamma_k(\alpha n+\beta+k)}$$

Using $\Gamma_k(x+k)=x\Gamma_k(x)$, we obtain

$$=\sum_{n=0}^{\infty}\frac{(\alpha n+\beta)(\gamma)_{\delta n,k}\,z^n}{\Gamma_k(\alpha n+\beta+k)}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\,z^n}{\Gamma_k(\alpha n+\beta)}$$

$$=\,_{\gamma,\delta}E_{k,\alpha,\beta}(z)$$

This completes the proof of the theorem 2.

Special cases

(i) For $k=1,$ the result in equation (28), reduces to the following result

$$_{\gamma,\delta}E_{\alpha,\beta}(z)=\beta\,_{\gamma,\delta}E_{\alpha,\beta+1}(z)+\alpha z\frac{d}{dz}\,_{\gamma,\delta}E_{\alpha,\beta+1}(z).$$

This result is analogous to the result of Garg *et al*. [4].

(ii) Assigning $k=1,\delta=0$, in equation (28), we get

$$E_{\alpha,\beta}(z)=\beta E_{\alpha,\beta+1}(z)+\alpha z\frac{d}{dz}E_{\alpha,\beta+1}(z).$$

This result is similar to the result earlier obtained by Haubold *et al.* [7].

Differential formulas

Theorem 3. Let $k\in R,\ \alpha,\ \beta,\ \gamma,\ \delta,\ w\in C,$

and (i) $\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\delta)>0,$

(ii) $\mathrm{Re}\,(\alpha)>\mathrm{Re}\,(\delta),\ m\in N,$

then

$$\left(\frac{d}{dz}\right)^m\left[z^{\frac{\beta}{k}-1}\,{}_{\gamma,\delta}E_{k,\alpha,\beta}\left(wz^{\frac{\alpha}{k}}\right)\right]=z^{\frac{\beta}{k}-m-1}(k)^{-m}\,{}_{\gamma,\delta}E_{k,\frac{\alpha}{k},\frac{\beta}{k}-m}\left(wz^{\frac{\alpha}{k}}\right). \tag{29}$$

Proof: Making use of the series (22) on the Left-hand side of the equation (29), we find

$$\left(\frac{d}{dz}\right)^m\left[z^{\frac{\beta}{k}-1}\,{}_{\gamma,\delta}E_{k,\alpha,\beta}\left(wz^{\frac{\alpha}{k}}\right)\right]=\left(\frac{d}{dz}\right)^m\left[z^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}}{\Gamma_k(\alpha n+\beta)}\left(wz^{\frac{\alpha}{k}}\right)^n\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}(w)^n}{\Gamma_k(\alpha n+\beta)}\left(\frac{d}{dz}\right)^m z^{\frac{\alpha n}{k}+\frac{\beta}{k}-1}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}(w)^n}{\Gamma_k\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}\frac{1}{k^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}}\left(\frac{d}{dz}\right)^m\left(z^{\frac{\alpha n}{k}+\frac{\beta}{k}-1}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}(w)^n}{\Gamma_k\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}\frac{z^{\frac{\alpha n}{k}+\frac{\beta}{k}-1-m}}{k^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}}\frac{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}-m\right)}$$

$$=z^{\frac{\beta}{k}-1-m}(k)^{-m}\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}}{\Gamma_k\left(\frac{\alpha}{k}n+\frac{\beta}{k}-m\right)}\frac{1}{k^{\frac{\alpha}{k}n+\frac{\beta}{k}-m-1}}\left(wz^{\frac{\alpha}{k}}\right)^n$$

$$=z^{\frac{\beta}{k}-m-1}(k)^{-m}\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}}{\Gamma_k\left(\frac{\alpha}{k}n+\frac{\beta}{k}-m\right)}\left(wz^{\frac{\alpha}{k}}\right)^n$$

$$=z^{\frac{\beta}{k}-m-1}(k)^{-m}\,{}_{\gamma,\delta}E_{k,\frac{\alpha}{k},\frac{\beta}{k}-1}\left(wz^{\frac{\alpha}{k}}\right).$$

This completes the proof of the Theorem 3.

Special cases

(i) For k = 1, the result in equation (29), reduces to the following result

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1}\,{}_{\gamma,\delta}E_{\alpha,\beta}\left(wz^{\alpha}\right)\right] = z^{\beta-m-1}\,{}_{\gamma,\delta}E_{\alpha,\,\beta-m}\left(wz^{\alpha}\right).$$

This result is analogous to the result of Garg *et al*. [4].

(ii) Assigning $k=1,\ \delta=0$, in equation (29), we obtain

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1}\,E_{\alpha,\beta}\left(wz^{\alpha}\right)\right] = z^{\beta-m-1}\,E_{\alpha,\,\beta-m}\left(wz^{\alpha}\right).$$

This result is similar to the result earlier obtained by Haubold *et al.* [7].

Theorem 4. Let $k\in R,\ \alpha,\ \beta,\ \gamma,\ \delta\in C,$

and (i) $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0, \mathrm{Re}(\delta)>0,$

(ii) $\mathrm{Re}\,(\alpha) > \mathrm{Re}\,(\delta),\ m\in N,$

then

$$\left(\frac{d}{dz}\right)^m {}_{\gamma,\delta}E_{k,\alpha,\beta}\left(z\right)=\left(1\right)_m\left(\gamma\right)_{\delta m,k}\sum_{n=0}^{\infty}\frac{\left(\gamma+\delta mk\right)_{\delta n,k}\left(1+m\right)_n}{\Gamma_k\left(\alpha n+\alpha m+\beta\right)}\frac{z^n}{n!}. \tag{30}$$

Proof: Making use of the series (22) on the Left-hand side of the equation (30), we find

$$\left(\frac{d}{dz}\right)^m {}_{\gamma,\delta}E_{k,\alpha,\beta}\left(z\right)=\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{\left(\gamma\right)_{\delta n,k}}{\Gamma_k\left(\alpha n+\beta\right)}z^n$$

$$=\sum_{n-m=0}^{\infty}\frac{\left(\gamma\right)_{\delta n,k}\quad z^{n-m}\quad n!}{\Gamma_k\left(\alpha n+\beta\right)\left(n-m\right)!}$$

$$=\sum_{n=0}^{\infty}\frac{\left(\gamma\right)_{\delta n+\delta m,k}}{\Gamma_k\left(\alpha n+\alpha m+\beta\right)}\frac{\left(n+m\right)!\,z^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{\left(1\right)_m\left(1+m\right)_n\left(\gamma\right)_{\delta n+\delta m,k}}{\Gamma_k\left(\alpha n+\alpha m+\beta\right)}\frac{z^n}{n!}$$

Using the relation $\left(\chi\right)_{n+m,k=}\left(\chi\right)_{m,k}\left(\chi+mk\right)_{n,k}$, we get

$$\left(\frac{d}{dz}\right)^m {}_{\gamma,\delta}E_{k,\alpha,\beta}\left(z\right)=\left(1\right)_m\left(\gamma\right)_{\delta m,k}\sum_{n=0}^{\infty}\frac{\left(\gamma+\delta mk\right)_{\delta n,k}\left(1+m\right)_n}{\Gamma_k\left(\alpha n+\alpha m+\beta\right)}\frac{z^n}{n!}\ .$$

This completes the proof of the Theorem 4.

Special cases

(i) For $k=1,$ the result in equation (30), reduces to the following result

$$\left(\frac{d}{dz}\right)^m {}_{\gamma,\delta}E_{\alpha,\beta}(z)=(1)_m(\gamma)_{\delta m}\sum_{n=0}^{\infty}\frac{(\gamma+\delta m)_{\delta n}(1+m)_n}{\Gamma(\alpha n+\alpha m+\beta)}\frac{z^n}{n!}.$$

(ii) Assigning $k=1,\delta=0$, the result in equation (30), reduces to the following simple result

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+m+1)}{\Gamma(\alpha n+\alpha m+\beta)}\frac{z^n}{n!}.$$

Integral representations

Theorem 5. Let $k\in R,\ \alpha,\ \beta,\ \gamma,\ \delta\in C,$

and (i) $\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,\mathrm{Re}(\delta)>0,$

(ii) $\mathrm{Re}\,(\alpha)>\mathrm{Re}\,(\delta),$

then

$$k\left[{}_{\gamma,\delta}E_{k,\alpha,\beta}(z)\right]=\frac{1}{\Gamma_k(\gamma)}\int_0^{\infty}e^{\frac{-u}{k}}\,u^{\frac{\gamma}{k}-1}E_{k,\alpha,\beta}(zu^{\frac{\delta}{k}})\,du. \tag{31}$$

Proof: By using the Right-hand side of the equation (31), we have

$$\frac{1}{\Gamma_k(\gamma)}\int_0^{\infty}e^{\frac{-u}{k}}\,u^{\frac{\gamma}{k}-1}E_{k,\alpha,\beta}(zu^{\frac{\delta}{k}})\,du$$

$$=\frac{1}{\Gamma_k(\gamma)}\int_0^{\infty}e^{\frac{-u}{k}}\,u^{\frac{\gamma}{k}-1}\left[\sum_{n=0}^{\infty}\frac{z^n u^{\frac{\delta n}{k}}}{\Gamma_k(\alpha n+\beta)}\right]du$$

$$=\frac{1}{\Gamma_k(\gamma)}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(\alpha n+\beta)}\int_0^{\infty}e^{\frac{-u}{k}}\,u^{\frac{\gamma}{k}+\frac{\delta n}{k}-1}du$$

$$=\frac{k}{\Gamma_k(\gamma)}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(\alpha n+\beta)}\Gamma_k(\delta n+\gamma)$$

$$=k\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\,z^n}{\Gamma_k(\alpha n+\beta)}$$

$$=k\left[{}_{\gamma,\delta}E_{k,\alpha,\beta}(z)\right].$$

This completes the proof of the theorem 5.

Special cases

(i) For k = 1, the result in (31), reduces to the following result

$$ {}_{\gamma,\delta}E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-u}\, u^{\gamma-1} E_{\alpha,\beta}(zt^{\delta})\, du. $$

This result is analogous to the result of Garg *et al*. [4].

Theorem 6. Let $k \in R,\ \alpha,\ \beta,\ \gamma,\ \delta \in C,$

and (i) $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0,$

(ii) $\mathrm{Re}\ (\alpha) > \mathrm{Re}\ (\delta),$

then

$$ k\left[{}_{\gamma,\alpha}E_{k,\alpha,\beta}(z)\right] = \frac{1}{\Gamma_k(\gamma)\,\Gamma_k(\beta-\gamma)} \int_0^1 u^{\frac{\gamma}{k}-1} (1-u)^{\frac{\beta}{k}-\frac{\gamma}{k}-1} \left(1 - zu^{\alpha/k}\right)^{-1} du. \tag{32} $$

Proof: Using the Right-hand side of the equation (32), we have

$$ \frac{1}{\Gamma_k(\gamma)\,\Gamma_k(\beta-\gamma)} \int_0^1 u^{\frac{\gamma}{k}-1} (1-u)^{\frac{\beta}{k}-\frac{\gamma}{k}-1} \left(1 - zu^{\alpha/k}\right)^{-1} du $$

$$ = \frac{1}{\Gamma_k(\gamma)\,\Gamma_k(\beta-\gamma)} \int_0^1 u^{\frac{\gamma}{k}-1} (1-u)^{\frac{\beta}{k}-\frac{\gamma}{k}-1} \sum_{n=0}^{\infty} \frac{(1)_n}{n!} \left(zu^{\alpha/k}\right)^n du $$

$$ = \frac{1}{\Gamma_k(\gamma)\,\Gamma_k(\beta-\gamma)} \sum_{n=0}^{\infty} z^n \int_0^1 u^{\frac{\alpha n}{k}+\frac{\gamma}{k}-1} (1-u)^{\frac{\beta}{k}-\frac{\gamma}{k}-1} du $$

$$ = \frac{k}{\Gamma_k(\gamma)\,\Gamma_k(\beta-\gamma)} \sum_{n=0}^{\infty} z^n \frac{\Gamma_k(\alpha n+\gamma)\,\Gamma_k(\beta-\gamma)}{\Gamma_k(\alpha n+\beta)} $$

$$ = k \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha n+\gamma)}{\Gamma_k(\gamma)} \frac{z^n}{\Gamma_k(\alpha n+\beta)} $$

$$ = k \sum_{n=0}^{\infty} \frac{(\gamma)_{\alpha n,k}\, z^n}{\Gamma_k(\alpha n+\beta)} $$

$$ = k\left[{}_{\gamma,\alpha}E_{k,\alpha,\beta}(z)\right]. $$

This completes the proof of the Theorem 6.

Special cases

(i) For $k = 1,$ the result in equation (32), reduces to the following result

$$ {}_{\gamma,\alpha}E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)\,\Gamma(\beta-\gamma)} \int_0^1 u^{\gamma-1} (1-u)^{\beta-\gamma-1} \left(1 - zu^{\alpha}\right)^{-1} du. $$

This result is analogous to the result of Garg *et al*. [4].

Integral transforms

Here, certain interesting integral transforms, including the Euler (Beta) transform, the Laplace transform, and the Whittaker transform, are described.

Euler (Beta) transform of ${}_{\gamma,\delta}E_{k,\alpha,\beta}(z)$

Theorem 7. Let $k \in R,\ a, b, \alpha,\ \beta,\ \gamma,\ \delta, \sigma \in C,$

and (i) $\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0, \text{Re}(\delta) > 0,$

(ii) $\text{Re}(\alpha) > \text{Re}(\delta), \text{Re}(a) > 0, \text{Re}(b) > 0, \text{Re}(\sigma) > 0,$

then

$$\int_0^1 u^{a-1}(1-u)^{b-1}\ {}_{\gamma,\delta}E_{k,\alpha,\beta}\left(zu^{\sigma}\right)du$$

$$= \frac{k^{1-\frac{\beta}{k}}\,\Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)}\ {}_3\psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},\delta\right),(a,\sigma),(1,1) \\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),(a+b,\sigma)\end{matrix}\ ;\left(k^{\delta-\frac{\alpha}{k}}\right)z\right]. \tag{33}$$

Proof: Applying series (22) on the left-hand side of the equation (33), we find

$$\int_0^1 u^{a-1}(1-u)^{b-1}\ {}_{\gamma,\delta}E_{k,\alpha,\beta}\left(zu^{\sigma}\right)du$$

$$= \int_0^1 u^{a-1}(1-u)^{b-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}}{\Gamma_k(\alpha n+\beta)}\left(zu^{\sigma}\right)^n du$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\ z^n}{\Gamma_k(\alpha n+\beta)}\int_0^1 u^{\sigma n+a-1}(1-u)^{b-1}du$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\ z^n}{\Gamma_k(\alpha n+\beta)}\,B(a+\sigma n,b)$$

$$= \sum_{n=0}^{\infty}\frac{k^{\delta n}\left(\frac{\gamma}{k}\right)_{\delta n}\ z^n}{k^{\left(\frac{\alpha n+\beta}{k}-1\right)}\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)}\ \frac{\Gamma(a+\sigma n)\,\Gamma(b)}{\Gamma(a+\sigma n+b)}$$

$$= \frac{k^{1-\frac{\beta}{k}}\,\Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma}{k}+\delta n\right)}{\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)}\ \frac{\Gamma(n+1)\,\Gamma(a+\sigma n)}{\Gamma(a+b+\sigma n)\ (n!)}\left(k^{\delta-\frac{\alpha}{k}}z\right)^n$$

$$= \frac{k^{1-\frac{\beta}{k}}\Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)}\, {}_3\psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},\delta\right),(a,\sigma),(1,1) \\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),(a+b,\sigma)\end{matrix};\left(k^{\delta-\frac{\alpha}{k}}\right)z\right].$$

This completes the proof of the theorem 7.

Special cases

(i) We find a result similar to a result obtained by Garg *et al.* [4] from equation (33) for $k=1$

$$\int_0^1 u^{a-1}(1-u)^{b-1}\, {}_{\gamma,\delta}E_{\alpha,\beta}\left(zu^{\sigma}\right)du = \frac{\Gamma(b)}{\Gamma(\gamma)}\, {}_3\psi_2\left[\begin{matrix}(\gamma,\delta),(a,\sigma),(1,1) \\ \\ (\beta,\alpha),(a+b,\sigma)\end{matrix};z\right].$$

(ii) A result corresponding to a result obtained by Garg *et al*. [4], when $k=1$, $a=\beta,\ b=\mu,\sigma=\alpha$ from equation (33)

$$\int_0^1 u^{\beta-1}(1-u)^{\mu-1}\, {}_{\gamma,\delta}E_{\alpha,\beta}\left(zu^{\alpha}\right)du = \Gamma(\mu)\, {}_{\gamma,\delta}E_{\alpha,\beta+\mu}(z).$$

Laplace transform of ${}_{\gamma,\delta}E_{k,\alpha,\beta}(z)$

Theorem 8. Let $k\in R,\ a,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \sigma, s\in C,$

and (i) $\left|\frac{z}{s^{\sigma}}\right|<1$, Re (α) > Re (δ) ,

(ii) $\text{Re}(\alpha)>0,\ \text{Re}(\beta)>0,\ \text{Re}(\gamma)>0,\ \text{Re}(\delta)>0,$ $\text{Re}(\sigma)>0,\ \text{Re}(s)>0,$

then

$$L\left\{u^{a-1}\, {}_{\gamma,\delta}E_{k,\alpha,\beta}\left(zu^{\sigma}\right);s\right\} = \int_0^{\infty} u^{a-1}e^{-su}\, {}_{\gamma,\delta}E_{k,\alpha,\beta}\left(zu^{\sigma}\right)du$$

$$= \frac{s^{-a}\, k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\, {}_3\Psi_1\left[\begin{matrix}\left(\frac{\gamma}{k},\delta\right),(a,\sigma),(1,1) \\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right)\end{matrix}\;;\; \frac{z}{s^{\sigma}}\left(k^{\delta-\frac{\alpha}{k}}\right)\right]. \quad (34)$$

Proof: Making use of the series (22) on the left-hand side of the equation (34), we get,

$$L\left\{u^{a-1}\,{}_{\gamma,\delta}E_{k,\alpha,\beta}\left(zu^{\sigma}\right);s\right\}$$

$$= \int_0^{\infty} u^{a-1}e^{-su}\,{}_{\gamma,\delta}E_{k,\alpha,\beta}\left(zu^{\sigma}\right)du = \int_0^{\infty} u^{a-1}e^{-su}\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}}{\Gamma_k(\alpha n+\beta)}\left(zu^{\sigma}\right)^n du$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\, z^n}{\Gamma_k(\alpha n+\beta)}\int_0^{\infty} u^{\sigma n+a-1}e^{-su}\,du = \sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}\, z^n}{\Gamma_k(\alpha n+\beta)}\frac{\Gamma(a+\sigma n)}{s^{a+\sigma n}}$$

$$= s^{-a}\sum_{n=0}^{\infty}\frac{k^{\delta n}\left(\frac{\gamma}{k}\right)_{\delta n}}{k^{\left(\frac{\alpha n+\beta}{k}-1\right)}\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)}\frac{\Gamma(a+\sigma n)\Gamma(n+1)}{n!}\left(\frac{z}{s^{\sigma}}\right)^n$$

$$= \frac{s^{-a}\, k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma}{k}+\delta n\right)}{\Gamma_k\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)}\frac{\Gamma(a+\sigma n)\Gamma(n+1)}{n!}\left(\frac{z\, k^{\delta-\frac{\alpha}{k}}}{s^{\sigma}}\right)$$

$$= \frac{s^{-a}\, k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\, {}_3\Psi_1\left[\begin{matrix}\left(\frac{\gamma}{k},\delta\right),(a,\sigma),(1,1) \\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right)\end{matrix}\;;\; \frac{z}{s^{\sigma}}\left(k^{\delta-\frac{\alpha}{k}}\right)\right].$$

This completes the proof of the Theorem 8.

Special cases

(i) For $k=1$, in equation (34), we get

$$L\left\{u^{a-1}\,{}_{\gamma,\delta}E_{\alpha,\beta}\left(zu^{\sigma}\right);s\right\}=\int_0^{\infty}u^{a-1}e^{-su}\,{}_{\gamma,\delta}E_{\alpha,\beta}\left(zu^{\sigma}\right)du$$

$$=\frac{s^{-a}}{\Gamma(\gamma)}\,{}_3\psi_1\left[\begin{matrix}(\gamma,\delta),(a,\sigma),(1,1)\\ \\(\beta,\alpha)\end{matrix};\frac{z}{s^{\sigma}}\right].,$$

which is the same result as obtained by Garg *et al.* [4].

Whittaker transform of ${}_{\gamma,\delta}E_{k,\alpha,\beta}(z)$

Theorem 9. Let $k\in R,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \eta,\ \lambda,\ \psi\in C$,

and (i) $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\delta)>0,$

(ii) $\mathrm{Re}(\lambda+\psi)>-1/2$

(iii) $\mathrm{Re}\,(\alpha)>\mathrm{Re}\,(\delta),$

then

$$\int_0^{\infty}t^{\xi-1}e^{\frac{-\phi t}{2}}W_{\lambda,\psi}(\phi t)\,{}_{\gamma,\delta}E_{k,\alpha,\beta}\left(\omega t^{\eta}\right)dt$$

$$=\frac{\phi^{-\xi}k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_4\psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},\delta\right),\left(\frac{1}{2}+\psi+\xi,\eta\right),\left(\frac{1}{2}-\psi+\xi,\eta\right),(1,1)\\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\lambda+\xi,\eta)\end{matrix};\left(\frac{\omega k^{\delta-\frac{\alpha}{k}}}{\phi^{\eta}}\right)\right].\qquad(35)$$

Proof: With the substitution $\phi t=v$ and applying series form, equation (22) on the left-hand side of equation (35), we get

$$\int_0^{\infty}\left(\frac{v}{\phi}\right)^{\xi-1}e^{\frac{-v}{2}}W_{\lambda,\psi}(v)\sum_{n=0}^{\infty}\frac{(\gamma)_{\delta n,k}}{\Gamma_k(\alpha n+\beta)}\left(\omega\left(\frac{v}{\phi}\right)^{\eta}\right)^n\frac{1}{\phi}dv$$

$$=\frac{\phi^{-\xi}k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma}{k}+\delta n\right)\left(k^{\delta-\frac{\alpha}{k}}\right)^n}{\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)}\left(\frac{\omega}{\phi^{\eta}}\right)^n\int_0^{\infty}v^{\xi+n\eta-1}e^{\frac{-v}{2}}W_{\lambda,\psi}(v)dv$$

$$=\frac{\phi^{-\xi}k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma}{k}+\delta n\right)\Gamma\left(\frac{1}{2}+\psi+\xi+n\eta\right)\Gamma\left(\frac{1}{2}-\psi+\xi+n\eta\right)\Gamma(n+1)}{\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)\Gamma(1-\lambda+\xi+n\eta)\ n!}\left(\frac{\omega}{\phi^{\eta}}k^{\delta-\frac{\alpha}{k}}\right)^n$$

$$=\frac{\phi^{-\xi}k^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_4\Psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},\delta\right),\left(\frac{1}{2}+\psi+\xi,\eta\right),\left(\frac{1}{2}-\psi+\xi,\eta\right),(1,1)\\ \\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\lambda+\xi,\eta)\end{matrix};\left(\frac{\omega k^{\delta-\frac{\alpha}{k}}}{\phi^{\eta}}\right)\right].$$

This completes the proof of the theorem 9.

Special cases

(i) Putting $k=1$, equation (35) reduces to the following form

$$\int_0^{\infty} t^{\xi-1}e^{\frac{-\phi t}{2}}W_{\lambda,\psi}(\phi t)\,{}_{\gamma,\delta}E_{\alpha,\beta}\left(\omega t^{\eta}\right)dt$$

$$=\frac{\phi^{-\xi}}{\Gamma(\gamma)}\,{}_4\Psi_2\left[\begin{matrix}(\gamma,\delta),\left(\frac{1}{2}+\psi+\xi,\eta\right),\left(\frac{1}{2}-\psi+\xi,\eta\right),(1,1)\\ \\ (\beta,\alpha),(1-\lambda+\xi,\eta)\end{matrix};\left(\frac{\omega}{\phi^{\eta}}\right)\right].$$

4. Conclusion

In this paper, we have established and defined a k-generalized Mittag-Leffler type function ${}_{\gamma,\delta}E_{k,\alpha,\beta}\left(z\right)$, which is a novel generalization of the Mittag-Leffler function. We have assessed various interesting and enlightening properties, such as the useful relationship between the Mittag-Leffler functions, recurrence relation, differential formulae, integral representations, and images of this function under the Euler (Beta), Laplace, and Whittaker transforms in the form of theorems. Certain new and known special cases have also been discussed.

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