On the Diophantine Equation \((p+6)^x - p^y = z^2\) where \(p\) is a Prime Number with \(p \equiv 1 \pmod{28}\)

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Abstract. This paper shows that the Diophantine equation \((p+6)^x - p^y = z^2\) where \(p\) is a prime number with \(p \equiv 1 \pmod{28}\), has a unique non-negative integer solution. The solution is \((x, y, z) = (0, 0, 0)\).

Keywords: Diophantine equation; integer solution; Mihailescu’s theorem

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In 2019, Thongnak, Chuayjan and Kaewong [8] proved that the Diophantine equation \(2^x - 3^y = z^2\) has three non-negative integer solutions \((x, y, z) \in \{(0,0,0),(1,0,1),(2,1,1)\}\).

In the same year, Burshtein [1] studied the Diophantine equation \((p+1)^x - p^y = z^2\) in which \(p\) is a prime number and \(x, y, z\) are positive integers with \(x+y = 2, 3, 4\). Burshtein [2] showed that the Diophantine equation \(6^x - 11^y = z^2\) has exactly one positive integer solution when \(x = 2\), and no positive integer solution when \(2 < x \leq 16\). Burshtein [3] found all positive integer solutions of the Diophantine equation \(p^x - p^y = z^2\), when \(p\) is a prime number.

In 2020, Burshtein [4] showed that the Diophantine equation \(13^x - 5^y = z^2\) has exactly one positive integer solution \((x, y, z) = (2, 2, 12)\) and the Diophantine equation \(19^x - 5^y = z^2\) has no positive integer solution. Elshahed and Kamarulhaidi [5] studied all non-negative integer solutions of the Diophantine equation \((4^n)^x - p^y = z^2\), where \(p\) is odd prime and \(n\) is a positive integer. In 2021, Thongnak, Chuayjan and Kaewong [9] showed that \((x, y, z) = (0, 0, 0)\) is the unique non-negative integer solution of the Diophantine equation \(7^x - 5^y = z^2\). In 2022, Tadee and Laomalaw [7] found all non-negative integer solutions of the Diophantine equation \(2^x - p^y = z^2\), for some prime \(p\).

In this paper, we solve the Diophantine equation of the form \((p+6)^x - p^y = z^2\), where \(p\) is a prime number with \(p \equiv 1 \pmod{28}\) and \(x, y, z\) are non-negative integers.
2. Main results
We begin this section by presenting an important theorem.

**Theorem 2.1.** (Mihailescu’s theorem) [6] The Diophantine equation $a^x - b^y = 1$ has the unique integer solution $(a, b, x, y) = (3, 2, 2, 3)$, where $a, b, x$ and $y$ are integers with $\min\{a, b, x, y\} > 1$.

**Lemma 2.1.** Let $p$ be an odd prime number. Then the Diophantine equation $1 - p^y = z^2$ has the unique non-negative integer solution $(y, z) = (0, 0)$.

**Proof:** Let $y$ and $z$ be non-negative integers and $(y, z)$ be a solution of the Diophantine equation $1 - p^y = z^2$. Then $(1 - z)(1 + z) = p^y$. Since $p$ is prime, we have $1 - z = p^v$ and $1 + z = p^{-v}$, for some non-negative integer $v$. Therefore $y \geq 2v$ and $2 = p^v(p^{-2v} + 1)$. Since $p \neq 2$, we have $v = 0$ and so $2 = p^y + 1$. Then $y = 0$. It implies that $z = 0$. Hence, $(y, z) = (0, 0)$ is the unique non-negative integer solution.

**Lemma 2.2.** Let $p$ be a prime number with $p \equiv 1 \pmod{4}$. Then the Diophantine equation $(p + 6)^x - 1 = z^2$ has the unique non-negative integer solution $(x, z) = (0, 0)$.

**Proof:** Let $x$ and $z$ be non-negative integers and $(x, z)$ be a solution of the Diophantine equation $(p + 6)^x - 1 = z^2$. We consider three following cases.

**Case 1.** $x = 0$. Then $z^2 = 0$. Hence, $(x, z) = (0, 0)$ is a solution.

**Case 2.** $x = 1$. Then $z^2 = p + 5$. Since $p \equiv 1 \pmod{4}$, we have $z^2 \equiv 2 \pmod{4}$, which contradicts the fact that $z^2 \equiv 0, 1 \pmod{4}$.

**Case 3.** $x > 1$. It is easy to check that $z > 1$. Therefore $\min\{p + 6, z, x, 2\} > 1$. Since $(p + 6)^x - z^2 = 1$ and Theorem 2.1, we have $p + 6 = 3$, a contradiction.

**Theorem 2.2.** Let $p$ be a prime number with $p \equiv 1 \pmod{28}$. Then $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of the Diophantine equation $(p + 6)^x - p^y = z^2$.

**Proof:** Let $x, y, z$ be non-negative integers and $(x, y, z)$ be a solution of the Diophantine equation $(p + 6)^x - p^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0, 0, 0)$, by Lemma 2.1 and 2.2, respectively. Now, we consider case $x > 0$ and $y > 0$. Since $p \equiv 1 \pmod{28}$, it implies that $p \equiv 1 \pmod{4}$ and $p \equiv 1 \pmod{7}$. Therefore $(p + 6)^x - p^y \equiv (-1)^x - 1 \pmod{4}$. Since $p \equiv 1 \pmod{4}$, we obtain that $p$ and $p + 6$ are odd. Thus, $z^2$ is even and so $z^2 \equiv 0 \pmod{4}$. Since $(p + 6)^x - p^y = z^2$, it follows that $(-1)^x - 1 \equiv 0 \pmod{4}$. We see that $x = 2k$, for some positive integer $k$. Therefore $((p + 6)^k - z)((p + 6)^k + z) = p^y$. Since $p$ is prime, there exists a non-negative integer $u$ such that $(p + 6)^k - z = p^u$ and $(p + 6)^k + z = p^{y-u}$. Then $y \geq 2u$ and $2(p + 6)^k = p^u(p^{y-2u} + 1)$.
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Assume that \(u > 0\). Then \(p \mid 2(p+6)^{x}\) and so \(p = 2\) or \(p = 3\). This is impossible since \(p \equiv 1 (mod\ 28)\). Thus \(u = 0\). Consequently, \(2(p+6)^{x} = p^{y} + 1\). Since \(p \equiv 1 (mod\ 7)\), we get \(2(p+6)^{x} \equiv 0 (mod\ 7)\) and \(p^{y} + 1 \equiv 2 (mod\ 7)\). Thus \(0 \equiv 2 (mod\ 7)\), a contradiction.

**Corollary 2.1.** The Diophantine equation \(35^{x} - 29^{y} = z^{2}\) has the unique non-negative integer solution \((x, y, z) = (0, 0, 0)\).

**Proof:** This corollary follows directly from Theorem 2.2.

**Corollary 2.2.** Let \(n\) be a positive integer and \(p\) be a prime number with \(p \equiv 1 (mod\ 28)\). Then the Diophantine equation \((p+6)^{x} - p^{y} = z^{2n}\) has the unique non-negative integer solution \((x, y, z) = (0, 0, 0)\).

**Proof:** Let \(a, b, c\) be non-negative integers such that \((p+6)^{a} - p^{b} = c^{2n}\). Then \((a, b, c^{n})\) is a non-negative integer solution \((x, y, z)\) of the Diophantine equation \((p+6)^{x} - p^{y} = z^{2n}\). By Theorem 2.2, we obtain that \((a, b, c^{n}) = (0, 0, 0)\). Then \(a = b = c = 0\). Hence, \((0, 0, 0)\) is the unique non-negative integer solution of the equation \((p+6)^{x} - p^{y} = z^{2n}\).

### 3. Conclusion

In this article, the Diophantine equation \((p+6)^{x} - p^{y} = z^{2}\), when \(p\) is a prime number and \(x, y, z\) are non-negative integers, is investigated. We found that \((x, y, z) = (0, 0, 0)\) is the unique non-negative integer solution of the equation in the following cases: 1) \(x = 0\) and \(p \neq 2\), 2) \(y = 0\) and \(p \equiv 1 (mod\ 4)\), and 3) \(p \equiv 1 (mod\ 28)\). For example, if \(p = 29\), then the Diophantine equation \(35^{x} - 29^{y} = z^{2}\) has the unique non-negative integer solution \((x, y, z) = (0, 0, 0)\).

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### Conflict of interest

The paper is written by single author so there is no conflict of interest.

### Authors’ Contributions

It is a single author paper. So, full credit goes to the author.

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