*Journal of Mathematics and Informatics*
*Vol. 24, 2023, 33-44*
*ISSN: 2349-0632 (P), 2349-0640 (online)*
*Published 19 February 2023*
*www.researchmathsci.org*
*DOI: http://dx.doi.org/10.22457/jmi.v24a04217*

Journal of
**Mathematics and Informatics**

# Blow-up for Semilinear Parabolic Equations with Memory Terms of Variable Exponents

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*Received 5 January 2023; accepted 17 February 2023*

***Abstract.*** In this paper, we concern with the blow-up problem of positive solutions to parabolic equations with reaction terms of local and nonlocal type involving a variable exponent. It is shown that under certain conditions on the nonlinearities and data, blow-ups will occur for the limited time.

***Keywords:*** Variable exponents; Blow-up in finite time; Semilinear parabolic equations

***AMS Mathematics Subject Classification (2010):*** 35B44, 35K40

## 1. Introduction

In this paper, we consider with the limited-time blow-up of solutions for the initial boundary value problem:

$$\begin{cases} u_t - \Delta u = u^q \int_0^t u^{p(x)}(x,s)ds, & x \in \Omega, t > 0; \\ u(x,t) = 0, & x \in \partial\Omega, t > 0; \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, $q \geq 0$, and $u_0$ is a nonnegative continuous function vanishing on $\partial\Omega$. $p(x)$ is a continuous and bounded function, we denote:

$$p^- = \inf_{x\in\Omega} p(x) > 1, \quad p^+ = \sup_{x\in\Omega} p(x) < +\infty. \tag{2}$$

This equation arises from a variety of mathematical models in engineering and physical sciences. As a physical motivation, the differential equation in (1) with $p = q = 1$ appears in the theory of nuclear reactor kinetics (see [1]). In this case, the non-linear term with time integral is called the **memory term.** Such equation model diffusion phenomena with memory effects have been widely considered by many authors (see [2]-[12]). For instance, in [9], Bellout considered the blow-up solution of the equation

$$u_t - \Delta u = \int_0^t (u+\lambda)^p ds + g(x), \quad x \in \Omega, t > 0,$$

where $g(x)\geq 0$ is a smooth function and $\lambda>0$. In [11], Yamada investigated the stability properties of the global solution of the following nonlocal Volterra equation

$$u_t-\Delta u=(a-bu)u-\int_0^t k(t-s)u(x,s)ds,\quad x\in\Omega, t>0.$$

In [12], Y. Li and C. Xie studied the equation (1) with $p(x)$ is a constant. They gave a complete answer to the existence and nonexistence of global solution to (1) and estimated the blow-up rate under some conditions.

In recent decades, with the development of hydrodynamics and elastodynamics, many authors began to pay attention to the problems of blow-up solutions of reaction-diffusion equations with variable exponential growth conditions (see [15]-[18]). In [18], Pinasco considered the following problem

$$\begin{cases} u_t=\Delta u+f(u), & x\in\Omega, t>0;\\ u(x,t)=0, & x\in\partial\Omega, t>0;\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}\tag{3}$$

where $f(u)=a(x)u^{p(x)}$ or $f(u)=a(x)\int_\Omega u^{q(y)}(y,t)dy$ and $p(x),q(x)>1$. The author demonstrated that the solution of (1) blow-up in finite time for sufficiently large initial value.

To our best acknowledges, problem with time-integral and variable exponent power don't seem to be studied. The variable $L^{p(x)}$ spaces are of interest for their applications to modelling in wide variety of physical problem. On the theoretical side, there are many interesting features of $L^{p(x)}$ spaces which present difficult challenges. For example, it is not clear whether the comparison principle holds when proving the existence of small global solutions. We will try to construct a new comparison principle for solving the problem presented above. Our main goal is to find the effects of weight function in the boundary condition and competitive relationship between nonlinear memory term and inner absorption term on whether determining blow-up of solutions or not for equation (1). Our conclusions are as follows:

- Assume $p^-+q>1$, then the solution to (1) $u$ blows up in finite time for sufficiently large $u_0$. As a more precise result, when $q<1$, the solution of (1) blows up in finite time for any nonnegative $u_0$.
- If $p^-+q>1$, $q\geq 1$, the solution of (1) exists globally for sufficiently small initial data $u_0$. Moreover $p^++q\leq 1$, the solution of (1) exists globally for all nonnegative $u_0$.

The rest of this article is organized as follows: In Section 2, we study the local existence of positive solution, and establish the comparison principle. The proofs of blows up results are given in Section 3. At last, in Section 4, the global existence theorems are proved. Section 5 contains conclusions.

## 2. Preliminaries

In this section, we show the existence theorem and comparison principle. Firstly, for problem (1), we consider the classical solution in $C(\overline{\Omega}_T)\cap C^{2,1}(\Omega_T)$. By using the fixed point theorem, we give the following existence theorem.

**Theorem 2.1.** If $\Omega \in R^N$ is an open bounded domain with smooth boundary $\partial\Omega$, $p(x)$ satisfies condition (2), then the solution of the equation (1) exists locally and is unique.

**Proof:** With an initial datum $u_0$ and homogeneous Dirichlet boundary conditions. (1) could be written as

$$u(x,t) = \int_\Omega G(x,z,t)u_0(z)dz + \int_0^t \int_\Omega G(x,z,t-s)(u^q \int_0^s u^{p(x)} d\sigma)dzds, \tag{4}$$

where $G(x,z,t)$ is the Green function. Now, the existence and uniqueness of solutions for a given $u_0(x)$ could be obtained by a fixed point argument.

We define inductively

$$u_1(x,t) = 0$$

$$u_{n+1} = \int_\Omega G(x,z,t)u_0(z)dz + \int_0^t \int_\Omega G(x,z,t-s)(u_n{}^q \int_0^s u_n{}^{p(x)} d\sigma)dzds,$$

and the convergence of the sequence $\{u_n\}$ follows by showing that

$$P(u) = \int_0^t \int_\Omega G(x,z,t-s)u^q \int_0^s u^{p(x)} d\sigma dzds,$$

is a contraction in

$$E = \left\{C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T) : \|u\|_\infty \le M\right\}$$

where $\Omega_T = \Omega \times (0,T)$, $M \ge \max\{\|u\|_\infty, 1\}$ is a fixed constant.

For any $u, v \in E$, we get

$$\left\| u^q \int_0^s u^{p(x)} d\sigma - v^q \int_0^s v^{p(x)} d\sigma \right\|_\infty$$

$$= \left\| (u^q - v^q) \int_0^s u^{p(x)} d\sigma + v^q \int_0^s v^{p(x)} d\sigma - v^q \int_0^s v^{p(x)} d\sigma \right\|_\infty$$

$$\le \left\| u^q - v^q \right\|_\infty \left\| \int_0^s u^{p(x)} d\sigma \right\|_\infty + \left\| v^q \right\|_\infty \left\| \int_0^s (u^{p(x)} - v^{p(x)}) d\sigma \right\|_\infty$$

For any $x \in \Omega$ fixed, according to Lagrange mean value theorem, we have

$$u^q - v^q = q\xi_1^{q-1}(u-v)$$

$$u^{p(x)} - v^{p(x)} = p(x)\xi_2^{p(x)-1}(u-v),$$

where $\xi_i = \theta_i u + (1-\theta_i)v$, $\theta_i \in (0,1)$. Although $\theta_i$ depends on $x$, we always have

$$\left\| q\xi_1^{q-1}(u-v) \right\|_\infty \le q(2M)^{q-1} \|u-v\|_\infty$$
$$\left\| p(x)\xi_2^{p(x)-1}(u-v) \right\|_\infty \le p(2M)^{q-1} \|u-v\|_\infty . \tag{5}$$

Now, let us define $Q(t)$ as

$$Q(t) = \sup_{x\in\Omega, 0\le\tau<t} \int_0^\tau \int_\Omega G(x,z,\tau-s)dzd\omega,$$

clearly $Q(t) \to 0$ as $t \to 0$.

It remains to prove that, for sufficiently small $Q(t)$, $P(u)$ is a contraction, that is , there exists $h < 1$ such that

$$\|P(u)-P(v)\|_\infty \le h\|u-v\|_\infty,$$

for every $u, v \in E$.

Since

$$\begin{aligned}
&\|P(u)-P(v)\|_\infty \\
&= \left\| \int_0^t \int_\Omega G(x,z,t-s) \left[ u_n^q \int_0^s u_n^{p(x)} d\sigma - v_n^q \int_0^s v_n^{p(x)} d\sigma \right] dzds \right\|_\infty \\
&\le Q(t) \left[ \|u^q - v^q\|_\infty \left\| \int_0^s u^{p(x)} d\sigma \right\|_\infty + \|v^q\|_\infty \left\| \int_0^s (u^{p(x)} - v^{p(x)} d\sigma \right\|_\infty \right] \\
&\le Q(t) \left[ M^{p^+} sq(2M)^{q-1} \|u-v\|_\infty + M^q sp^+ (2M)^{p^+-1} \|u-v\|_\infty \right] \\
&= Q(t) sM^{p^{|+}+q-1} (q2^{q-1} + p^+ 2^{p^+-1}) \|u-v\|_\infty .
\end{aligned}$$

We choose a sufficiently small initial data $\varepsilon$, when $0 \le t \le \varepsilon$, $Q(t)$ is small enough. Such that

$$Q(t) sM^{p^{|+}+q-1} (q2^{q-1} + p^+ 2^{p^+-1}) \|u-v\|_\infty < 1$$

Base on the Banach's fixed point theorem, this implies there exists a unique local solution $u$ in $\Omega_T$.

Next, we establish a modified comparison principle for problem (1). We begin with a lemma. Although its proof is standard. For completeness, we give the details:

Let $S_T = \partial\Omega \times (0,T)$, $0 < T < \infty$.

**Lemma 2.1.** Assume that $a$, $b$, and $c$ satisfy

$$a(x,t), b(x,t), c(x,t) \in C(\overline{\Omega}_T) \quad and \quad b(x,t), c(x,t) \ge 0,$$

suppose that $u \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ satisfies

$$\begin{cases} u_t - \Delta u \ge au + b\int_0^t c(s)u(s)ds, & in\ \Omega_T, \\ u(x,t) \ge 0 & on\ S_T, \\ u(x,0) = u_0(x) \ge 0 & in\ \Omega, \end{cases} \tag{6}$$

then $u(x,t) \ge 0$ for $(x,t) \in \Omega_T$.

**Proof:** We shall prove the lemma by contradiction. Firstly, define the constants $N_1$, $N_2$ and $N_3$ by

$$N_1 = \max_{\Omega_T} |a(x,t)|, \quad N_2 = \max_{\Omega_T} |b(x,t)|, \quad N_3 = \max_{\Omega_T} |c(x,t)|.$$

Set $u(x,t) = e^{\lambda t}\omega'(x,t)$, where $\lambda$ is an arbitrary positive constant to be chosen later. Obviously, $\omega'$ satisfies:

$$\omega'_t - \Delta\omega' + (\lambda - a)\omega' - be^{-\lambda t}\int_0^t e^{\lambda s} c(s)\omega'(s)ds \ge 0. \tag{7}$$

Suppose $\omega'$ achieves its negative minimum at $(x_0, t_0) \in \overline{\Omega}_{T'}$, for some $T' < T$, then

$$\omega'(x_0,t_0) < 0, \quad \Delta\omega'(x_0,t_0) \ge 0, \quad \omega'_t(x_0,t_0) \le 0.$$

If we choose $\lambda$ large enough such that

$$\lambda > \frac{N_1 + \sqrt{N_1^2 + 4N_2N_3}}{2}$$

Thus

$$\begin{aligned}
&\left[\omega'_t - \Delta\omega' + (\lambda - a)\omega' - be^{-\lambda t}\int_0^t e^{\lambda s} c(s)\omega'(s)ds\right](x_0, t_0) \\
&\le (\lambda - N_1)\omega'(x_0, t_0) - \frac{1}{\lambda} N_2 N_3 e^{-\lambda t_0}(e^{\lambda t_0} - 1)\omega'(x_0, t_0) \qquad (8)\\
&\le -\omega'(x_0, t_0)(-(\lambda - N_1) + \frac{1}{\lambda} N_2 N_3) < 0,
\end{aligned}$$

since the left-hand side of (7) is strictly non-negative, which contradict that assumption.

From the above Lemma, it follows the comparison theorem.

**Theorem 2.2.** Let $f, g : [0, \infty) \to [0, \infty)$ be non-decreasing nonnegative locally Lipschitz functions on $(0, \infty)$. Suppose that $u, v \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ and $u \ge \delta > 0$, $v \ge 0$ satisfy

$$\begin{aligned}
&u_t - \Delta u \ge g(u)\int_0^t f(u)ds, \\
&v_t - \Delta v \ge g(v)\int_0^t f(v)ds, \quad in\ \Omega_T, \\
&u \ge v, \quad in\ S_T, \\
&u(x,0) \ge v(x,0) \qquad in\ \Omega.
\end{aligned}$$

Then $u \ge v$ in $\Omega_T$.

**Proof:** Let $a, b, c$ be the continuous functions defined by

$$\begin{aligned}
&a(x,t) = \int_0^t f(u)ds \times \begin{cases} \dfrac{g(u) - g(v)}{u - v} & u \ne v \\ g'(u) & u = v, \end{cases} \\
&b(x,t) = g(v), \\
&c(x,t) = \begin{cases} \dfrac{f(u) - f(v)}{u - v} & u \ne v \\ f'(u) & u = v. \end{cases}
\end{aligned}$$

Since $u \ge \delta > 0$, $a, c$ are bounded, and $b, c \ge 0$. We have

$$\begin{aligned}
(u - t)_t - \Delta(u - v) &= u_t - v_t - \Delta u + \Delta v \\
&\ge g(u)\int_0^t f(u)ds - g(v)\int_0^t f(v)ds \\
&= g(u)\int_0^t f(u)ds - g(v)\int_0^t f(u)ds + g(v)\int_0^t f(u)ds - g(v)\int_0^t f(v)ds \\
&= \frac{g(u) - g(v)}{u - v}(u - v)\int_0^t f(u)ds + g(v)\int_0^t \frac{f(u) - f(v)}{u - v}(u - v)ds \\
&= a(u - v) + b\int_0^t c(s)(u - v)ds.
\end{aligned}$$

Then the theorem follows from Lemma 2.1.

Based on the above comparison principle, we obtain the theorem.

**Theorem 2.3.** If $u \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ satisfies

$$\begin{cases} u_t - \Delta u \ge u^q \int_0^t u^{p(x)}(x,s)ds, & in\ \Omega_T; \\ u(x,t) \ge 0, & on\ S_T; \\ u(x,0) \ge u_0(x), & in\ \Omega. \end{cases}$$

Then $u \ge 0$ in $\overline{\Omega}_T$ . Furthermore, $u > 0$ in $\overline{\Omega}_T$ .

## 3. Blow-up of the solution

In this section, we discuss the finite-time blow-up of the solution by subsolution method based on the comparison principle.

The main point about the time-integral non-local problems is that only when the time is large, the time-integral terms plays a dominant role in the evolution of the solutions. Based on this idea, we construct a blowing-up subsolution of the form

$$\underline{u} = \frac{\omega(r)}{z^l(r,t)}, \quad 0 \le r < R, \quad 0 < t < T, \tag{9}$$

where $\omega(r) = \cos^2(\frac{\pi r}{2R})$ , $z(r,t) = 4T^2 - \omega^\delta (t-T)^2 = z_1 z_2$ , and

$$z_1 = 2T - \omega^{\delta/2}(t-T),$$
$$z_2 = 2T + \omega^{\delta/2}(t-T).$$

Obviously, $\underline{u}(r,t)$ blows up at $t = 0$ as $t$ approaches $3T$ .

By direct calculation, we have

$$-\Delta\omega = \tfrac{\pi}{2R}(\frac{\pi}{R}\cos\frac{\pi r}{R} + \frac{n-1}{r}\sin\frac{\pi r}{R}).$$

Hence, there exists a unique $r_0 \in (0,R)$ such that

$$-\Delta\omega \le 0 \quad for \quad r_0 \le r < R, \tag{10}$$

$$0 \le -\Delta\omega \le \frac{n\pi^2}{2R^2} \quad and \quad \omega \ge \cos^2\frac{\pi r_0}{2R}, \quad for \quad 0 \le r < r_0. \tag{11}$$

**Theorem 3.1.** If $p^- + q > 1$,then the solution of (1) blows up in finite time for sufficiently large $u_0$ .

**Proof:** Without loss of generality, we assume that $0 \in \Omega$ , and take a ball $B_R(0) \subset\subset \Omega$ . Consider the problem

$$\begin{cases} v_{1t} - \Delta u = v_1^q \int_0^t v_1^{p(x)}(x,s)ds, & x \in B_R(0), t > 0; \\ v_1(R,t) = u(R,t), & \\ v_1(x,0) = u_0(x), & x \in B_R(0), \end{cases} \tag{12}$$

where $u_0(x)$ is the same as in (1). We assume $u_0 > 0$ in $\overline{B_R}(0)$ . From the uniqueness of \$u\$, it follows that $v = u|_{B_R(0)}$ .

We will show that $\underline{u}$ defined by (9) is a subsolution of (12), which implies by Theorem 2.2 that $v_1$ blows up in finite time.
Define

$$Lu = u_t - \Delta u - u^q \int_0^t u^{p(x)} ds.$$

Choose $\sigma > 2, l \geq 1$ and $T \leq \frac{1}{2}$ such that $z \leq 1$. By a series of computations yields, we obtain

$$\begin{aligned} L\underline{u} &= \frac{2l\omega^{\sigma+1}(t-T)}{z^{l+1}} - \frac{\Delta\omega}{z^l} - \frac{2l\sigma\omega^{\sigma-1}|\Delta\omega|^2(t-T)^2}{z^{l+1}} - \frac{l\sigma(\sigma-1)\omega^{\sigma-1}|\Delta\omega|^2(t-T)^2}{z^{l+1}} \\ &\quad - \frac{l\sigma\omega^{\sigma}\Delta\omega(t-T)^2}{z^{l+1}} - \frac{l(l+1)\omega|\nabla z|^2}{z^{l+2}} - \frac{\omega^{p(x)+q}}{z^{lq}} \int_0^t \frac{ds}{z^{lp(x)}} \\ &\leq \frac{2l\omega^{\sigma+1}(t-T)}{z^{l+1}} - \frac{\Delta\omega}{z^l} - \frac{l\sigma\omega^{\sigma}\Delta\omega(t-T)^2}{z^{l+1}} - \frac{\omega^{p(x)+q}}{z^{lq}} \int_0^t \frac{ds}{z^{lp(x)}}. \end{aligned} \tag{13}$$

In the proof, we divide the interval $(0,3T)$ into two intervals as interval $(0,T/2]$ and interval $(T/2,3T)$.

Firstly, we consider the case $\mathbf{0 < t \leq \frac{T}{2}}$. In this case, the linear term is the main factors. We will show that

$$\frac{2l\omega^{\sigma+1}(t-T)}{z^{l+1}} - \frac{\Delta\omega}{z^l} - \frac{l\sigma\omega^{\sigma}\Delta\omega(t-T)^2}{z^{l+1}} \leq 0,$$

for sufficient small $T$. In fact, this inequality is equivalent to

$$-\Delta\omega \leq \frac{2l\omega^{\sigma+1}(T-t)}{4T^2 + (l\sigma-1)\omega^{\sigma}(t-T)^2} \tag{14}$$

Since $\frac{1}{l} < \frac{\sigma}{2}$ and $0 \leq \omega \leq 1$, we get

$$\frac{2l\omega^{\sigma+1}(T-t)}{4T^2 + (l\sigma-1)\omega^{\sigma}(t-T)^2} \geq \frac{l\omega^{\sigma+1}T}{4T^2 + (l\sigma-1)\omega^{\sigma}T^2} = \frac{l\omega^{\sigma+1}}{4T + (l\sigma-1)\omega^{\sigma}T} \geq \frac{l\omega^{\sigma+1}}{4T\frac{1}{l} + \sigma T} \geq \frac{\omega^{\sigma+1}}{3\sigma T}.$$

Then (14) can be replaced by

$$-\Delta\omega = \frac{\omega^{\sigma+1}}{3\sigma T}$$

From (9), for $r_0 \leq r < R$, it is trivial. For $0 \leq r < r_0$, it is satisfied if

$$T \leq \frac{2R^2}{3n\sigma\pi^2}(\cos\frac{\pi r_0}{2R})^{2(\sigma+1)}.$$

Now, we study the case $\mathbf{\frac{T}{2} < t < 3T}$, we will find in this case the time-integral term takes into main actions. Choose $lp^- > 2$, we have

$$\underline{u}^q\int_0^t \underline{u}^{p(x)}ds=\frac{\omega^{p(x)+q}}{z^{lq}}\int_0^t\frac{ds}{z(s)^{lp(x)}}\ge\frac{\omega^{p^++q}}{z^{lq}}\int_0^t\frac{ds}{z(s)^{lp^-}}=\frac{\omega^{p^++q}}{z^{lq}}\int_0^t\frac{ds}{z_1(s)^{lp^-}z_2(s)^{lp^-}}$$

$$\ge\frac{\omega^{p^++q}}{z^{lq}z_2{}^{lp^-}}\int_0^t\frac{ds}{z_1(s)^{lp^-}}=\frac{\omega^{p^++q}}{z^{lq}z_2{}^{lp^-}}\frac{1}{\omega^{\frac{\sigma}{2}}(lp^--1)}\left(\frac{1}{z_1{}^{lp^-}}-\frac{1}{z_1(0)^{lp^-}}\right) \tag{15}$$

$$=\frac{\omega^{p^++q-\frac{\sigma}{2}}}{z^{l(p^-+q-1)}z_2(lp^--1)}-\frac{\omega^{p^++q-\frac{\sigma}{2}}}{(lp^--1)z^{lq}z_2{}^{lp^-}z_1(0)^{lp^-}}.$$

Hence, from (13) and (15), we have

$$L\underline{u}\le\frac{4l\omega^{\sigma+1}T}{z^{l+1}}+\frac{4T^2(-\Delta\omega)}{z^{l+1}}+\frac{(l\sigma-1)\omega^{\sigma}(-\Delta\omega)(t-T)^2}{z^{l+1}}$$
$$-\frac{\omega^{p^++q-\frac{\sigma}{2}}}{z^{l(p^-+q-1)}z_2(lp^--1)}+\frac{\omega^{p^++q-\frac{\sigma}{2}}}{(lp^--1)z^{lq}z_2{}^{lp^-}z_1(0)^{lp^-}}. \tag{16}$$

**Case 1:** If $r_0\le r<R$, since $-\Delta\omega\le 0$, we have

$$L\underline{u}\le\frac{4l\omega^{\sigma+1}T}{z^{l+1}}-\frac{\omega^{p^++q-\frac{\sigma}{2}}}{z^{l(p^-+q-1)}z_2(lp^--1)}+\frac{\omega^{p^++q-\frac{\sigma}{2}}}{(lp^--1)z^{lq}z_2{}^{lp^-}z_1(0)^{lp^-}}.$$

Choose $\sigma\ge p^++q-1$, $l\ge\max\{2(p^-+q-1)^{-1},2(p^-)^{-1}\}$ and put $a'=\frac{3\sigma}{2}-(p^++q-1)$, then we get

$$L\underline{u}\le\frac{\omega^{p^++q-\frac{\sigma}{2}}}{z^{l(p^-+q-1)}z_2{}^{lp^-}(lp^--1)}\left(4(lp^--1)lT\omega^{\sigma}z_2z^{l(p^-+q-1)-2}+\frac{z_1^{lp^--1}}{z_1(0)^{z_1^{lp^--1}}}-1\right).$$

Since $lp^--1>1$ and $0\le\frac{z_1}{z_1(0)}\le\frac{2T-\omega^{\sigma/2}(T/2-T)}{2T+\omega^{\sigma/2}T}$, then

$$\frac{z_1^{lp^--1}}{z_1(0)^{lp^--1}}\le\frac{z_1}{z_1(0)}\le\frac{2+\frac{\omega^{\sigma/2}}{2}}{2+\omega^{\sigma/2}}$$

and we have

$$L\underline{u}\le\frac{\omega^{p^++q-\frac{\sigma}{2}}}{z^{l(p^-+q-1)}z_2{}^{lp^-}(lp^--1)}\left(8(lp^--1)lT\omega^{\sigma}+\frac{2+\frac{\omega^{\sigma/2}}{2}}{2+\omega^{\sigma/2}}-1\right)$$
$$\le\frac{\omega^{p^++q-\frac{\sigma}{2}}}{z^{l(p^-+q-1)}z_2{}^{lp^-}(lp^--1)(2+\omega^{\sigma/2})}\left(24(lp^--1)lT\omega^{\sigma}-\frac{\omega^{\sigma/2}}{2}\right). \tag{17}$$

Since $\omega\in[0,1]$, the last term of (17) is nonpositive, if

$$T\le\frac{1}{48(lp^--1)l}.$$

**Case 2:** If $0\le r<r_0$ . By (11) and (16), we have

$$L\underline{u} \le \frac{4lT + 2nl\sigma\pi^2 / R^2}{z^{l+1}} - \frac{\omega^{p^+ + q - \frac{\sigma}{2}}}{z^{l(p^- + q - 1)} z_2 (lp^- - 1)} + \frac{\omega^{p^+ + q - \frac{\sigma}{2}}}{(lp^- - 1) z^{lq} {z_2}^{lp^-} z_1(0)^{lp^-}}$$

$$= \frac{4lT + 2nl\sigma\pi^2 / R^2}{z^{l+1}} - \frac{\omega^{p^+ + q - \frac{\sigma}{2}} / (lp^- - 1)}{z^{l(p^- + q - 1)} z_2}\left(1 - \frac{z_1^{lp^- - 1}}{z_1(0)^{lp^- - 1}}\right).$$

Put $C = \frac{z_1}{z_1(0)}$, obviously, then $0 < C < 1$, $4T^2(1-\omega^\sigma) \le z(r,t) \le 4T^2$ and $2T < z_2 < 4T$. Hence, we have

$$L\underline{u} \le \frac{2l + nl\sigma\pi^2 / 2R^2}{4^{l+1} T^{2(l+1)} (1-\omega^\sigma)^{l+1}} - \frac{\omega^{p^+ + q - \frac{\sigma}{2}} / 2(lp^- - 1)}{4^{l(p^- + q) - 1} T^{2(l(p^- + q) - 1)}} (1 - C^{lp^- - 1}). \qquad (18)$$

If we choose $l \ge 2(p^- + q - 1)^{-1}$, the right-hand side of (18) is non-positive for sufficiently small $T$.

In conclusion, if we choose that

$$l \ge \max\{1, 2(p^-)^{-1}, 2(p^- + q - 1)^{-1}\}, \quad \max\{2, p^+ + q - 1\} < \sigma < p^+ + q,$$

We have

$$\underline{u}_t - \Delta\underline{u} - \underline{u}^q \int_0^t \underline{u}^{p(x)} ds \le 0$$

for $T$ is sufficiently small. Therefore, $\underline{u}(r,t)$ is the subsolution of the problem (12) in $B_R(0)$, if $u_0(x) \ge \underline{u}(x,0)$. So the solution of (1) blows up in limited time. The proof of Theorem 3.1 is completed.

**Theorem 3.2.** If $p^- + q > 1$, $q < 1$, then the solution of (1) blows up in limited time for any nonnegative nontrivial $u_0$.

**Proof:** Since $u > 0$ in $\Omega_T$ by Lemma 2.1, we denote $U = u^{1-q}$. Noting $0 < q < 1$, then hold $\frac{1}{1-q} > 1$. Hence, we obtain

$$\begin{aligned}
0 &= (U^{1/(1-q)})_t - \Delta(U^{1/(1-q)}) - U^{q/(1-q)} \int_0^t U^{p(x)/(1-q)} ds \\
&= (1-q)^{-1} U^{q/(1-q)} U_t - \left[(1-q)^{-1} U^{q/(1-q)} \Delta U + q(1-q)^{-2} U^{(2q-1)/(1-q)} |\nabla U|^2\right] - U^{q/(1-q)} \int_0^t U^{p(x)/(1-q)} ds \\
&\le (1-q)^{-1} U^{q/(1-q)} (U_t - \Delta U - (1-q)\int_0^t U^{p(x)/(1-q)} ds) \le (1-q)^{-1} U^{q/(1-q)} (U_t - \Delta U - (1-q)\int_0^t (U^{p^-/(1-q)} - 1) ds) \\
&= (1-q)^{-1} U^{q/(1-q)} (U_t - \Delta U - (1-q)\int_0^t (U^{p^-/(1-q)} ds + (1-q)t).
\end{aligned}$$

Let $\lambda$ be the first eigenvalue of Laplacian in $\Omega' \subset \Omega$ with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta\psi = \lambda\psi & in\ \Omega' \\ \psi(x) = 0 & on\ \partial\Omega', \end{cases}$$

and let $\psi$ be the corresponding positive eigenfunction. We can choose $\psi$ satisfies

$$\int_{\Omega'} \psi dx = 1$$

We will show that

$$U_t - \Delta U \ge (1-q)\int_0^t U^{p^-/(1-q)} ds - (1-q)t, \tag{19}$$

for any nonnegative nontrivial $U$.

Multiplying equation (19) by $\psi(x)$, integrating this by parts over $\Omega'$, and using Jensen's inequality, we get

$$\begin{aligned} \frac{d}{dt}\int_{\Omega'} U\psi dx - \int_{\Omega'} \Delta U \psi dx &\ge (1-q)\int_{\Omega'} \psi \int_0^t U^{p^-/(1-q)} ds dx - (1-q)t\int_{\Omega'} \psi dx \\ &\ge (1-q)\int_0^t (\int_{\Omega'} U\psi dx)^{p^-/(1-q)} ds - (1-q)t \end{aligned} \tag{20}$$

Put $Z = \int_{\Omega'} U\psi dx$, (20) can be written as

$$Z' + \lambda Z \ge (1-q)\int_0^t Z^{p^-/(1-q)} ds - (1-q)t.$$

According to the Comparison principle of ordinary differential equations (Theorem 5.1 in [13]), we conclude that $u = U^{1/(1-q)}$ blows up in finite time for any nonnegative nontrivial $u_0$.

## 4. Global existence of the solution

In this section, we shall prove the global existence of the solutions of (1).

**Theorem 4.1.** If $p^- + q > 1$, $q \ge 1$, then the solution of (1) exists globally for sufficiently small initial data $u_0$.

**Proof:** Let $\varphi(x)$ be the first eigenfunction of the following eigenvalue problem:

$$\begin{cases} -\Delta\varphi = \lambda_1 \varphi & in\ \Omega \\ \varphi(x) = 0 & on\ \partial\Omega, \end{cases}$$

where $\lambda_1$ is the first eigenvalue. Then it is known that $\varphi(x)$ is a non-negative smooth function on $\overline{\Omega}$ and $\varphi(x)$ is positive in $\Omega$. In particular, we shall normalize $\varphi(x)$ in sup-norm, that is,

$$\max_{x\in\Omega} \varphi(x) = 1,$$

For $q \ge 1$, put

$$V(x,t) = \frac{1}{(D+t)^{2\delta}} \varphi(x),$$

where $\delta > 0$ and $D > 1$ is a sufficiently large constant to be chosen later. We obtain

$$\begin{aligned} LV &\ge -2\delta(D+t)^{-2\delta-1}\varphi + \lambda_1 (D+t)^{-2\delta}\varphi - \varphi^{p(x)+q}(D+t)^{-2\delta q}\int_0^t (D+t)^{-2\delta p(x)} ds \\ &\ge -2\delta(D+t)^{-2\delta-1}\varphi + \lambda_1 (D+t)^{-2\delta}\varphi - \varphi^{p^-+q}(D+t)^{-2\delta q}\int_0^t (D+t)^{-2\delta p^-} ds \\ &\ge \varphi(D+t)^{-2\delta}\left[-2\delta(D+t)^{-1} + \lambda_1 - (D+t)^{-2\delta(q-1)}\int_0^t (D+t)^{-2\delta p^-} ds\right]. \end{aligned} \tag{21}$$

Since $q \geq 1$, we choose $\delta = (2(p^- + q - 1)^{-1}) > 0$ in (21). If $2p^-\delta < 1$ we have

$$LV \geq \varphi(D+t)^{-2\delta}\left[-2\delta(D+t)^{-1} + \lambda_1 - \frac{1}{-2\delta p^- + 1}(D+t)^{-2\delta(p^-+q-1)+1}\right]$$
$$\geq 0.$$

If $2p^-\delta = 1$,we have

$$LV \geq \varphi(D+t)^{-2\delta}\left[-2\delta(D+t)^{-1} + \lambda_1 - (D+t)^{-2\delta(q-1)}\ln(1+\frac{t}{D})\right]$$
$$\geq 0.$$

Combining the two cases, and the Theorem 2.1. This completes the proof.

**Theorem 4.2.** If $p^+ + q \leq 1$, the solution of (1) exists globally for all nonnegative $u_0$.

**Proof:** Let $V' = \beta e^{\alpha t}$, where $\alpha, \beta > 0$ are to be chosen lateral. By direct calculation, we get

$$\begin{aligned} LV' &= \alpha\beta e^{\alpha t} - \beta^{q+p(x)} e^{\alpha q t}\int_0^t e^{\alpha p(x) s} ds \\ &\geq \alpha\beta e^{\alpha t} - \beta^{q+p(x)} e^{\alpha q t}\int_0^t e^{\alpha p^+ s} ds \\ &\geq \alpha\beta e^{\alpha t} - \frac{\beta^{q+p(x)}}{\alpha p^+} e^{\alpha(p^+ + q)t} \\ &\geq \alpha\beta e^{\alpha t} - \frac{\beta^{q+p^+} + 1}{\alpha p^+} e^{\alpha(p^+ + q)t} \end{aligned}$$

Since $p^+ + q \leq 1$, if we choose $\beta = \max_{x\in\Omega} u_0$ and $\alpha = (p^+)^{-1/2}(\beta^{p^+ + q - 1} + \beta^{-1})^{1/2}$, then $V' = \beta e^{\alpha t}$ is a supersolution of (1). According to Theorem 2.2, the theorem is proved.

## 5. Conclusion

Semilinear parabolic equation with memory is widely studied, but parabolic equation with a variable exponent is difficult to study. In this paper, we use the comparison principle and some inequalities to discuss the blow-up of solutions or not for equation (1) in Dirichlet boundary. Next, we will continue to study the blow-up rate and blow-up time of the solution of the equation.

***Acknowledgement:*** The authors are thankful to the referees for their valuable comments for the improvement of the paper.

**Conflict of interest.** There is no conflict of interest among the authors.

**Authors' Contributions.** All authors equally contributed in the paper.

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