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Mex-Related Partition Identities

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Abstract. Recently, several authors have considered the smallest positive part missing from an integer partition, known as the minimal excludant or mex-function. After that, many properties and identities about the extended function $p_{A,a}(n)$ and $\overline{p}_{A,a}(n)$ appeared. In this paper, we deduce some new results about $p_{A,a}(n)$ and $\overline{p}_{A,a}(n)$. Moreover, we generalize the definition of minimal excludant to overpartitions, and obtain some identities.

Keywords: partition, mex-function, overpartition

AMS Mathematics Subject Classification (2010): 05A15, 05A17, 05A19

1. Introduction

The main purpose of this paper is to present some new identities related to minimal excludant functions on partitions. To this end, we first introduce some definitions and notations.

A *partition* [1] of n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. We write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ and call λ_i's the *parts* of λ. The *size* of λ is the sum of all parts, which is denoted by $|\lambda|$,and the *length* of λ is the number of parts, which is denoted by $\ell(\lambda)$. We also write a partition λ of n as $(1^{f_1} 2^{f_2} \dots n^{f_n})$ if a part i has multiplicity f_i for $1 \le i \le n$, where the superscript f_i can be neglected provided $f_i = 1$. The *conjugate* of λ is partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$, where $\lambda'_i = |\{\lambda_j : \lambda_j \ge i, 1 \le j \le \ell\}|$ for $1 \le i \le \lambda_1$, or λ' can be equivalently expressed as $(1^{\lambda_1 - \lambda_2} 2^{\lambda_2 - \lambda_3} \dots (\ell - 1)^{\lambda_{\ell-1} - \lambda_\ell} \ell^{\lambda_\ell})$. For example, the conjugate of partition (5,4,3,3) is (4,4,4,2,1).

The *Young diagram* of λ is a graphical interpretation of λ, which is defined by a left-justified collection of n boxes in l rows with λ_i boxes in row i. The Young diagram of partition $(6,6,3,2)$ is given in Figure 1.
One can also check that taking the conjugate of partition λ is equivalent to transpose the Young diagram of λ.

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is called a *distinct partition* if $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$, and an *odd* (resp. *even*) *partition* if λ_i is odd (resp. even) for all $1 \le i \le \ell$. The definition of the overpartition is introduced by Corteel and Lovejoy [9]. An *overpartition* is a partition of in which the final occurrence of a part may be overlined. For example, the 8 overpartitions of 3 are $(3), (\overline{3}), (2,1), (\overline{2},1), (2,\overline{1}), (\overline{2},\overline{1}), (1,1,1), (1,1,\overline{1})$.

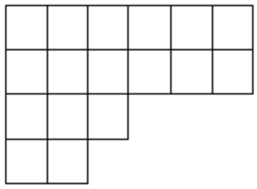

Figure 1:

The *minimal excludant* of a set of integers, abbreviated as *mex*, is the smallest positive integer not in the set. The following lists some examples of the mex applied to partitions:

$$mex(4,3,2) = 1, \quad mex(2,2,1) = 3.$$

The concept of "the smallest part that is not a summand" is first proposed by Andrews [2]. He and Newman [5] started using the term mex from combinatorial game theory. Given a partition λ, Andrews and Newman [6] defined $mex_{A,a}(\lambda)$ as the smallest positive integer congruent to a modulo A that is not a part of λ, and denote by $p_{A,a}(n)$ the number of partitions λ of n satisfying

$$mex_{A,a}(\lambda) \equiv a \pmod{2A},$$

and denote by $\overline{p}_{A,a}(n)$ the number of partitions λ of n satisfying

$$mex_{A,a}(\lambda) \equiv a + A \pmod{2A}.$$

Example 1.1. *Consider* $n = 4$, $A = 3$, *and* $a = 1$. *There are 5 partitions of 4, and the* $mex(\lambda)$ *and* $mex_{A,a}(\lambda)$ *are*

$$mex\big((4)\big) = 1, mex\big((3,1)\big) = 2, mex\big((2,2)\big) = 1, mex\big((2,1,1)\big) = 3,$$
$$mex((1,1,1,1)) = 2,$$
$$mex_{3,1}((4)) = 1, mex_{3,1}((3,1)) = 4, mex_{3,1}((2,2)) = 1, mex_{3,1}((2,1,1)) = 4,$$
$$mex_{3,1}((1,1,1,1)) = 4.$$

Since two partitions (4) and $(2,2)$ have $mex_{3,1}$ congruent to 1 modulo 6, and three partitions $(3,1), (2,1,1)$ and $(1,1,1,1)$ have $mex_{3,1}$ congruent to 4 modulo 6, we see $p_{3,1}(4) = 2$ and $\overline{p}_{3,1}(4) = 3$, respectively.

Through out this paper, we use $p(n)$ to denote the number of partitions of n, and set $p_{A,a}(0) = p(0) = 1$ and $p(n) = p_{A,a}(n) = \overline{p}_{A,a}(n) = 0$ for negative integer n. It is clear that

$$p(n) = p_{A,a}(n) + \overline{p}_{A,a}(n).$$

In [6, lemma 9] and [6, lemma 8], for $k > 0$, Andrews and Newman found that the generating functions of $p_{k,k}(n)$ and $p_{2k,k}(n)$ are

$$\sum_{n=0}^{\infty} p_{k,k}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \, q^{kn(n+1)/2}, \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} p_{2k,k}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{kn^2}, \tag{1.2}$$

where the q-*series* [11] notations are defined by

$$(a;q)_0 = 1,$$
$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad \text{for } n \geq 1,$$
$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad \text{for } |q|<1.$$

Recently, Dhar, Mukhopadhyay and Sarma [10] obtained a more general result that gives the generating functions of $p_{A,a}(n)$ and $\overline{p}_{A,a}(n)$ for all A and a.

Theorem 1.1. *(Dhar, Mukhopadhyay and Sarma [10]) For positive integers A and a, we have*

$$\sum_{n=0}^{\infty} p_{A,a}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\frac{An(n-1)}{2}+an}, \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_{A,a}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{An(n+1)}{2}+a(n+1)}, \tag{1.4}$$

Furthermore, they [10] obtained the generating function of the difference of $p_{2k,2k-i}(n)$ and $\overline{p}_{2k,i}(n)$.

Theorem 1.2. *(Dhar, Mukhopadhyay and Sarma [10]) For $2k > i > 0$, we have*

$$\sum_{n\geq 0}(p_{2k,2k-i}(n) - \overline{p}_{2k,i}(n))q^n = \frac{(q^i;q^{2k})_\infty(q^{2k-i};q^{2k})_\infty(q^{2k};q^{2k})_\infty}{(q;q)_\infty}.$$

Based on the works of Andrews-Newman [6] and Dhar-Mukhopadhyay-Sarma [10], we present the following two theorems that gives the identities related to $p_{A,a}(n)$ and $\overline{p}_{A,a}(n)$ for certain values of a and A.

Theorem 1.3. *For $n > 0$, we have*

$$p_{3,2}(n) = \overline{p}_{3,1}(n).$$

Theorem 1.4. *For $k > 0, n > 0$ and $r \geq s > 0$, we have*

$$\overline{p}_{rk,sk}(n) = \sum_{t=0}^{\infty} p(n - k(rt+s)(2t+1)) - \sum_{t=1}^{\infty} p(n - kt(2rt+2s-r)).$$

We also get some results that gives the identities related to the difference between $p_{A,a}(n)$ and $\overline{p}_{A,a}(n)$ for certain values of a and A, which we state as the following theorems.

Theorem 1.5. *We get*

$$\sum_{n=0}^{\infty} p_{10,3}(n)q^n - \sum_{n=0}^{\infty} p_{10,7}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(q;q^2)_n}.$$

Theorem 1.6. *Let $p_d(n)$ denote the number of partitions of n with distinct parts, and*

$p_d(0)=1$, *then for any nonnegative* n, *we obtain*

$$p_{6,4}(n)-\overline{p}_{6,2}(n)=p_d(n).$$

Theorem 1.7. *For* $k>0$, *we have*

$$\sum_{n=0}^{\infty}(p_{2k,k}(n)-\overline{p}_{2k,k}(n))q^n=\frac{\chi(-q^k)^2f(-q^{2k})}{f(-q)}.$$

Theorem 1.8. *We have*

$$\sum_{n=0}^{\infty}(p_{12,7}(n)-\overline{p}_{12,5}(n))q^n+\sum_{n=0}^{\infty}(p_{12,11}(n)-\overline{p}_{12,1}(n))q^{n+1}=\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

Except for the identities related to ordinary partitions, in this paper, we also generalize the definition of $\mathrm{mex}_{A,a}(\lambda)$ to overlined parts of overpartitions, and we get some interesting results.

For an overpartition λ, let $\widehat{mex}_{A,a}(\lambda)$ denote the smallest positive integer congruent to a modulo A that is not an overlined part of λ. Let $O_{A,a}(n)$ denote the number of overpartitions λ of n such that $\widehat{mex}_{A,a}(\lambda)\equiv a\ (mod\ 2A)$ and the non-overlined parts of λ are not congruent to a modulo A, we define $O_{A,a}(0)=1$.

Example 1.1. *The overlined parts of the overpartition* $\lambda=(\overline{6},5,4,\overline{3},2,2,1,\overline{1})$ *are* $\overline{6},\overline{3},\overline{1}$, *thus* $\widehat{mex}_{2,1}(\lambda)=\overline{5}$. *Let* $n=3$, $A=2$, *and* $a=1$ *in* $O_{A,a}(n)$, *then we have*

$$\widehat{mex}_{2,1}(3)=\overline{1},\quad \widehat{mex}_{2,1}(\overline{3})=\overline{1},\quad \widehat{mex}_{2,1}(2,1)=\overline{1},\quad \widehat{mex}_{2,1}(\overline{2},1)=\overline{1},$$
$$\widehat{mex}_{2,1}(2,\overline{1})=\overline{3},\quad \widehat{mex}_{2,1}(\overline{2},\overline{1})=\overline{3},\quad \widehat{mex}_{2,1}(1,1,1)=\overline{1},\quad \widehat{mex}_{2,1}(1,1,\overline{1})=\overline{3}.$$

Only $\widehat{mex}_{2,1}(\overline{3})=\overline{1}$ satisfies the condition, hence $O_{2,1}(3)=1$.

Theorem 1.9. *For* $n\geq 0$, *we get*

$$p_{1,1}(n)\equiv O_{1,1}(n)\ (mod\ 2).$$

Theorem 1.10. *Let* $E_{A,a}(n)$ *denote the number of even partitions* π *of* n *such that* $mex_{A,a}(\pi)\equiv a(mod\ 2A)$, *and define* $E_{A,a}(0)=1$. *For* $n\geq 0$, *we get*

$$E_{2,2}(n)\equiv O_{2,2}(n)\ (mod\ 2).$$

In addition to generalizing the minimal excludant function to the overlined parts of overpartitions, we also extend this definition to non-overlined parts. Let $\overline{mex}_{A,a}(\lambda)$ be the smallest integer congruent to a modulo A that is not a non-overlined part in the overpartition λ, and define

$$op_{A,a}(n)=\left|\{\lambda|\overline{mex}_{A,a}(\lambda)\equiv a\ \ (\mathrm{mod}\ 2A)\}\right|,$$
$$\overline{op}_{A,a}(n)=\left|\{\lambda|\overline{mex}_{A,a}(\lambda)\equiv a+A\ \ (\mathrm{mod}\ 2A)\}\right|.$$

Therefore, it is easy to get

$$op_{A,a}(n)+\overline{op}_{A,a}(n)=\overline{p}(n),\tag{1.5}$$

where $\overline{p}(n)$ is the number of overpartitions of n, and we set $\overline{p}(0)=1, op_{A,a}(0)=1, \overline{op}_{A,a}(0)=0$. Let $D_{A,a}(n)$ to be the number of overpartitions λ of n, where $\overline{mex}_{A,a}(\lambda)$ are congruent to a modulo $2A$ and the overlined parts are not congruent to

a modulo A, we define $D_{A,a}(0) = 1$.

Example 1.2. *The non-overlined parts of the overpartition* $\lambda = (\overline{6}, 5,4, \overline{3}, 2,2,1, \overline{1})$ *are* 5,4,2,2,1, *thus* $\overline{mex}_{2,1}(\lambda) = 3$. *Let* $n = 3$, $A = 2$, *and* $a = 1$, *we have*

$\overline{mex}_{2,1}((3)) = 1$, $\overline{mex}_{2,1}((\overline{3})) = 1$, $\overline{mex}_{2,1}((2,1)) = 3$, $\overline{mex}_{2,1}((\overline{2}, 1)) = 3$,
$\overline{mex}_{2,1}((2, \overline{1})) = 1$, $\overline{mex}_{2,1}((\overline{2}, \overline{1})) = 1$, $\overline{mex}_{2,1}((1,1,1)) = 3$, $\overline{mex}_{2,1}((1,1, \overline{1})) = 3$.

Hence $op_{2,1}(3) = 4$, $\overline{op}_{2,1}(3) = 4$, and $D_{2,1}(3) = 1$.

After generalizing the definition minimal excludant function to overpartitions, we also get some identities related to $op_{A,a}(n)$.

Theorem 1.11. *For* $n \geq 0, A > 0, a > 0$, *we obtain*

$$op_{A,a}(n) \equiv \begin{cases} 1 \pmod{2}, & if\ n = \frac{Ak(k-1)}{2} + ak\ for\ some\ k \geq 0, \\ 0 \pmod{2}, & otherwise. \end{cases}$$

Theorem 1.12. *For* $n \geq 0$, *we obtain*

$$op_{3,1}(n) + op_{3,2}(n) = p_d(n) + \overline{p}(n).$$

Theorem 1.13. *Let* $f_k(n)$ *be the number of the partitions of* n *whose parts congruent to* $\pm k$, $2k$ *modulo* $4k$, *and we define* $f_k(0) = 1$. *For* $n \geq 0, k > 0$, *we obtain*

$$op_{k,k}(n) \equiv f_k(n) \quad (mod\ \ 2).$$

Theorem 1.14. *For* $n \geq 0, k > 0$, *we obtain*

$$D_{k,k}(n) \equiv O_{k,k}(n) \quad (mod\ \ 2).$$

This paper is organized as follows. Section 2 is dedicated to state the theorems that we frequently use in the proof. In Section 3, we will give the proof of the main theorems related to $p_{A,a}(n)$.

2. Preliminary results

In this section, we introduce some theorems that will be used to prove the main results of this paper.

Theorem 2.1. *(Euler [1, p.5]) We have*

$$\sum_{n=0}^{\infty} p_d(n) q^n = (-q; q)_\infty = \frac{1}{(q; q^2)_\infty}.$$

Theorem 2.2. *(Euler's pentagonal number theorem [1, p.11]) We have*

$$(q; q)_\infty = 1 + \sum_{n=1}^{\infty} (-1)^n q^{n(3n-1)/2}(1 + q^n)$$
$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

Theorem 2.3. *(Euler [1, p.19]) We have*

$$\sum_{n\geq 0}\frac{t^n q^{n(n-1)/2}}{(q;q)_n}=(-t;q)_\infty.$$

Theorem 2.4. *(Gauss[1, p.23]) We have*

$$\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Theorem 2.5. *([3, p.158]) We have*

$$\sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n}.$$

Theorem 2.6. *(The quintuple product identity, [7, p.18,Theorem 1.3.17]).*

$$\sum_{n=-\infty}^{\infty} q^{3n^2+n}(z^{3n}q^{-3n}-z^{-3n-1}q^{3n+1})$$
$$=(q^2;q^2)_\infty(qz;q^2)_\infty(q/z;q^2)_\infty(z^2;q^4)_\infty(q^4/z^2;q^4)_\infty.$$

3. Proofs of the main results

In this section, we present the proofs of the main theorems of this paper. For convenience, in the rest of this paper, we let $\mathcal{P}$ and $\mathcal{D}$ denote the set of partitions and the set of distinct partitions, respectively. Specifically, let $\mathcal{P}(n)$ and $\mathcal{D}(n)$ denote the set of partitions of n and the set of distinct partitions of n, respectively.

3.1. Proofs of Theorems 1.3–1.10

By setting $A\to rk$ and $a\to sk$ in (1.3) and (1.4) respectively, we obtain the following lemmas.

Lemma 3.1. *For $k>0$, $r\geq s>0$, the generating function of $p_{rk,sk}(n)$ is*

$$\sum_{n=0}^{\infty} p_{rk,sk}(n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{k(nr+2s-r)n/2}.$$

Lemma 3.2. *For $k>0$, $r\geq s>0$, the generating function of $\overline{p}_{rk,sk}(n)$ is*

$$\sum_{n=0}^{\infty}\overline{p}_{rk,sk}(n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{k(nr+2s)(n+1)/2}.$$

In the following part, we will give several proofs to Theorem 1.3.

First proof of Theorem 1.3. In [10, Theorem3.1], we know $p_{3,1}(n)+p_{3,2}(n)=p(n)$ for $n>0$, and by the definition of $p_{A,a}(n)$ and $\overline{p}_{A,a}(n)$, one has $p_{3,1}(n)+\overline{p}_{3,1}(n)=p(n)$ for $n>0$, then

$$p_{3,2}(n)=\overline{p}_{3,1}(n),$$

for $n > 0$.

Second proof of Theorem 1.3. In [3, p.232,Entry 9.4.1], we know

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(-aq;q)_n} = \sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}). \tag{3.1}$$

Let $k \to 1, r \to 3, s \to 2$ in Lemma 3.1, and $k \to 1, r \to 3, s \to 1$ in Lemma 3.2, respectively, and $p_{3,2}(0) = 1$, $\overline{p}_{3,1}(0) = 0$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{3,2}(n)q^n - \sum_{n=0}^{\infty} \overline{p}_{3,1}(n)q^n \\
&= \frac{1}{(q;q)_\infty}\left(\sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} - \sum_{n=0}^{\infty} (-1)^n q^{(3n+2)(n+1)/2}\right) \\
&= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2}(1-q^{2n+1}) \\
&= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} \quad \big(\text{let } a=-1 \text{ in } (3.1)\big) \\
&= \frac{1}{(q;q)_\infty}(q;q)_\infty \quad (\text{let } t=-q \text{ in Theorem 2.3 }) \\
&= 1.
\end{aligned} \tag{3.2}$$

Comparing the coefficients of q^n on both sides, we complete the proof.

Third proof of Theorem 1.3. According to (3.2), we get

$$\sum_{n=0}^{\infty} p_{3,2}(n)q^n - \sum_{n=0}^{\infty} \overline{p}_{3,1}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q;q)_n}. \tag{3.3}$$

Now, we will prove that the right-hand side of (3.3) equals 1 by a combinatorial involution.

Denote by $\mathcal{S}$ the set of pairs (λ,μ) such that $\lambda \in \mathcal{P}$, $\mu \in \mathcal{D}$, we obtain

$$\sum_{(\lambda,\mu)\in\mathcal{S}} (-1)^{\ell(\mu)} q^{|\lambda|+|\mu|} = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n}. \tag{3.4}$$

Let λ_m (resp.μ_m) be the largest part in λ (resp.μ). For convenience, we define $\lambda_m = 0$ (resp. $\mu_m = 0$) if there is no parts in λ (resp.μ). We compare λ_m and μ_m.

Case 1: If $\lambda_m \le \mu_m$ and $\mu_m > 0$, we obtain μ^* and λ^* by removing μ_m from μ and adding it to λ. Obviously, the new pair $(\lambda^*,\mu^*) \in \mathcal{S}$ with $|\lambda| + |\mu| = |\lambda^*| + |\mu^*|$, and $\ell(\mu^*) = \ell(\mu) - 1$, which endows (λ^*,μ^*) with the opposite sign compared to (λ,μ).

Case 2: If $\lambda_m > \mu_m$, we remove λ_m from λ and add it to μ to get new partitions λ^* and μ^* respectively. This new pair (λ^*,μ^*) also inherits the size of (λ,μ) but have a different sign to (λ,μ).

Case 3: If $\lambda_m = 0, \mu_m = 0$, we do nothing.

Consequently, the partition pairs in Case 1 and Case 2 cancel each other out, and there remains only Case 3, which contains partition pair $(\lambda,\mu) = (\emptyset,\emptyset) \in \mathcal{S}$. Then the right-hand side of (3.4) is 1. Thus, we know $p_{3,2}(n) = \overline{p}_{3,1}(n)$ for $n > 0$.

Proof of Theorem 1.4. In Lemma 3.2, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{rk,sk}(n)q^n$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{k(rn+2s)(n+1)/2} = \sum_{n=0}^{\infty} p(n)q^n \sum_{n=0}^{\infty} (-1)^n q^{k(rn+2s)(n+1)/2}$$
$$= \sum_{n=0}^{\infty} p(n)q^n \left(\sum_{t=0}^{\infty} q^{k(rt+s)(2t+1)} - \sum_{t=1}^{\infty} q^{kt(2rt+2s-r)} \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{t=0}^{\infty} p\big(n - k(rt+s)(2t+1)\big) - \sum_{t=1}^{\infty} p\big(n - kt(2rt+2s-r)\big) \right) q^n,$$

Comparing the coefficients of q^n on both sides, we complete the proof.

In the next part, we proceed to give the proofs of Theorems 1.5–1.10.

Proof of Theorem 1.5. In [4, p.90], we know

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(q;q^2)_n (1+aq^{4n+1}) a^{3n} q^{5n^2+n}}{(-aq^2;q^2)_n}. \tag{3.5}$$

Let $a \to -q^{-1}$ in (3.5), we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q^2)_n} = \sum_{n=0}^{\infty} (1-q^{4n})(-1)^n q^{5n^2-2n}. \tag{3.6}$$

Next, let $k \to 1, r \to 10, s \to 3$ and $k \to 1, r \to 10, s \to 7$ in Lemma 3.1, respectively, we see

$$\sum_{n=0}^{\infty} p_{10,3}(n)q^n - \sum_{n=0}^{\infty} p_{10,7}(n)q^n$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(5n-2)} (1-q^{4n})$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q^2)_n}. \quad \text{(by (3.6))}$$

Proof of Theorem 1.6. Let $z = -q$ in Theorem 2.6 , we have

$$\text{LHS} = \sum_{n=-\infty}^{\infty} q^{3n^2+n}((-q)^{3n}q^{-3n} - (-q)^{-3n-1}q^{3n+1})$$
$$= 2\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+n}$$
$$= 2\sum_{n=0}^{\infty} (-1)^n q^{3n^2+n} + 2\sum_{n=1}^{\infty} (-1)^n q^{3n^2-n},$$
$$\text{RHS} = (q^2;q^2)_\infty(-q^2;q^2)_\infty(-1;q^2)_\infty(q^2;q^4)_\infty(q^2;q^4)_\infty$$
$$= 2(q^2;q^2)_\infty(-q^2;q^2)_\infty^2(q^2;q^4)_\infty^2$$
$$= 2(q;q)_\infty(-q;q)_\infty^3(q;q^2)_\infty^2$$
$$= 2(q;q)_\infty(-q;q)_\infty, \quad \text{(by Theorem 2.1)}$$

thus

$$\sum_{n=0}^{\infty}(-1)^n q^{3n^2+n}+\sum_{n=1}^{\infty}(-1)^n q^{3n^2-n}=(q;q)_\infty(-q;q)_\infty. \tag{3.7}$$

Let $k=2, r=3, s=2$ in Lemma 3.1 and $k=2, r=3, s=1$ in Lemma 3.2, respectively, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{6,4}(n)q^n-\sum_{n=0}^{\infty}\overline{p}_{6,2}(n)q^n\\
&=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+n}-\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{3n^2-n}\\
&=\frac{1}{(q;q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n q^{3n^2+n}+\sum_{n=1}^{\infty}(-1)^n q^{3n^2-n}\right)\\
&=\frac{1}{(q;q)_\infty}(q;q)_\infty(-q;q)_\infty \qquad \text{(by (3.7))}\\
&=(-q;q)_\infty\\
&=\sum_{n=0}^{\infty}p_d(n)q^n.
\end{aligned}$$

Proof of Theorem 1.7. In [3, p.21], we have

$$\chi(-q)=(q;q^2)_\infty \tag{3.8}$$

and

$$f(-q)=(q;q)_\infty. \tag{3.9}$$

Let $i=k$ in Theorem 1.2, one has

$$\begin{aligned}
&\sum_{n=0}^{\infty}(p_{2k,k}(n)-\overline{p}_{2k,k}(n))q^n\\
&=\frac{(q^{2k};q^{2k})_\infty(q^k;q^{2k})_\infty(q^k;q^{2k})_\infty}{(q;q)_\infty}\\
&=\frac{\chi(-q^k)^2 f(-q^{2k})}{f(-q)}.\left(\text{let } q\to q^k \text{ in (3.8) and } q\to q^{2k} \text{ in (3.9)}\right)
\end{aligned}$$

Corollary 3.3. *We have*

$$\sum_{n=0}^{\infty}\left(p_{6,3}(n)-\overline{p}_{6,3}(n)\right)q^n=\frac{\phi(-q^3)}{f(-q)}.$$

Proof: In [3, p,21], we get

$$\phi(q)=(-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{3.10}$$

let $i=k=3$ in Theorem 1.2, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(p_{6,3}(n)-\overline{p}_{6,3}(n)\right)q^n&=\frac{(q^3;q^6)_\infty(q^3;q^6)_\infty(q^6;q^6)_\infty}{(q;q)_\infty}\\
&=\frac{\phi(-q^3)}{f(-q)}. \qquad \left(\text{let } q\to -q^3 \text{ in (3.10)}\right)
\end{aligned}$$

Corollary 3.4. *We obtain*

$$\sum_{n=0}^{\infty}(p_{8,4}(n)-\overline{p}_{8,4}(n))q^n=\frac{\chi(q^2)}{\chi(-q)}.$$

Proof: Let $i=k=4$ in Theorem 1.2, one has

$$\sum_{n=0}^{\infty}(p_{8,4}(n)-\overline{p}_{8,4}(n))q^n=\frac{(q^4;q^8)_\infty(q^4;q^8)_\infty(q^8;q^8)_\infty}{(q;q)_\infty}$$
$$=\frac{(-q^2;q^4)_\infty}{(q;q^2)_\infty}$$
$$=\frac{\chi(q^2)}{\chi(-q)}.\qquad (\text{let } q\to -q^2 \text{ in } (3.8))$$

Proof of Theorem 1.8. In [8], we know

$$\sum_{n=-\infty}^{\infty}x^{(6n+1)n}-x\sum_{n=-\infty}^{\infty}x^{(6n+5)n}=(x;x)_\infty. \tag{3.11}$$

By Lemma 3.1 and Lemma 3.2, we have

$$\sum_{n=0}^{\infty}p_{12,7}(n)q^n-\sum_{n=0}^{\infty}\overline{p}_{12,5}(n)q^n+q\sum_{n=0}^{\infty}p_{12,11}(n)q^n-q\sum_{n=0}^{\infty}\overline{p}_{12,1}(n)q^n$$
$$=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{(6n+1)n}-\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{(6n-1)n}$$
$$+\frac{q}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{(6n+5)n}-\frac{q}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{(6n-5)n}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{(6n+1)n}+\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{-1}(-1)^nq^{(6n+1)n}$$
$$+\frac{q}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{(6n+5)n}+\frac{q}{(q;q)_\infty}\sum_{n=-\infty}^{-1}(-1)^nq^{(6n+5)n}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{(6n+1)n}+\frac{q}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{(6n+5)n}$$
$$=\frac{(-q;-q)_\infty}{(q;q)_\infty}\qquad \big(\text{let } x=-q \text{ in } (3.11)\big)$$
$$=\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

Proof of Theorem 1.9. Let $k=1$ in (1.1), we have

$$\sum_{n=0}^{\infty}p_{1,1}(n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)/2}$$
$$\equiv\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{n(n+1)/2}\pmod 2$$

$$= \frac{1}{(q;q)_\infty} \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \quad \text{(by Theorem 2.4)}$$
$$= \frac{(-q;q)_\infty}{(q;q^2)_\infty}$$
$$= (-q;q)_\infty^2 \quad \text{(by Theorem 2.1)}$$
$$\equiv (q;q)_\infty^2 \pmod 2. \tag{3.12}$$

By the definition of $O_{k,k}(n)$, we get

$$\sum_{n=0}^{\infty} O_{k,k}(n)q^n$$
$$= \frac{(-q;q)_\infty (q^k;q^k)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{kn(2n+1)}}{(-q^k;q^k)_{2n+1}}$$
$$= \frac{(-q;q)_\infty (q^k;q^k)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{kn(n+1)/2}}{(-q^k;q^k)_n} \quad \text{(let } q \to q^k \text{ in Theorem 2.5)}$$
$$\equiv (q^k;q^k)_\infty \sum_{n=0}^{\infty} \frac{q^{kn(n+1)/2}}{(q^k;q^k)_n} \pmod 2.$$
$$= (q^k;q^k)_\infty (-q^k;q^k)_\infty \quad \text{(let } t \to q^k \text{ and } q \to q^k \text{ in Theorem 2.3)}$$
$$\equiv (q^k;q^k)_\infty^2 \pmod 2. \tag{3.13}$$

Let $k = 1$ in (3.13), we get

$$\sum_{n=0}^{\infty} O_{1,1}(n)q^n \equiv (q;q)_\infty^2 \pmod 2. \tag{3.14}$$

Combining (3.12) and (3.14), we get

$$p_{1,1}(n) \equiv O_{1,1}(n) \pmod 2,$$

which completes the proof.

Proof of Theorem 1.10. By the definition of $E_{2,2}(n)$, we have

$$\sum_{n=0}^{\infty} E_{2,2}(n)q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{2+4+6+\cdots+4n}(1-q^{4n+2})$$
$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \equiv \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)} \pmod 2$$
$$= \frac{1}{(q^2;q^2)_\infty} \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty} \quad \text{(let } q \to q^2 \text{ in Theorem 2.4)}$$
$$\equiv \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^2} \pmod 2$$
$$\equiv (q;q)_\infty^4 \pmod 2. \quad \text{(by Theorem 2.1)} \tag{3.15}$$

Let $k = 2$ in (3.13), we obtain

$$\sum_{n=0}^{\infty} O_{2,2}(n)q^n \equiv (q^2;q^2)_\infty^2 \pmod 2$$
$$\equiv (q;q)_\infty^4 \pmod 2. \tag{3.16}$$

Combining (3.15) and (3.16), we get

$$E_{2,2}(n) \equiv O_{2,2}(n) \pmod{2}.$$

which completes the proof.

3.2. Proofs of Theorems 1.11– 1.14

In this subsection, we shall prove some identities related to overpartitions.

Comparing the definitions of $op_{A,a}(n)$ and $\overline{op}_{A,a}(n)$ with the definitions of $p_{A,a}(n)$ and $\overline{p}_{A,a}(n)$, we can directly deduce the following lemma.

Lemma 3.5. *We have*

$$\sum_{n=0}^{\infty} op_{A,a}(n)q^n = (-q;q)_\infty \sum_{n=0}^{\infty} p_{A,a}(n)q^n \tag{3.17}$$

and

$$\sum_{n=0}^{\infty} \overline{op}_{A,a}(n)q^n = (-q;q)_\infty \sum_{n=0}^{\infty} \overline{p}_{A,a}(n)q^n. \tag{3.18}$$

By (1.5), we have

$$\sum_{n=0}^{\infty} op_{A,a}(n)q^n + \sum_{n=0}^{\infty} \overline{op}_{A,a}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

where $\frac{(-q;q)_\infty}{(q;q)_\infty}$ is the generating function of overpartition [9].

Proof of Theorem 1.11. By (3.17), we get

$$\begin{aligned}\sum_{n=0}^{\infty} op_{A,a}(n)q^n &= (-q;q)_\infty \sum_{n=0}^{\infty} p_{A,a}(n)q^n \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} (-1)^n q^{\frac{An(n-1)}{2}+an} \qquad (\text{by } (1.3)) \\ &\equiv \sum_{n\geq 0} q^{\frac{An(n-1)}{2}+an} \pmod{2},\end{aligned}$$

which completes the proof.

Proof of Theorem 1.12. Let $A=3,\ a=1$ and $A=3, a=2$ in (3.17), respectively, and let $k=1, r=3, s=1$ and $k=1, r=3, s=2$ in Lemma 3.1, respectively, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} op_{3,1}(n)q^n + \sum_{n=0}^{\infty} op_{3,2}(n)q^n \\ &= (-q;q)_\infty \left(\sum_{n=0}^{\infty} p_{3,1}(n)q^n + \sum_{n=0}^{\infty} p_{3,2}(n)q^n\right) \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(3n-1)/2} + \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} + 1\right)\end{aligned}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty}((q;q)_\infty + 1) \quad \text{(by Theorem 2.2)}$$
$$= (-q;q)_\infty + \frac{(-q;q)_\infty}{(q;q)_\infty}$$
$$= \sum_{n=0}^{\infty} p_d(n)q^n + \sum_{n=0}^{\infty} \overline{p}(n)q^n.$$

Comparing the coefficients of q^n on both sides, we complete the proof.

Proof of Theorem 1.13. Let $A = k,\ a = k$ in (3.17), we get

$$\sum_{n=0}^{\infty} op_{k,k}(n)q^n = (-q;q)_\infty \sum_{n=0}^{\infty} p_{k,k}(n)q^n$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{kn(n+1)/2} \quad \text{(by (1.1))}$$
$$\equiv \sum_{n=0}^{\infty} q^{kn(n+1)/2} \pmod 2$$
$$= \frac{(q^{2k};q^{2k})_\infty}{(q^k;q^{2k})_\infty} \left(\text{let } q \to q^k \text{ in Theorem 2.4}\right)$$
$$= (q^{2k};q^{2k})_\infty(-q^k;q^k)_\infty \quad \text{(let } q \to q^k \text{ in Theorem 2.1}$$
$$\equiv (q^k;q^k)_\infty^3 \pmod 2. \tag{3.19}$$

By the definition of $f_k(n)$, we get

$$\sum_{n=0}^{\infty} f_k(n)q^n = \frac{1}{(q^k;q^{4k})_\infty(q^{2k};q^{4k})_\infty(q^{3k};q^{4k})_\infty}$$
$$= \frac{(q^{4k};q^{4k})_\infty}{(q^k;q^k)_\infty}$$
$$\equiv \frac{(q^k;q^k)_\infty^4}{(q^k;q^k)_\infty} \pmod 2$$
$$= (q^k;q^k)_\infty^3. \tag{3.20}$$

Combining (3.19) and (3.20), we arrive at

$$op_{k,k}(n) \equiv f_k(n) \pmod 2,$$

which completes the proof.

Proof of Theorem 1.14. For $k > 0$, by the definition of $D_{k,k}(n)$, we get

$$\sum_{n=0}^{\infty} D_{k,k}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{k+2k+\cdots+2nk}\left(1 - q^{(2n+1)k}\right)}{(q;q)_\infty} \frac{(-q;q)_\infty}{(-q^k;q^k)_\infty}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty(-q^k;q^k)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{kn(n+1)/2}$$
$$\equiv \frac{1}{(-q^k;q^k)_\infty} \sum_{n=0}^{\infty} q^{kn(n+1)/2} \pmod 2$$
$$= \frac{1}{(-q^k;q^k)_\infty} \frac{(q^{2k};q^{2k})_\infty}{(q^k;q^{2k})_\infty} \quad \left(\text{let } q \to q^k \text{ in Theorem 2.4}\right)$$

$$= (q^{2k};q^{2k})_\infty \quad (\text{let } q \to q^k \text{ in Theorem 2.1})$$
$$\equiv (q^k;q^k)_\infty^2 \pmod{2}. \tag{3.21}$$

Combining (3.21) and (3.13), we get

$$D_{k,k}(n) \equiv O_{k,k}(n) \pmod{2},$$

which completes the proof.

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