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The Shape Analysis of DTB-like and DT B-spline-like Curves

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Abstract. In the field of computer-aided design and related applications, the free curve with parameters shows its strong function. But an excellent curve with parameters need to eliminate unnecessary cusps and inflection points, so shape analysis is needed for some specific curves. We have already shown many excellent properties of the DTB-like curves and DT B-spline-like curves. Thus the shape features like convexity, loops, inflection points, and cusps of the DTB-like curves, and DT B-spline-like curves are further discussed in this paper. And the necessary and sufficient conditions for the existence of shape features of the corresponding DTB-like curves and DT B-spline-like curves are given. And the shape features distribution all be generalized into tables and diagrams, which are useful for industrial design. In addition, the effect of shape parameters on the shape features diagram and its adjustment ability to the shape of corresponding curves are analyzed, respectively. The work in this paper enables users to determine how to set parameter values so that the generated curve could be a global convex or local convex curve, has a required inflection point, or eliminate the unnecessary one, and could be adjusted to another aimed shape.

Keywords: DTB-like Curve, Inflection Points, Convexity, Shape Features, Feature Distribution

AMS Mathematics Subject Classification (2010): 41A15

1. Introduction

Curve construction with shape parameters is always an important part of CAGD. Often a kind of practical curve method plays an important role in promoting the growth of the geometry industry. So researchers have done a lot of work. The simplest method is to add shape parameters to the classical Bernstein and B-spline basis [1-5]. But for sake of effectively amending the curve, a lot of work revolves around the tension parameters [6-11]. Based on the C^4 splines, the $C^2 \cap FC^3$ spline curve with tension parameters is constructed [9]. On this basis, a group of $C^2 \cap G^3$ spline curves with three shape parameters is constructed [12]. Han constructed spline curves, which possess

$C^2 \cap FC^{2k+3}$ continuity [13]. Given the shape-preserving property of the curve, Costantini [14] proposed a variable degree polynomial basis in $\{1, t, (1-t)^p, t^q\}$, which is a Quasi Extended Chebyshev (QEC) space [15,16]. From that, $\{1, t, t^2, \cdots, t^{n-2}, (1-t)^p, t^q\}$ is also verified to be a QEC space [17]. Lately, with the theory of Canonical Complete Chebyshev space, it is proved that the variable degree splines basis possesses a total positivity property [18]. Zhu constructed a kind of quasi-Bernstein basis, which is proved a B-basis, and the related B-spline curve possesses $C^2 \cap FC^{k+3}$ continuous [19]. The changeable spline basis is given [20]. Wang constructed a group of DTB-like basis, and the associated B-spline basis possesses C^{2n-1} continuity when the shape parameters are global [21]. In addition, the corresponding DTB-like curves and DT B-spline-like curves can accurately describe the elliptic arc and arc while retaining all the good properties of the traditional Bézier and B-spline curve.

For geometric design, the presentation of curves with good properties is only the first step. In many applications, good shape design should remove unnecessary cusps and inflection points; convexity is also an indispensable element in shape design. The determination conditions of the shape features are very important for the shape control and adjustment of the curve with parameters [22-27]. For example, these geometric properties directly affect the dynamic performance of the shape design, the complexity of the algorithm, and the operability of the processing. Therefore, it is particularly important to analyze the shape features of the proposed curve. However, there are few kinds of literature about the proposed method for shape analysis. For the planar cubic Bézier curve, an exhaustive study is proposed [28, 29] and the related rational case is given [30]. And the related general parametric curve form is given [31]. In addition, the shape features of cusps, inflection points, and loops are given and the associated shape diagram is shown as well [32]. However, the global and local convexity of the proposed curve is not explained [32]. Han analyzed the shape features of TB curves, and he also gave the shape diagram [33]. Zhu gave the shape diagram of $\alpha\beta$- like curve [19].

By using envelop theory and topological mapping, we discuss the shape features of the DTB-like and DT B-spline-like curves which are given in [21]. We also give the distribution conditions for cusps, inflection points, loops, and convexity, which are summarized into tables. And we also give the shape diagram of the DTB-like curves and DT B-spline-like curves, which are similar to the one in [19, 29, 33]. Further, we discuss the influence of parameters on the shape diagram and the ability of curve adjustment.

The other work arrangements of this study are as follows: Section 2 shows the related definition of background knowledge. Sections 3 and 4 discuss the shape analysis of the DTB-like curve and DT B-spline-like curve, respectively. Section 5 puts forward the conclusion.

2. Preliminaries

Definition 1. Given control points $P_i (i = 0,1,2,3)$ in R^2 / R^3, we call the following formula

$$Q(t) = \sum_{i=0}^{3} T_i(t) P_i, t \in [0, \pi/2], \alpha, \beta \in [2, +\infty) \tag{1}$$

as the cubic DTB-like curve [21], where

$$\begin{cases} T_0(t) = \dfrac{(1-\sin t)^2}{1+(\alpha-2)\sin t}, \\ T_1(t) = 1-\sin^2 t - \dfrac{(1-\sin t)^2}{1+(\alpha-2)\sin t}, \\ T_2(t) = 1-\cos^2 t - \dfrac{(1-\cos t)^2}{1+(\beta-2)\cos t}, \\ T_3(t) = \dfrac{(1-\cos t)^2}{1+(\beta-2)\cos t}. \end{cases} \quad (2)$$

Definition 2. Given $P_i(i=0,1,\ldots,n)$ in R^2/R^3 and a non-uniform knot vector U, for any $\alpha_i, \beta_i \in [2,+\infty), n \geq 3, u \in [u_3, u_{n+1}]$, a non-uniform DT B-spline-like curve[21] can be shown as follows:

$$Q(u) = \sum_{i=0}^{n} B_i(u) P_i \quad (3)$$

where the basis function $B_i(u)$ is defined as Definition 3:

Definition 3. Given a sequence of knots $u_0 < u_1 < \cdots < u_{n+4}$ and a knot vector $U = (u_0, u_1, \cdots, u_{n+4})$. Let $h_j = u_{j+1} - u_j$, and $t_j(u) = \pi(u-u_j)/2h_j, j = 0,1,\cdots,n+3$, for any $\alpha_i, \beta_i \in [2,+\infty)$, $i = 0,1,\cdots,n$, the $B_i(u)$ could be concluded as follows:

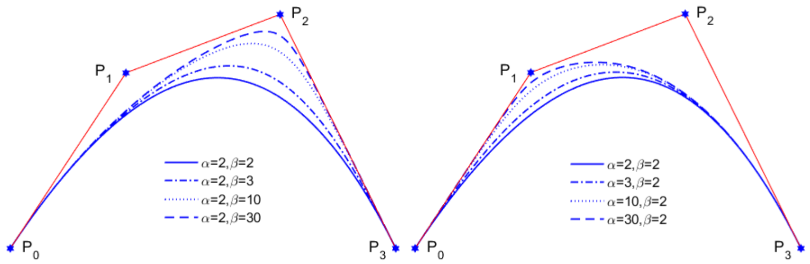
$$B_i(u) = \begin{cases} B_{i,0}(t_i) = d_i T_3(t_i;\beta_i), & u \in [u_i, u_{i+1}), \\ B_{i,1}(t_{i+1}) = \sum_{j=0}^{3} c_{i+1,j} T_j(t_{i+1}, \alpha_{i+1}, \beta_{i+1}), & u \in [u_{i+1}, u_{i+2}), \\ B_{i,2}(t_{i+2}) = \sum_{j=0}^{3} b_{i+2,j} T_j(t_{i+2}, \alpha_{i+2}, \beta_{i+2}), & u \in [u_{i+2}, u_{i+3}), \\ B_{i,3}(t_{i+3}) = a_{i+3} T_0(t_{i+3};\alpha_{i+3}), & u \in [u_{i+3}, u_{i+4}), \\ 0, & u \notin [u_i, u_{i+4}), \end{cases} \quad (4)$$

where the related coefficient values $a_i, b_{i,j}, c_{i,j}, d_i$ are defined as follows:

$$\begin{aligned} &\lambda_i = (\alpha_{i+1}-1)^2 \beta_i h_i + \alpha_{i+1}(\beta_i - 1)^2 h_{i+1}, \quad \mu_i = \alpha_{i+1} h_i + \beta_i h_{i+1}, \\ &\varphi_i = \frac{\lambda_i h_i + \mu_i h_{i+1}}{\beta_i h_{i+1}^2}, \qquad \phi_i = \frac{\lambda_{i-1} h_i + \mu_{i-1} h_{i-1}}{\alpha_i h_{i-1}^2} \\ &a_i = \frac{\beta_{i-1}\lambda_{i-2} h_i^2}{\lambda_{i-1}\mu_{i-2} h_{i-2} + \lambda_{i-2}\lambda_{i-1} h_{i-1} + \lambda_{i-2}\mu_{i-1} h_i}, \\ &d_i = \frac{\alpha_{i+1}\lambda_{i+1} h_i^2}{\lambda_{i+1}\mu_i h_i + \lambda_i \lambda_{i+1} h_{i+1} + \lambda_i \mu_{i+1} h_{i+2}}, \\ &b_{i,0} = \frac{\alpha_i \varphi_i h_{i-1}}{\mu_{i-1}} a_{i+1} + \frac{\beta_{i-1}\phi_{i-1} h_i}{\mu_{i-1}} d_{i-2}, \quad c_{i,0} = d_{i-1}, \\ &b_{i,1} = \varphi_i a_{i+1}, \qquad c_{i,1} = \frac{\mu_{i-1}}{\alpha_i h_{i-1}} d_{i-1}, \\ &b_{i,2} = \frac{\mu_i}{\beta_i h_{i+1}} a_{i+1}, \qquad c_{i,2} = \phi_i d_{i-1}, \\ &b_{i,3} = a_{i+1}, \qquad c_{i,3} = \frac{\alpha_{i+1}\varphi_{i+1} h_i}{\mu_i} a_{i+2} + \frac{\beta_i \phi_i h_{i+1}}{\mu_i} d_{i-1}. \end{aligned} \quad (5)$$

For $u \in [u_i, u_{i+1}], i = 3, 4, \ldots, n$, we rewrite (2) as follows:

$$\begin{aligned} Q_i(u) &= \sum_{j=i-3}^{i} B_j(u) P_j \\ &= (a_i P_{i-3} + b_{i0} P_{i-2} + c_{i0} P_{i-1}) A_0(t_i; \alpha_i) + (b_{i1} P_{i-2} + c_{i1} P_{i-1}) A_1(t_i; \alpha_i) \\ &+ (b_{i2} P_{i-2} + c_{i2} P_{i-1}) A_2(t_i; \beta_i) + (b_{i3} P_{i-2} + c_{i3} P_{i-1} + d_i P_i) A_3(t_i; \beta_i). \end{aligned} \tag{5}$$



(a) (b)

Figure 1: DTB-like curve with different parameters.

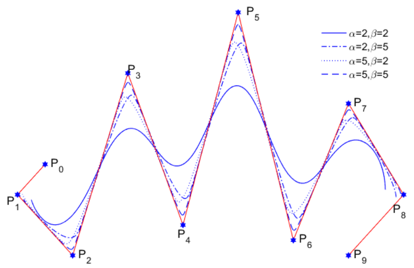


Figure 2: Uniform DT B-spline-like curve with different parameters.

3. The shape analysis of the DTB-like curve

In this section, we will discuss the shape features of the curve, given in [21], by using envelope theory and topological mapping. A detailed knowledge of shape features can be found in [28, 31, 34-36].

3.1. The spatial QCT-Bézier curve

Theorem 1. When $\alpha, \beta \in [2, +\infty), t \in [0, \pi/2]$, if $P_i (i = 0, 1, 2, 3)$ are not coplanar, the DTB-like curve is spatial, it does not have cusps, loops, and inflection points, and it has the same rotation direction as $P_i (i = 0, 1, 2, 3)$.

Proof: Let $a_i = P_i - P_{i-1}, i = 1, 2, 3$, we rewrite (1) as

$$Q(t) = P_0 + [1 - T_0(t;\alpha)] a_1 + [T_2(t;\beta) + T_3(t;\beta)] a_2 + T_3(t;\beta) a_3. \tag{7}$$

Thus, we have

$$Q'(t) = -T_0'(t;\alpha) a_1 + [T_2'(t;\beta) + T_3'(t;\beta)] a_2 + T_3'(t;\beta) a_3. \tag{8}$$

When $0 < t < \pi/2$, according (2), we have $T_2'(t;\beta) + T_3'(t;\beta) = 2\sin t\cos t \neq 0$. Since P_i are not coplanar, we can know that the vector a_i are linear independent. Thus, $Q(t) \neq 0$, and it has no cusps.

Next, we assume that $Q(t)$ possess loops, when $0 \le t_1 < t_2 \le \pi/2$, we have $Q(t_1) - Q(t_2) = 0$. Thus,

$$\begin{aligned} &[T_0(t_2;\alpha) - T_0(t_1;\alpha)]a_1 + [T_2(t_1;\beta) \\ &+ T_3(t_1;\beta) - T_2(t_2;\beta) - T_3(t_2;\beta)]a_2 \\ &+ [T_3(t_1;\beta) - T_3(t_2;\beta)]a_3 = 0. \end{aligned} \tag{9}$$

According to the previous discussion, we have known that a_i are linearly independent, therefore, it follows that

$$T_0(t_1;\alpha) = T_0(t_2;\alpha); T_2(t_1;\beta) = T_2(t_2;\beta); T_3(t_1;\beta) = T_3(t_2;\beta).$$

Then, we let $T_0'(t;\alpha) = 0$, we have

$$-\frac{\cos t(1-\sin t)[\alpha + (\alpha-2)\sin t]}{[1+(\alpha-2)\sin t]^2}, \tag{10}$$

we can obtain $t = \pi/2 (\times)$ and $t = \arcsin(\alpha/(\alpha-2))$. According to

$$0 < \arcsin(\alpha/(\alpha-2)) < \pi/2, \text{ we have } \alpha < 2.$$

Thus, when $\alpha \in [2, +\infty)$, $T_0(t;\alpha)$ is monotonically decreasing on $[0, \pi/2]$, and $T_0(t;\alpha)$ has no loops. Thus, when $\alpha \in [2, +\infty)$, $Q(t)$ has no loops.

Then, let $G(t) = \det(Q'(t), Q''(t), Q'''(t))$, we have

$$\begin{aligned} G(t) &= \det\left[\sum_{i=0}^{3} P_i T_i'(t) \quad \sum_{i=0}^{3} P_i T_i''(t) \quad \sum_{i=0}^{3} P_i T_i'''(t)\right] \\ &= \begin{vmatrix} \sum_{i=0}^{3} T_i(t) & \sum_{i=0}^{3} T_i'(t) & \sum_{i=0}^{3} T_i''(t) & \sum_{i=0}^{3} T_i'''(t) \\ \sum_{i=0}^{3} P_i T_i(t) & \sum_{i=0}^{3} P_i T_i'(t) & \sum_{i=0}^{3} P_i T_i''(t) & \sum_{i=0}^{3} P_i T_i'''(t) \end{vmatrix} \\ &= \left| \begin{bmatrix} 1 & 1 & 1 & 1 \\ P_0 & P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} T_0(t) & T_0'(t) & T_0''(t) & T_0'''(t) \\ T_1(t) & T_1'(t) & T_1''(t) & T_1'''(t) \\ T_2(t) & T_2'(t) & T_2''(t) & T_2'''(t) \\ T_3(t) & T_3'(t) & T_3''(t) & T_3'''(t) \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} 1 & 0 & 0 & 0 \\ P_0 & a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} T_0(t) & T_0'(t) & T_0''(t) & T_0'''(t) \\ T_1(t) & T_1'(t) & T_1''(t) & T_1'''(t) \\ T_2(t) & T_2'(t) & T_2''(t) & T_2'''(t) \\ T_3(t) & T_3'(t) & T_3''(t) & T_3'''(t) \end{bmatrix} \right| \\ &= (a_1, a_2, a_3) D(t). \end{aligned} \tag{11}$$

where (a_1, a_2, a_3) is a mixed product of vector edge a_1, a_2, a_3, by directly computing, we have $D(t) > 0$. For any $0 < t < \pi/2$, from $(a_1, a_2, a_3) \neq 0$, we have $G(t) \neq 0$, and it has the same positive and negative property as (a_1, a_2, a_3). Thus, $Q(t)$ has no inflection points, and it has the same rotation direction as the control points.

3.2. The planar DTB-like curve

If P_i are coplanar, $Q(t)$ is planar curve. At this time, we have $(a_1, a_2, a_3) = 0$. Firstly, we consider $a_1 \nparallel a_3$, we have $a_2 = ua_1 + va_3$. Substituting it into (7), it follows that

$$Q(t) = P_0 + [1 - T_0(t) + u(T_2(t) + T_3(t))]a_1 + [T_3(t) + v(T_2(t) + T_3(t))]a_3. \tag{12}$$

3.2.1. Cusps

The necessary condition of the planar DTB-like curve $Q(t)$ has cusps is
$Q'(t) = 0 (0 < t < \pi / 2)$. Thus, we can get

$$[-T_0'(t) + u(T_2'(t) + T_3'(t))]a_1 + [T_3'(t) + v(T_2'(t) + T_3'(t))]a_3 = 0. \tag{13}$$

Because a_1 and a_3 are linearly independent, from (13) and (2), we have

$$C: \begin{cases} u = -\dfrac{(1 - \sin t)[\alpha + (\alpha - 2)\sin t]}{2\sin t[1 + (\alpha - 2)\sin t]^2} \\ v = -\dfrac{(1 - \cos t)[\beta + (\beta - 2)\cos t]}{2\cos t[1 + (\beta - 2)]^2} \end{cases}, 0 < t < \pi / 2. \tag{14}$$

We analyze the shape of C, from (14), we have

$$\lim_{t \to 0} u = -\infty, \lim_{t \to 0} v = 0, \lim_{t \to \pi/2} u = 0, \lim_{t \to \pi/2} v = -\infty. \tag{15}$$

This shows that the two curves $u = 0$ and $v = 0$ all are asymptotes for curve C. For any $\alpha, \beta \in [2, +\infty)$, we have $du / dv < 0,\ d^2u / dv^2 < 0,$. This indicates that the C is a strictly convex and monotonically decreasing curve. For any $(u_0, v_0) \in C$, we have $Q'(t) = 0$ and $Q''(t) \neq 0$. In fact, similar to the discussion of (13) and (14), we have

$$[-T_0''(t) + u(T_2''(t) + T_3''(t))]a_1 + [T_3''(t) + v(T_2''(t) + T_3''(t))]a_3 = 0, \tag{16}$$

$$\begin{cases} u = \dfrac{\begin{Bmatrix} 2(\alpha - 1)^2 \cos^2 t + \\ (1 - \sin t)\sin t[\alpha + (\alpha^2 - \alpha - 2)\sin t + (\alpha - 2)^2 \sin^2 t] \end{Bmatrix}}{2\cos 2t[1 + (\alpha - 2)\sin t]^3}, \\ v = \dfrac{\begin{Bmatrix} 2(\beta - 1)^2 \sin^2 t + \\ (1 - \cos t)\cos t[\beta + (\beta^2 - \beta - 2)\cos t + (\beta - 2)^2 \cos^2 t] \end{Bmatrix}}{2\cos 2t[1 + (\beta - 2)\cos t]^3}, \end{cases} 0 < t < \pi / 2, \tag{17}$$

For any $\alpha, \beta \in [2, +\infty)$, we can easily verify that (14) and (17) cannot be established at the same time. It indicates $Q'(t) = 0$ and $Q''(t) \neq 0$. Therefore, we have

$$Q'(t) = Q''(t)(t - t_0) + o(t - t_0). \tag{18}$$

When the parameter t passes t_0, the direction of $Q'(t)$ will changes. We can easily conclude that $Q'(t_0)$ is a cusp. Therefore, the planar DTB-like curve defined by (12) possesses cusps that are equivalent to $(u, v) \in C$.

3.2.2. Inflection points

We can easily get $Q'(t) \times Q''(t) = f(t, u, v)a_1 \times a_3$, where

$$f(t; u, v) = -\begin{vmatrix} T_0'(t) & T_3'(t) \\ T_0''(t) & T_3''(t) \end{vmatrix} + u\begin{vmatrix} T_2'(t) & T_3'(t) \\ T_2''(t) & T_3''(t) \end{vmatrix} + v\begin{vmatrix} T_0'(t) & T_1'(t) \\ T_0''(t) & T_1''(t) \end{vmatrix}. \tag{19}$$

$Q(t_0)(0<t_0<\pi/2)$ is an inflection point when $f(t;u,v)$ changes sign at t_0. In the $uv-$plane, the possible region of inflection points must be covered by a family of straight lines $f(t;u,v)=0$. According to [11], the curve C is just the envelope of the family of straight lines $f(t;u,v)=0$. Therefore, it can be known that C is a strictly convex continuous curve. Thus, the region swept by the tangent line of the curve C is $S\cup D\cup C$, that is, the region where the inflection point may occur. In Figure 3., 'S' represents the region where the DTB-like curve only has one inflection point; 'N_0,N_1,N_2' represents the region where the DTB-like curve without inflection points and loops; and 'D' represents the region where the DTB-like curve with two inflection points; and 'L' represents the region where the DTB-like curve has loops.

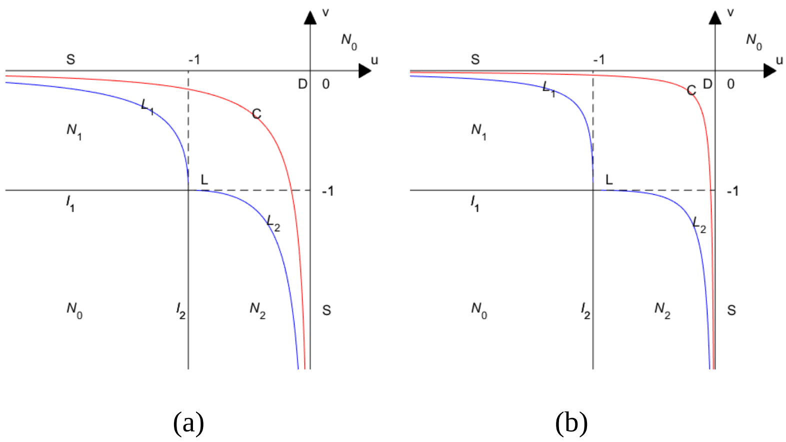


Figure 3: Shape diagram of DTB-like curve((a): $\alpha=\beta=2$; (b): $\alpha=\beta=5$).

There is one tangent line $f(t_0;u,v)=0$ of curve C that passes through at any point $(u_0,v_0)\in S\cup D\cup C$. When $(u_0,v_0)\in C$, from the expansion below, we have

$$f(t;u_0,v_0)=\frac{1}{2}f_{tt}''(t_0;u_0,v_0)(t-t_0)^2+o((t-t_0)^2), \tag{20}$$

where $f''(t;u_0,v_0)\neq 0$. Thus, we can easily conclude that $f(t,u_0,v_0)$ does not change sign at t_0, and the planar DTB-like curve has no inflection points. When $(u_0,v_0)\in S\cup D$, we assume that $f(t,u_0,v_0)$ is one of the tangent lines of C, which passes (u_0,v_0), according to Taylor expansion, we have

$$f(t;u_0,v_0)=\frac{1}{2}f_t'(t_0;u_0,v_0)(t-t_0)+o(t-t_0). \tag{21}$$

It easily gets $f_t'(t_0,u_0,v_0)\neq 0$ (if $f_t'(t_0,u_0,v_0)=0$, from the definition of the envelope, we can know that $(u_0,v_0)\in C$). Thus, $f(t;u_0,v_0)$ will change sign at t_0, the planar DTB-like curve $Q(t)$ has an inflection point at t_0. In addition, when $(u_0,v_0)\in S$, the curve C only has one tangent line which passes (u_0,v_0), the related curve $Q(t)$ only has one inflection point; When $(u_0,v_0)\in D$, C has two tangent lines which passes (u_0,v_0), thus, $Q(t)$ has two inflection points.

3.2.3. Loops

The planar DTB-like curve $Q(t)$ has loops if and only if there possesses $0\le t_1<t_2\le \pi/2$ such that $Q(t_1)=Q(t_2)$. This is equivalent to the parameter u,v,t_1,t_2 satisfying the following equations:

$$\begin{cases} u=\dfrac{T_0(t_2)-T_0(t_1)}{T_2(t_2)+T_3(t_2)-T_2(t_1)-T_3(t_1)}, \\ v=\dfrac{T_3(t_1)-T_3(t_2)}{T_2(t_2)+T_3(t_2)-T_2(t_1)-T_3(t_1)}, \end{cases} (t_1,t_2)\in\Omega. \tag{22}$$

where $\Omega=\{(t_1,t_2)\in R^2\,|\,0\le t_1<t_2<\pi/2\}$. Easy to verify the expression (22) defines a topological mapping $F:\Omega\subset R^2\to F(\Omega)\subset R^2$ Thus, $L=F(\Omega)$ is a simply connected region. The three boundary lines of L correspond to the three boundary lines $t_1=t_2$, $t_1=0$ and $t_2=1$ of Ω, i.e. the C (without L), L_1 and L_2 (both belonging to L) (see Figure 3). The planar DTB-like curve corresponding to the point in L has only one loop, where the expression of L_1 and L_2 are as follows:

$$L_1:\begin{cases} u=\dfrac{(1-\sin t)^2-[1+(\alpha-2)\sin t]}{[1+(\alpha-2)\sin t]\sin^2 t}, \\ v=-\dfrac{(1-\cos t)^2}{[1+(\beta-2)\cos t]\sin^2 t}, \end{cases} 0\le t<\pi/2, \tag{23}$$

$$L_2:\begin{cases} u=-\dfrac{(1-\sin t)^2}{[1+(\alpha-2)\sin t]\cos^2 t}, \\ v=\dfrac{(1-\cos t)^2-[1+(\beta-2)\cos t]}{[1+(\alpha-2)\sin t]\cos^2 t}, \end{cases} 0\le t<\pi/2, \tag{24}$$

From the mathematical analysis, it can be deduced that the curve L_1 and L_2 are both strictly convex curves and monotonically decreasing. L_1 and L_2 both intersect at point $(-1,-1)$, the asymptote of L_1 is the u axis, the asymptote of L_2 is the v axis as the asymptotic line. The curve C does not intersect either L_1 or L_2.

3.2.4. Convexity

Let $N=R^2\setminus(C\cup S\cup D\cup L)$, where the upper left part enclosed by the curve L_1,L_2 (excluding the boundary of L_1,L_2) is N_1, the lower right part is N_2, and $N_0=N\setminus(N_1\cup N_2)$, see Figure 3. From the previous discussion, when $(u,v)\in N$, the planar DTB-like curve $Q(t)$ has no cusps, inflection points and loops. Next, we consider the following vector:

$$\begin{cases} M(t)=Q'(0)\times[Q(t)-Q(0)], \\ N(t)=[Q(t)-Q(0)]\times Q'(t). \end{cases} \tag{25}$$

According to (12) and (25), by direct computing, we have

$$\begin{cases} M(t) = \xi(t;u,v)(a_1 \times a_3), \\ N(t) = \zeta(t;u,v)(a_1 \times a_3). \end{cases} \tag{26}$$

where,

$$\xi(t;u,v) = \alpha\{T_3(t) + v[T_2(t) + T_3(t)]\}, \tag{27}$$

$$\begin{aligned} \zeta(t;u,v) = {} & [1-T_0(t)]T_3'(t) + T_3(t)T_0'(t) + u[T_2(t)T_3'(t) - T_2'(t)T_3(t)] \\ & + v\{[1-T_0(t)][T_2'(t) + T_3'(t)] + T_0'(t)[T_2(t) + T_3(t)]\}. \end{aligned} \tag{28}$$

When $v_0 = -T_3(t_0)/[T_2(t_0) + T_3(t_0)]$, $\xi(t;u,v)$ changes the direction at t_0. We can easily get $-1 < v_0 < 0$. Thus, when $(u,v) \in N_1$ (see Figure 3), the curve $Q(t)$ is locally convex [11]. Happens, N_1 happens to be part of N of the region covered by the tangent line to L_2. According to the literature [11], the envelope of the family of lines $\zeta(t;u,v) = 0$ is exactly curve L_1, and the region swept by the tangent line of L_1 in N is N_2 (see Figure 3). Therefore, when $(u,v) \in N_2$, $Q(t)$ is locally convex. When $(u,v) \in N_0 = N \setminus (N_1 \cup N_2)$, $Q(t)$ is globally convex.

Finally, when $a_1 \parallel a_3$, the curve $Q(t)$ has no cusps and loops; when a_1 has the same direction of a_3 (Does not include the 4-point collinear singular case), the curve $Q(t)$ only has one inflection point.

Theorem 2. When $a_1 \parallel a_3$, the planar DTB-like curve $Q(t)$ has no cusps and loops; If and only if a_1 has the same direction of a_3, $Q(t)$ only has one inflection point; When $a_1 \nparallel a_3$, let $a_2 = ua_1 + va_3$, the distribution of points (u,v) can determine shape feature of $Q(t)$ in the $uv-$plane (see Table 1), ie

Table 1: Distribution of shape features of the planar DTB-like curve.

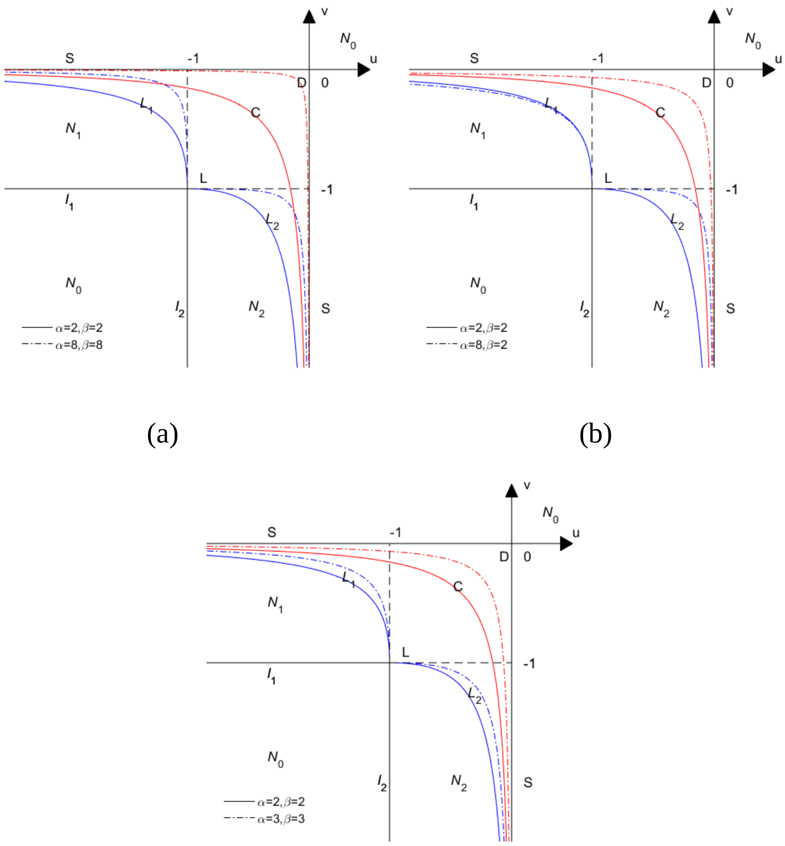
(U,V)	Convexity	Cusps	Loops	Inflection points
C		1	No	No
L		No	1	No
S		No	No	1
D		No	No	2
N_0	Globally	No	No	No
$N_1 \cup N_2$	Locally	No	No	No

3.2.5. Influence of shape parameters on shape diagram and adjustment of curve shape

Through the above analysis of the shape of the DTB-like curve, the following conclusions can be drawn

As shown in Figure 4, the change of the shape parameter α and β does not affect the single inflection point region S and the global convex region N_0, so when $Q(t)$ is global convex, it cannot be eliminated by adjusting parameters; and when it is globally convex, the shape parameters $(\alpha, \beta \geq 2)$ are modified anyway, the curve is still globally convex.

As the shape parameters α and β increase, the curve C is stretched toward the point $(0,0)$, L_1 is stretched toward the point (-1,0), L_2 is stretched toward the point (0,-1), double inflection points region D shrinks, loops region L expands correspondingly, and locally convex regions N_1 and N_2 expand, as shown in Figure 4.



(a) (b)

(c)

Figure 4: Shape diagram of the DTB-like curve.

When $(u,v) \in \{(u,v) \mid -1 \le u, v < 0\} \setminus \{(-1,-1)\}$, i.e. the first and last two edges of the polygon are intersected (except the first and last points coincide), the curve $Q(t)$ may have cusps and loops or inflection points. The curve $Q(t)$ cannot be made into a locally convex curve simply by modifying the shape parameter, as shown in Figure 5.

When $(u,v) \in \{(u,v) \mid u < -1, -1 < v\} \cup \{(u,v) \mid -1 < u, v < -1\}$, i.e. when the control polygon is locally convex, the loops, cusps and inflection points of the curve $Q(t)$ can be eliminated by modifying the shape parameter, and the curve $Q(t)$ can be adjusted to a locally convex curve, as shown in Figure 6.

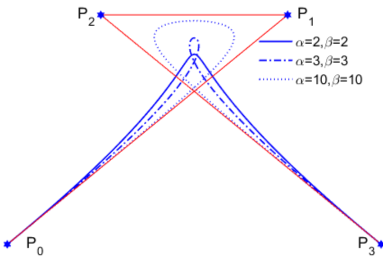


Figure 5: The curve must have a cusp and loop or a double inflection point.

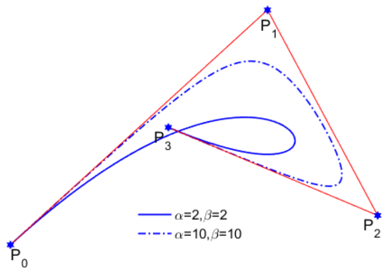


Figure 6: Eliminate double inflection points, cusp, or loop to adjust the curve to local convex.

4. Shape analysis of the DT B-spline curve

In this section, we will use the same method as Section 3 to analyze the shape features of non-uniform DT B-spline curve segments given in (6). From the expression (6), we can easily find that each segment possesses two shape parameters that will influence its shape for a fixed i. Thus, we only discuss the situation about the interval $[u_i, u_{i+1}]$, other situations can be discussed similarly.

4.1. The spatial DT B-spline curve

Theorem 3. When $\alpha_i, \beta_i \in [2, +\infty), u \in [u_i, u_{i+1}], i = 3, 4, \cdots, n$, if the four control points $P_j (j = i-3, i-2, i-1, i)$ are not coplanar, then the DT B-spline-like curve segment is a spatial curve, it has the same rotation direction as the control points and it does not have cusps, loops, and inflection points, .

Proof: Let $b_j = P_j - P_{j-1} (j = i-2, i-1, i)$, the DT B-spline-like curve segment given in (6) can be rewritten as

$$Q_i(u) = P_{i-3} + [1 - B_{i-3}(u)]b_{i-2} + [B_{i-1}(u) + B_i(u)]b_{i-1} + B_i(u)b_i. \tag{29}$$

We have $Q_i'(u) = -B_{i-3}'(u)b_{i-2} + [B_{i-1}'(u) + B_i'(u)]b_{i-1} + B_i'(u)b_i$. Since the four control points P_j are not coplanar, the knot vectors b_j are linearly independent, and $Q_i'(u) = 0 (u_i < u < u_{i+1})$. Therefore, we can easily get $B_{i-3}'(u) = 0$ and $B_i'(u) = 0$. From $B_i'(u) = 0$, we have $a_i T_0'(t_i, \alpha_i) = 0 (0 < t_i < \pi / 2)$, i,e. $T_0'(t_i, \alpha_i) = 0$. Obviously, this can lead to contradictions. Thus, we can conclude $Q_i'(u) \neq 0$, and we also can easily conclude that the curve segment $Q_i(u)$ have no cusps and loops.

Next, we consider the inflection points. Let $H(t) = \det(Q_i'(u), Q_i''(u), Q_i'''(u))$, we have

$$H(u)=\det\begin{bmatrix}\sum_{j=i-3}^{i}B'_j(u)P_j & \sum_{j=i-3}^{i}B''_j(u)P_j & \sum_{j=i-3}^{i}B'''_j(u)P_j\end{bmatrix}$$

$$=\begin{vmatrix}\sum_{j=i-3}^{i}B_j(u) & \sum_{j=i-3}^{i}B'_j(u) & \sum_{j=i-3}^{i}B''_j(u) & \sum_{j=i-3}^{i}B'''_j(u)\\ \sum_{j=i-3}^{i}B_j(u)P_j & \sum_{j=i-3}^{i}B'_j(u)P_j & \sum_{j=i-3}^{i}B''_j(u)P_j & \sum_{j=i-3}^{i}B'''_j(u)P_j\end{vmatrix}$$

$$=\left|\begin{bmatrix}1 & 1 & 1 & 1\\ P_{i-3} & P_{i-2} & P_{i-1} & P_i\end{bmatrix}\begin{bmatrix}B_{i-3}(u) & B'_{i-3}(u) & B''_{i-3}(u) & B'''_{i-3}(u)\\ B_{i-2}(u) & B'_{i-2}(u) & B''_{i-2}(u) & B'''_{i-2}(u)\\ B_{i-1}(u) & B'_{i-1}(u) & B''_{i-1}(u) & B'''_{i-1}(u)\\ B_i(u) & B'_i(u) & B''_i(u) & B'''_i(u)\end{bmatrix}\right| \tag{30}$$

$$=\left|\begin{bmatrix}1 & 0 & 0 & 0\\ P_{i-3} & b_{i-2} & b_{i-1} & b_i\end{bmatrix}\begin{bmatrix}B_{i-3}(u) & B'_{i-3}(u) & B''_{i-3}(u) & B'''_{i-3}(u)\\ B_{i-2}(u) & B'_{i-2}(u) & B''_{i-2}(u) & B'''_{i-2}(u)\\ B_{i-1}(u) & B'_{i-1}(u) & B''_{i-1}(u) & B'''_{i-1}(u)\\ B_i(u) & B'_i(u) & B''_i(u) & B'''_i(u)\end{bmatrix}\right|$$

$$=(b_{i-2},b_{i-1},b_i)M_i(u).$$

where, (b_{i-2},b_{i-1},b_i) is a mixed product of vector edge b_{i-2},b_{i-1},b_i, by directly computing, we have $M_i(u)>0$. For any $u\in[u_i,u_{i+1}]$, since $(b_{i-2},b_{i-1},b_i)\neq 0$, we can get $H(u)\neq 0$, and it has same positive and negative property as (b_{i-2},b_{i-1},b_i). Thus, $Q_i(u)$ has the same rotation direction as the control points and has no inflection points.

4.2. The planar DT B-spline-like curve

If the four control points $(P_j\in R^2, j=i-3,i-2,i-1,i)$ are coplanar, the segment $Q_i(u)$ is a planar curve, at this time, there are the following theorems about its shape features.

Theorem 4. When $b_i \parallel b_{i-2}$, the planar DT B-spline-like curve segment $Q_i(u)$ has no cusps and loops; If and only if b_i has the same direction of b_{i-2}, $Q_i(u)$ only has one inflection point; When $b_i \nparallel b_{i-2}$, let $b_{i-1}=Ub_{i-2}+Vb_i$, where $(U,V)=(b_{i-1}\times b_i, b_{i-2}\times b_{i-1})/(b_{i-2}\times b_i)$, then the points (U,V) can determine the shape features of $Q_i(u)$ in the $UV-$plane (see Table 2), ie

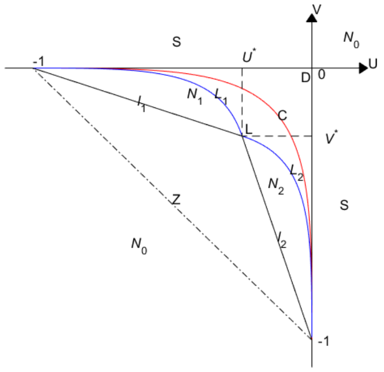


Figure 7: The complete shape analysis diagram of the DT-B spline-like curve with parameters.

Table 2: Distribution of shape features of the planar DT B-spline-like curve.

(U,V)	Convexity	Cusps	Loops	Inflection points
C		1	No	No
L		No	1	No
S		No	No	1
D		No	No	2
N_0	Globally	No	No	No
$N_1 \cup N_2$	Locally	No	No	No

where, the description of each distribution area is as follows: $S=\{(U,V)\,|\,UV<0\}\cup\{(U.0)\,|\,-1<U<0\}\ \cup\{(0,V)\,|\,-1<V<0\}$; D is an open region surrounded by a coordinate axis U,V and curve C. L is a region surrounded curve L_1,L_2 and C, where $L_1\subset L, L_2\subset L$, but $C\not\subset L$; N_1 is an open region surrounded curve L_1 and l_1; N_2 is an open region surrounded curve L_2 and l_2; $N_0=R^2\setminus(S\cup D\cup C\cup L\cup N_1\cup N_2)$ which include the boundaries $\{(U,0)\,|\,U(U+1)\geq 0\}\cup\{(0,V)\,|\,V(V+1)\geq 0\}\cup l_1\cup l_2$. The parametric equations for the relevant curves are as follows:

$$C:\begin{cases} U=\dfrac{d_i T_3'(t_i;\beta_i)}{(a_i+b_{i0}-b_{i1})T_0'(t_i;\alpha_i)+(b_{i3}-b_{i2})T_3'(t_i;\beta_i)+2(b_{i2}-b_{i1})\sin t\cos t} \\ V=-\dfrac{a_i T_0'(t_i;\alpha_i)}{(a_i+b_{i0}-b_{i1})T_0'(t_i;\alpha_i)+(b_{i3}-b_{i2})T_3'(t_i;\beta_i)+2(b_{i2}-b_{i1})\sin t\cos t} \end{cases} \quad 0<t_i<\pi/2 \tag{31}$$

$$L_1:\begin{cases} U=\dfrac{d_i T_3(t_i;\beta_i)}{(a_i+b_{i0})T_0(t_i;\alpha_i)+b_{i1}T_1(t_i;\alpha_i)+b_{i2}T_2(t_i;\beta_i)+b_{i3}T_3(t_i;\beta_i)-b_{i0}-a_i} \\ V=\dfrac{a_i-a_i T_0(t_i;\alpha_i)}{(a_i+b_{i0})T_0(t_i;\alpha_i)+b_{i1}T_1(t_i;\alpha_i)+b_{i2}T_2(t_i;\beta_i)+b_{i3}T_3(t_i;\beta_i)-b_{i0}-a_i} \end{cases} \quad 0<t_i<\pi/2 \tag{32}$$

$$L_2:\begin{cases} U=\dfrac{d_i-d_iT_3(t_i;\beta_i)}{b_{i3}-[(a_i+b_{i0})T_0(t_i;\alpha_i)+b_{i1}T_1(t_i;\alpha_i)+b_{i2}T_2(t_i;\beta_i)+b_{i3}T_3(t_i;\beta_i)]} \\ V=\dfrac{a_iT_0(t_i;\alpha_i)}{b_{i3}-[(a_i+b_{i0})T_0(t_i;\alpha_i)+b_{i1}T_1(t_i;\alpha_i)+b_{i2}T_2(t_i;\beta_i)+b_{i3}T_3(t_i;\beta_i)]} \end{cases} 0<t_i<\pi/2 \tag{33}$$

$$l_1:V=\frac{V^*(U+1)}{U^*+1},-1<U<U^*, \tag{34}$$

$$l_2:V=-1+\frac{U(V^*+1)}{U^*},U^*<U<0, \tag{35}$$

where $U^*=V^*=\dfrac{d_i}{b_{i3}-b_{i0}-a_i}$.

Proof: We first consider the situation $b_i \parallel b_{i-2}$, we can easily get $b_{i-1}=Ub_{i-2}+Vb_i$, then we combine (29), we have

$$\begin{aligned} Q_i(u)&=P_{i-3}+\{1-B_{i-3}(u)+U[B_{i-1}(u)+B_i(u)]\}b_{i-2} \\ &+\{B_i(u)+V[B_{i-1}(u)+B_i(u)]\}b_i. \end{aligned} \tag{36}$$

Below we discuss the cusps, inflection points, loops, and convexity.

4.2.1. Cusps

The necessary condition that the planar DT-B spline-like segment curve $Q_i(u)$ has cups is $Q_i'(u)=0 \quad (u_i<u<u_{i+1})$. From (36), we have

$$\{U[B_{i-1}'(u)+B_i'(u)]-B_{i-3}'(u)\}b_{i-2}+\{B_i'(u)+V[B_{i-1}'(u)+B_i'(u)]\}b_i=0. \tag{37}$$

Since b_{i-2} and b_i are linearly independent, according to (37), we can get the curve C:

$$C:\begin{cases} U=\dfrac{B_{i-3}'(u)}{B_{i-1}'(u)+B_i'(u)}, \\ V=-\dfrac{B_i'(u)}{B_{i-1}'(u)+B_i'(u)}, \end{cases} u_i<u<u_{i+1}. \tag{38}$$

We analyze the shape of the curve C given in (38) and (31), we have

$$\lim_{u\to u_i}U=-1, \lim_{v\to u_i}V=0, \lim_{u\to u_i}U=0, \lim_{v\to u_{i+1}}V=-1. \tag{39}$$

This shows that the curve C has two asymptotes $U=0$ and $V=0$, respectively. By direct calculation, we have

$$\frac{dU}{dV}<0,\frac{d^2U}{dV^2}>0. \tag{40}$$

This indicates that the curve C is a monotonically decreasing and strictly convex curve, and the curve is tangent to the axis U at $(-1,0)$, and it is tangent to the axis V axis at $(0,-1)$.

For any $(U_0,V_0)\in C$, we have $Q_i'(u)=0$ and $Q_i''(t)\neq 0$. In fact, similar to the discussion in (37) and (38), we have

$$\begin{aligned}&\{U[B_{i-1}''(u)+B_i''(u)]-B_{i-3}''(u)\}b_{i-2}\\&+\{B_i''(u)+V[B_{i-1}''(u)+B_i''(u)]\}b_i=0.\end{aligned}\tag{41}$$

In fact, similar to the discussion of 3.2.1, there are no parameters $u\in[u_i,u_{i+1}]$ such that (37) and (38) hold together, which indicates $Q_i'(u)=0$ and $Q_i''(u)\neq 0$. Thus, we have

$$Q_i'(u)=Q_i''(u)(u-u_0)+o(u-u_0).\tag{42}$$

We can know that $Q_i'(u)$ change the sign at u_0, thus, $Q_i(u_0)$ is a cusp. Therefore, the planar DT-B spline-like curve segment defined by (6) possesses cusps that are equivalent to $(u,v)\in C$.

4.2.2. Inflection points

$Q_i(u_0)$ is an inflection point of the DT-B spline-like curve means that $Q_i'(u)\times Q_i''(u)=g(u;U,V)(b_{i-3}\times b_i)$ change its sign when the curve passes the point u_0. where,

$$\begin{aligned}&g(u;U,V)=\\&-\begin{vmatrix}Q_{i-3}'(u) & Q_i'(u)\\ Q_{i-3}''(u) & Q_i''(u)\end{vmatrix}+U\begin{vmatrix}Q_{i-1}'(u) & Q_i'(u)\\ Q_{i-1}''(u) & Q_i''(u)\end{vmatrix}+V\begin{vmatrix}Q_{i-3}'(u) & Q_{i-2}'(u)\\ Q_{i-3}''(u) & Q_{i-2}''(u)\end{vmatrix}.\end{aligned}\tag{43}$$

Since $b_{i-3}\times b_i\neq 0$, we only consider the sign of $g(u;U,V)$. On the $UV-$planar, the possible region that makes the curve $Q_i(u)$ has an inflection point must be covered by the line family $g(u;U,V)$. From the previous discussion, it can be seen that the envelope of the line family $g(u;U,V)$ is curve C, and the curve C is globally convex, so the region swept by the tangent of the curve is $S\cup D\cup C$, which also the possible region of the inflection points, where $S=\{(U,V)\,|\,UV<0\}\cup\{(U,0)\,|\,-1<U<0\}\cup\{(0,V)\,|\,-1<V<0\}$, D is the open region enclosed by the curve C and the coordinate axis U,V (see Figure 7). At any point $(U_0,V_0)\in S\cup D\cup C$ has at least one straight line on the $UV-$plane that is tangent to the curve C. When $(U_0,V_0)\in C$, according to Taylor expansion, we have

$$g(u;U,V)=0.5g_{uu}''(u_0;U_0,V_0)(u-u_0)^2+o(u-u_0)^2,$$

where $g_{uu}''(u_0;U_0,V_0)\neq 0$. We can get that $g(u;U,V)$ will not change the sign when the curve passes the parameter u_0. Thus, $Q_i(u)$ has no inflection points; When $(U_0,V_0)\in S\cup D$, let $g(u_0;U,V)$ is one of the tangent lines of the curve C. Thus, we have

$$g(u;U_0,V_0)=g_u'(u_0;U_0,V_0)(u-u_0)+o(u-u_0)\ (\text{where } g_u'(u_0;U_0,V_0)\neq 0,$$

otherwise $(U_0,V_0)\in C$), we can get that $g(u;U_0,V_0)$ change sign at u_0; Further, when $(U_0,V_0)\in S$, there is only one tangent of curve C can be made through it, and $Q_i(u)$ has only one inflection point; When $(U_0,V_0)\in D$, there are two tangent of curves C can be made, and $Q_i(u)$ has two inflection point.

4.2.3. Loops

The DT-B spline curve segment has loops means that $Q_i(u_1^*)=Q_i(u_2^*)$ when $u_i \le u_1^* < u_2^* \le u_{i+1}$. This is equivalent to U,V,u_1^*,u_2^* satisfying the following equations:

$$\begin{cases} U=\dfrac{B_{i-3}(u_2^*)-B_{i-3}(u_1^*)}{B_{i-1}(u_2^*)+B_i(u_2^*)-B_{i-1}(u_1^*)-B_i(u_1^*)} \\ V=\dfrac{B_i(u_1^*)-B_i(u_2^*)}{B_{i-1}(u_2^*)+B_i(u_2^*)-B_{i-1}(u_1^*)-B_i(u_1^*)} \end{cases},(u_1^*,u_2^*)\in\Delta. \tag{44}$$

where $\Delta=\{(u_1^*,u_2^*)\in R^2 \mid u_i \le u_1^* < u_2^* \le u_{i+1}\}$. The equations given in (44) define a topological mapping $G:\Delta\subset R^2 \to G(\Delta)\subset R^2$. Thus, the image $L=F(\Omega)$ is a simply connected region in $UV-$plane. The three boundary lines of L correspond to the three boundary lines $u_1^*=u_2^*$, $u_1^*=u_i$ and $u_2^*=u_{i+1}$ of Δ, i.e. the curve C (not belonging to L), L_1 and L_2 (both belonging to L). The curve $Q_i(u)$ corresponding to point (U,V) in L only has one loop. From the mathematical analysis, it can be inferred that both the curves L_1 and L_2 are continuous curves of monotonically decreasing and strictly convex, and when $u\to u_i$, the U axis is tangent of the curve L_1 at the point $(-1,0)$, and when $u\to u_{i+1}$, the V axis is tangent of the curve L_2 at the point $(0,-1)$. L_1 and L_2 intersect at point (U^*,V^*), the tangent line l_2 of L_1 at point (U^*,V^*) crosses point $(0,-1)$, and the tangent line l_1 of L_2 at point (U^*,V^*) crosses point $(-1,0)$ (see Figure 7), where the equation of l_1 and l_2 are given in (34) and (35), respectively.

4.2.4. Convexity

The following is the case for $(U,V)\in N=R^2\setminus(C\cup S\cup D\cup L)$. The curve segments $Q_i(u)$ has not cusps, inflection points, and loops. Next, we consider $R(u)=Q'(u_i)\times[Q(u)-Q(u_i)]$ and $S(u)=[Q(u)-Q(u_i)]\times Q'(u)$, from (37), we have

$$\begin{cases} R(u)=\Psi(u;U,V)(b_{i-3}\times b_i), \\ S(u)=\Phi(u;U,V)(b_{i-3}\times b_i). \end{cases} \tag{45}$$

where,

$$\Psi(u;U,V)= \\ \alpha_i\{U[(b_{i1}-b_{i0})B_i+a_i(B_{i-1}-b_{i1})]+B_{i-3}[a_i-V(c_{i1}-c_{i0})]\}, \tag{46}$$

$$\begin{aligned}\Phi(u;U,V) &= [B_{i-1}(u_i) - B_{i-3}(u)]B_i'(u) + B_{i-3}'(u)B_i(u) \\ &+U\{[B_{i-1}(u) + B_i(u) - B_{i-1}(u_i)]B_i'(u) - [B_{i-1}'(u) + B_i'(u)]B_i(u)\} \\ &+V\{[B_{i-1}(u_i) - B_{i-3}(u)][B_{i-1}'(u) + B_i'(u)] \\ &+B_{i-3}'(u)[B_{i-1}(u) + B_i(u) - B_{i-1}(u_i)]\}.\end{aligned} \tag{47}$$

According to (46), $\Psi(u;U,V)=0$ determines a family of straight lines passing $(-1,0)$ on the UV -plane. Its slope is

$$K(u) = \frac{(b_{i1} - b_{i0})B_i(u) + a_i(B_{i-1}(u) - b_{i1})}{B_{i-3}(u)}, \tag{48}$$

Through direct computing, we have

$$\frac{V(c_{i1} - c_{i0}) - a_i}{U} < K(u) < 0, \tag{49}$$

therefore, the region swept by the line family $\Psi(u;U,V)=0$ in N happens to be the part enclosed by the curve L_1 and the straight line segment l_1, which we record as N_1 (excluding the boundary lines L_1 and l_1, as shown in Figure 7). If $(U_0,V_0) \in N_1$, we have $\Psi'(u;U,V) \neq 0$ (otherwise. $U=-1, V=0$) Thus, from the expansion $\Psi(u;U_0,V_0) = \Psi'(u_0;U_0,V_0)(u-u_0) + o(u-u_0)$, we can know that $\Psi(u;U_0,V_0)$ will change sign at point u_0. The region N_1 is exactly the portion of the tangent of the curve L_2 swept in N.

Solve the following equations about UV:

$$\begin{cases}\Phi(u;U,V)=0 \\ \Phi'(u;U,V)=0\end{cases} \tag{50}$$

We can easily check that the solution is exactly the equation of the parameters curve L_1. The region where the tangent of L_1 is swept in N is N_2. Then N_2 is surrounded by the curve L_2 and the straight line segment l_2 (where $L_2 \not\subset N_2, l_2 \not\subset N_2$, see Figure 7). If $(U_0,V_0) \in N_2$, we have $\Phi'(u_0;U_0,V_0) \neq 0$ (otherwise $(U_0,V_0) \in L_1$). Thus, from $\Phi(u;U_0,V_0) = \Phi'(u_0;U_0,V_0)(u-u_0) + o(u-u_0)$, we can easily check that $\Phi(u;U_0,V_0)$ will change the sign at the point u_0.

Let $N_0 = N \setminus (N_1 \cup N_2)$, when $(U,V) \in N_0$, $Q_i'(u) \times Q_i''(u)$, $R(u)$ and $S(u)$ all doesn't change significantly. Thus, the curve $Q_i(u)$ is globally convex; When $(U,V) \in N_1$, $Q_i'(u) \times Q_i''(u)$ and $S(u)$ doesn't change significantly, but $R(u)$ change sign one time, at this time, the curve $Q_i(u)$ is locally convex; When $(U,V) \in N_2$, $Q_i'(u) \times Q_i''(u)$ and $R(u)$ doesn't change significantly, but $S(u)$ change sign one time, at this time, the curve $Q_i(u)$ is locally convex [11].

Finally, when $b_{i-2} \nparallel b_i$, the curve $Q_i(u)$ has no cusps and loops; If and only if when b_{i-2} has the same direction of b_i, $Q_i(u)$ only has one inflection point.

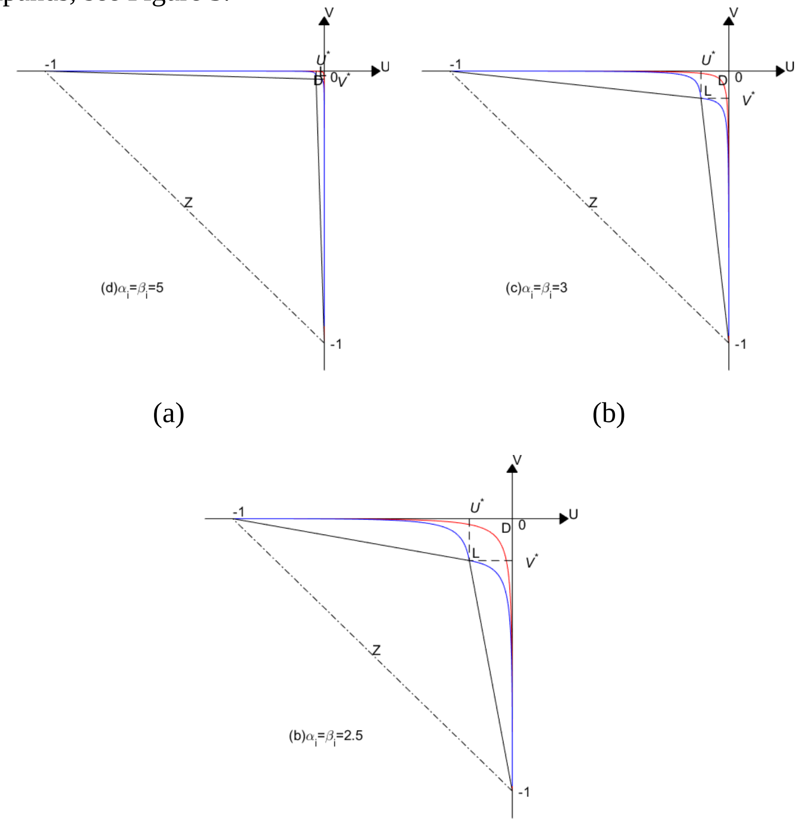
4.2.5. Adjustment effect of shape parameters

According to the equation and region division of the curve in the shape diagram, the following conclusions can be drawn:

The change of shape parameters α_i and β_i does not affect the inflection point region S, so when there is only one inflection point on $Q_i(u)$, shape parameters cannot be adjusted to eliminate it.

The change of shape parameters α_i and β_i does not affect the region N_0/Z, at this time, $Q_i(u)$ is globally convex, where Z is the triangular region enclosed by $(-1,0),(0,-1),(U^*,V^*)$ (including the boundary line l_1 and l_2, excluding the straight line connecting two points $(-1,0)$ and $(0,-1)$).

With the increase of shape parameters $\alpha_i,\beta_i \in [2,+\infty)$, (U^*,V^*) tends to $(0,0)$, and with the decrease of shape parameters, (U^*,V^*) tends to $(-1/4,-1/4)$. The region of double inflection points and loops gradually shrinks, and the global convex region Z gradually expands, see Figure 5.



(a) (b)

(c)

Figure 8: The effect of parameters α_i and β_i on L,D,Z

For $\alpha_i,\beta_i \in [2,+\infty)$, when $\alpha_i,\beta_i \to +\infty$, $N_1 \cup N_2 \to 0$, see Figure 8.

By adjusting the value of the shape parameter α_i, β_i, the cusp, loop and double inflection point on the curve can be eliminated, and the curve can be adjusted to global convexity, see Figure 9.

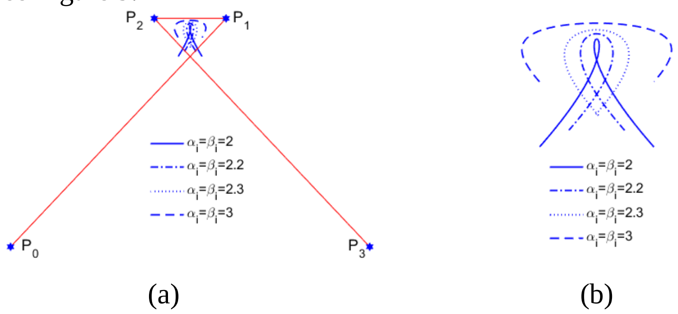


Figure 9: The unnecessary shape features are avoided by adjusting shape parameters((a): the original image, (b): enlarged image).

5. Conclusion

We have given many excellent properties of the DTB-like curves and DT B-spline-like curves [21], which play an important role in the design of curves and surfaces. Especially the high-order continuity, which is completely absent in many existing methods. And this article is a further discussion of the literature [21]. The key point is that the shape features of DTB-like curves and DT B-spline-like curves with two denominator parameters are studied, including the convexity, inflection points, cusps, and loops. And the existence conditions of corresponding features are given, which could avoid unnecessary feature points in the obtained curve. The work in this paper has a certain guidance value for further application of the DTB-like curve and the DT B-spline-like curve.

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