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The Diophantine Equations $6^x + 4^y = z^2$ and $24^x + 4^y = z^2$

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Abstract. This research aims to study and find all solutions of two Diophantine equations $6^x + 4^y = z^2$ and $24^x + 4^y = z^2$ where *x*, *y* and *z* are non-negative integers, by using elementary concepts of number theory and Mihăilescu's theorem. The research results found that the Diophantine equation $6^x + 4^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (2,3,10)$. The Diophantine equation $24^x + 4^y = z^2$ has exactly two non-negative integer solutions (x, y, z), which are $(1, 0, 5)$ and $(2, 5, 40)$.

Keywords: Diophantine equation; Mihăilescu's theorem; non-negative integer solution

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In 2011, Suvarnamani, Singta and Chotchaisthit [1] showed that the Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. In 2012, Chotchaisthit [2] presented all solutions of the Diophantine equation $4^x + p^y = z^2$, where x, y, z are non-negative integers and p is a prime number. In the same year, Peker and Cenberci [3] studied the Diophantine equation $\left(4^n\right)^x + p^y = z^2$, where x, y, z are non-negative integers, p is odd prime and n is a positive integer. In 2014, Sroysang [4] proved that the Diophantine equation $4^x + 10^y = z^2$ has no non-negative integer solution. In 2020, Kambheera and Kumpapan [5] showed that $(x, y, z) = (1,0,4)$ is a non-negative integer solution of the Diophantine equation $15^x + 4^y = z^2$.

After that, in 2021, Behera and Panda [6] proved that the Diophantine equation $4^x + 12^y = z^2$ has no non-negative integer solution. Orosram, Niratsrok and Sukkharin [7] proved that the Diophantine equation $4^x + n^y = z^2$, where n is a positive integer with $n \equiv 1 \pmod{15}$, has no solution in non-negative integers x, y and z. Meanwhile, Saranya and Yashvandhini [8] proved that $(x, y, z) = (1,1,7)$ is a solution of the Diophantine equation $25^x + 24^y = z^2$. Borah and Dutta [9] proved that the Diophantine equation

$5^x + 24^y = z^2$ has exactly one positive integer solution $(x, y, z) = (2,1,7)$. Dutta and Borah [10] also studied the non-negative integer solutions of the Diophantine equation $n^x + 24^y = z^2$, where n is a positive integer with $n \equiv 3 \pmod 4$. Recently, Butsan, Phrommarat and Kanchai [11] proved that the Diophantine equation $21^x + 4^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1,1,5)$.

From the above research study, we are interested in finding all non-negative integer solutions (x, y, z) of two Diophantine equations $6^x + 4^y = z^2$ and $24^x + 4^y = z^2$, by using elementary concepts of number theory and Mihăilescu's theorem.

Theorem 1.1. (Mihăilescu's theorem) [12] The Diophantine equation $a^x - b^y = 1$ has the unique integer solution $(a, b, x, y) = (3, 2, 2, 3)$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.

2. Main results

In this section, we present our results.

Theorem 2.1. The Diophantine equation $6^x + 4^y = z^2$ has a unique non-negative integer solution (x, y, z), which is $(2, 3, 10)$.

Proof: Let x, y and z are non-negative integers such that

$$6^x + 4^y = z^2. \tag{1}$$

Therefore $(z - 2^y)(z + 2^y) = 2^x \cdot 3^x$ and so there exist two non-negative integers u and v such that

$$z - 2^y = 2^u \cdot 3^v \tag{2}$$

and

$$z + 2^y = 2^{x-u} \cdot 3^{x-v}. \tag{3}$$

From Equations (2) and (3), we have

$$2^{y+1} = 2^{x-u} \cdot 3^{x-v} - 2^u \cdot 3^v. \tag{4}$$

Assume that $x - v > 0$ and $v > 0$. Then $3 \mid (2^{x-u} \cdot 3^{x-v} - 2^u \cdot 3^v)$. From Equation (4), we obtain that $3 \mid 2^{y+1}$, which is impossible. Thus, $x - v = 0$ or $v = 0$.

Case 1. $x - v = 0$. Then $x = v$. From Equation (4), we get

$$2^{y+1} = 2^u (2^{x-2u} - 3^x). \tag{5}$$

Since $\gcd(2^u, 2^{x-2u} - 3^x) = 1$, we have $y + 1 = u$ and

$$2^{x-2u} - 3^x = 1. \tag{6}$$

Case 1.1 $x = 0$. From Equation (6), we get $2^{-2u} = 2$ and so $-2u = 1$, which is impossible.

Case 1.2 $x = 1$. From Equation (6), we obtain that $2^{1-2u} = 2^2$ and so $1 - 2u = 2$, which is impossible.

Case 1.3 $x > 1$. From Equation (6), it easy to check that $x - 2u > 1$. It contradicts Theorem 1.1.

Case 2. $v = 0$. From Equation (4), we get

$$2^{y+1} = 2^{x-u} \cdot 3^x - 2^u. \quad (7)$$

Case 2.1 $x - u < u$. From Equation (7), we have

$$2^{y+1} = 2^{x-u}\left(3^x - 2^{2u-x}\right). \quad (8)$$

Since $\gcd\left(2^{x-u}, 3^x - 2^{2u-x}\right) = 1$, it follows that $y + 1 = x - u$ and

$$3^x - 2^{2u-x} = 1. \quad (9)$$

Then $x > 1$ and $2u - x > 1$. By Theorem 1.1, we have $x = 2$ and $2u - x = 3$. Thus $2u = 5$. This is impossible.

Case 2.2 $x - u > u$. From Equation (7), we obtain that $2^{y+1} = 2^u\left(2^{x-2u} \cdot 3^x - 1\right)$. Since $\gcd\left(2^u, 2^{x-2u} \cdot 3^x - 1\right) = 1$, it follows that $y + 1 = u$ and $2^{x-2u} \cdot 3^x = 2$. Then $x - 2u = 1$ and $x = 0$. It implies that $-2u = 1$. This is impossible.

Case 2.3 $x - u = u$. From Equation (7), we have

$$3^x - 2^{y-u+1} = 1. \quad (10)$$

Then $x > 1$ and $y - u + 1 > 1$. By Theorem 1.1, we get $x = 2$ and $y - u + 1 = 3$. It follows that $u = 1$ and $y = 3$. From Equation (2), we have $z = 10$. Hence $(x, y, z) = (2, 3, 10)$.

Theorem 2.2. The Diophantine equation $24^x + 4^y = z^2$ has exactly two non-negative integer solutions (x, y, z), which are $(1, 0, 5)$ and $(2, 5, 40)$.

Proof: Let x, y and z are non-negative integers such that

$$24^x + 4^y = z^2. \quad (11)$$

Case 1. $x \le y$. From Equation (11), we have

$$6^x + 4^{y-x} = \left(\frac{z}{2^x}\right)^2. \quad (12)$$

By Theorem 2.1, we have $\left(x, y - x, \frac{z}{2^x}\right) = (2, 3, 10)$. Then $x = 2, y - x = 3$ and $\frac{z}{2^x} = 10$. Hence $(x, y, z) = (2, 5, 40)$.

Case 2. $x > y$. From Equation (11), we have

$$4^{x-y} \cdot 6^x + 1 = \left(\frac{z}{2^y}\right)^2. \quad (13)$$

Let $w = \frac{z}{2^y}$. From Equation (13), we have

$$(w-1)(w+1) = 2^{3x-2y} \cdot 3^x. \quad (14)$$

Then there exist two non-negative integers u and v such that

$$w-1 = 2^u \cdot 3^v \quad (15)$$

and

$$w+1 = 2^{3x-2y-u} \cdot 3^{x-v}. \quad (16)$$

From Equations (15) and (16), we get

$$2 = 2^{3x-2y-u} \cdot 3^{x-v} - 2^u \cdot 3^v. \quad (17)$$

Assume that $x-v>0$ and $v>0$. It implies that $3 \mid \left(2^{3x-2y-u} \cdot 3^{x-v} - 2^u \cdot 3^v\right)$. From Equation (17), we get $3 \mid 2$. This is impossible. Thus $x-v=0$ or $v=0$.

Case 2.1 $x-v=0$. From Equation (17), we have

$$2 = 2^u\left(2^{3x-2y-2u} - 3^x\right). \quad (18)$$

Since $\gcd\left(2^u, 2^{3x-2y-2u} - 3^x\right) = 1$, we obtain that $u=1$ and

$$2^{3x-2y-2u} - 3^x = 1. \quad (19)$$

Case 2.1.1 $x=0$. From Equation (19), we have $2^{-2y-2u} = 2$. Then $2(-y-u) = 1$ and so $2 \mid 1$. This is impossible.

Case 2.1.2 $x=1$. From Equation (19), we have $2^{3-2y-2u} = 2^2$. Then $3-2y-2u=2$ or $3 = 2(y+u+1)$. Therefore $2 \mid 3$. This is impossible.

Case 2.1.3 $x>1$. From Equation (19), it easy to check that $3x-2y-2u>1$. It contradicts Theorem 1.1.

Case 2.2 $v=0$. From Equation (17), we have

$$2 = 2^{3x-2y-u} \cdot 3^x - 2^u. \quad (20)$$

Case 2.2.1 $3x-2y-u=u$. From Equation (20), we have

$$2 = 2^u\left(3^x - 1\right). \quad (21)$$

Then $u=0$ or $u=1$.

If $u=0$, then form Equation (21), we have $3 = 3^x$ and so $x=1$. Therefore $3=2y$. Thus $2 \mid 3$. This is impossible.

If $u=1$, then form Equation (21), we have $2 = 3^x$. This is impossible.

Case 2.2.2 $3x-2y-u<u$. From Equation (20), we have

$$2 = 2^{3x-2y-u}\left(3^x - 2^{2u-3x+2y}\right). \quad (22)$$

Since $\gcd\left(2^{3x-2y-u}, 3^x - 2^{2u-3x+2y}\right) = 1$, we get

$$3x - 2y - u = 1 \tag{23}$$

and

$$3^x - 2^{2u-3x+2y} = 1. \tag{24}$$

Case 2.2.2.1 $x = 0$. From Equation (24), we get $2^{2u+2y} = 0$. This is impossible.

Case 2.2.2.2 $x = 1$. From Equation (24), we get $2^{2u-3+2y} = 2$ and so $2u - 3 + 2y = 1$. From Equations (23) and (15), we obtain that $y = 0$, $u = 2$ and $z = 5$. Hence $(x, y, z) = (1,0,5)$.

Case 2.2.2.3 $x > 1$.

If $2u - 3x + 2y = 0$, then from Equation (24), we get $3^x = 2$. This is impossible.

If $2u - 3x + 2y = 1$, then from Equation (24), we have $3^x = 3$ and so $x = 1$. This is impossible.

If $2u - 3x + 2y > 1$, then from Equation (24) and Theorem 1.1, we get $x = 2$ and $2u - 3x + 2y = 3$. Then $2(u + y) = 9$ and so $2 \mid 9$. This is impossible.

Case 2.2.3 $3x - 2y - u > u$. From Equation (20), we have

$$2 = 2^u\left(2^{3x-2y-2u} \cdot 3^x - 1\right). \tag{25}$$

Since $\gcd\left(2^u, 2^{3x-2y-2u} \cdot 3^x - 1\right) = 1$ and Equation (25), we get $u = 1$ and $2 = 2^{3x-2y-2u} \cdot 3^x$. Then $x = 0$ and $2(-y - u) = 1$. Therefore $2 \mid 1$. This is impossible.

3. Conclusion

In this article, we show that all solutions of two Diophantine equations $6^x + 4^y = z^2$ and $24^x + 4^y = z^2$, where x, y and z are non-negative integers, are $(x, y, z) = (2,3,10)$ and $(x, y, z) \in \{(1,0,5),(2,5,40)\}$, respectively.

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