Journal of Mathematics and Informatics
Vol. 28, 2025, 45-58
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 6 July 2025
www.researchmathsci.org
DOI:http://dx.doi.org/10.22457/jmi.v28a05257

Journal of
Mathematics and
Informatics

Some Results on Picture Fuzzy Multirelations

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Received 24 May 2025; accepted 5 July 2025

Abstract. In this paper, the concept of picture fuzzy multirelation is introduced. Some basic operations and the inverse of picture fuzzy multirelations, together with their properties, were established. Additionally, the study examined Arithmetic, Geometric, and Harmonic mean operators, providing examples to illustrate both operations and the application of these operators to a picture fuzzy multirelation. Finally, the composition of picture fuzzy multirelations was defined and associated properties were established.

Keywords: Multiset, Fuzzy multiset, Multirelation, Composite relation

AMS Mathematics Subject Classification (2010): 03E72, 08A72

1. Introduction

The introduction of the concept of fuzzy relation (FR) came into play as a result of the work of Zadeh [18]. Atanassov [1] initiated the idea of intuitionistic fuzzy sets to extend Zadeh's work, which served as the basis for the generalisation of fuzzy relations into intuitionistic fuzzy relations (IFR) established by Bustince and Burillo [2]. Varghese and Kurrikose [13] extended the concept of intuitionistic fuzzy relations (IFRs) to relations between intuitionistic fuzzy sets (IFSs).

Cuong and Kreinovich [3], introduced the concept of picture fuzzy sets (PFSs) in order to generalise both FSs and IFSs and also introduced PFRs as a generalisation of fuzzy relations and intuitionistic fuzzy relations. Hasan et al [6] defined max-min composition and min-max composition for picture fuzzy relations and investigated some of their properties, and also discussed an application of picture fuzzy relations in decision making. In [7], Hasan et al. also defined PFRs over PFSs, establishing numerous characteristics related to picture fuzzy relations and discussing some operations with examples.

In [17], Yagar put forward the idea of fuzzy multisets (FMs) as an extension of fuzzy sets. In [12], Shinoj and Sunil initiated the concept of intuitionistic fuzzy multisets (IFMSs) as an extension of IFSs and FSs. In [11], Cao et al. introduced the notion of picture fuzzy multisets (PFMS) as a generalisation of IFMS and FMS, and established some basic operations of picture fuzzy multisets.

In this paper, the Picture Fuzzy MultiRelation (PFMR) is presented as an extension of PFR and a generalisation of IFMR. Investigation of some operations and the inverse of PFMR is carried out. Also, some operators, the Arithmetic mean operator, the Geometric mean operator and the Harmonic mean operator, were studied. Lastly, the composition of PFMR was defined, and some of its properties were obtained.

2. Preliminaries

This section gives basic definitions and preliminaries that are needed in the sequel.

Definition 2.1. *[18] Let Z be a nonempty set. A FR U on Z is a Fuzzy set, defined as*
$$\{\langle(r_1,r_2),\sigma_U(r_1,r_2)\rangle|(r_1,r_2)\in Z\times Z\}$$
where $\sigma_U: Z\times Z\longrightarrow[0,1]$.

Definition 2.2. *[3] Let X be a universe. A PFS Z of X is an object of the form*
$$Z=\{\langle \mathrm{r},\sigma_Z(\mathrm{r}),\tau_Z(\mathrm{r}),\eta_Z(\mathrm{r}))|\mathrm{r}\in X\rangle\},$$
such that $\sigma_Z(\mathrm{r})\in[0,1]$ is referred to as the degree of positive membership, $\tau_Z(\mathrm{r})\in[0,1]$ is called degree of neutral membership and $\eta_Z(\mathrm{r})\in[0,1]$ is called degree of negative membership of $\mathrm{r}\in Z$ and for all $\mathrm{r}\in X$,
$$\sigma_Z(\mathrm{r})+\tau_Z(\mathrm{r})+\eta_Z(\mathrm{r})\leq 1$$
and the degree of refusal membership of $\mathrm{r}\in Z$ is $1-(\sigma_Z(\mathrm{r})+\tau_Z(\mathrm{r})+\eta_Z(\mathrm{r}))$.

Definition 2.3. *[3] Let Z_1 and Z_2 be nonempty sets. Then, a picture fuzzy relation (PFR) U is a PFS over $Z_1\times Z_2$, defined as*
$$U=\{\langle(r_1,r_2),\sigma_U(r_1,r_2),\tau_U(r_1,r_2),\eta_U(r_1,r_2)\rangle|(r_1,r_2)\in Z_1\times Z_2\}$$
with $\sigma_U: Z_1\times Z_2\to[0,1],\tau_U: Z_1\times Z_2\to[0,1],\eta_U: Z_1\times Z_2\to[0,1]$, such that $0\leq \sigma_U(r_1,r_2)+\tau_U(r_1,r_2)+\eta_U(r_1,r_2)\leq 1$ for every $(r_1,r_2)\in Z_1\times Z_2$.

Definition 2.4. *[3] Let U be a PFR between Z_1 and Z_2. The inverse relation of U, U^{-1} between Z_2 and Z_1 is defined as*
$$\sigma_{U^{-1}}(r_2,r_1)=\sigma_U(r_1,r_2),\tau_{U^{-1}}(r_2,r_1)=\tau_U(r_1,r_2),\eta_{U^{-1}}(r_2,r_1)=\eta_U(r_1,r_2),$$
$\forall\,(r_1,r_2)\in(Z_1\times Z_2)$.

Definition 2.5. *[3] Let U and V be two PFRs between Z_1 and Z_2. Then,*

- $U\leq V\Leftrightarrow(\sigma_U(r_1,r_2)\leq\sigma_V(r_1,r_2)),(\tau_U(r_1,r_2)\leq\tau_V(r_1,r_2))$ and $(\eta_U(r_1,r_2)\geq \eta_V(r_1,r_2))$
- $U\cup V=\{((r_1,r_2),\sigma_U(r_1,r_2)\vee\sigma_V(r_1,r_2),\tau_U(r_1,r_2)\wedge\tau_V(r_1,r_2),\eta_U(r_1,r_2)\wedge \eta_V(r_1,r_2))|(r_1,r_2)\in Z_1\times Z_2\}$
- $U\cap V=\{((r_1,r_2),\sigma_U(r_1,r_2)\wedge\sigma_V(r_1,r_2),\tau_U(r_1,r_2)\wedge\tau_V(r_1,r_2),\eta_U(r_1,r_2)\vee \eta_V(r_1,r_2))|(r_1,r_2)\in Z_1\times Z_2\}$
- $U^c=\{((r_1,r_2),\eta_U(r_1,r_2),\tau_U(r_1,r_2),\sigma_U(r_1,r_2))|(r_1,r_2)\in Z_1\times Z_2\}$

  for every $(r_1,r_2)\in(Z_1\times Z_2)$.

Definition 2.6. *[11] Let Y be a nonempty set. A PFMS Z in Y is characterised by three functions namely positive membership count function pmc, neutral membership count function n_emc and negative membership count function nmc such that $pmc: Y\to W$, $n_emc: Y\to W$ and $nmc: Y\to W$, respectively, where W is the set of all crisp multisets drawn from* $[0,1]$. *Thus, for any $r\in Y$, pmc is the crisp multiset from* $[0,1]$ *whose positive membership sequence is defined by $(\sigma_Z^1(r),\sigma_Z^2(r),\cdots,\sigma_Z^n(r))$ such that $\sigma_Z^1(r)\geq\sigma_Z^2(r)\geq \cdots\geq\sigma_Z^n(r)$, n_emc is the crisp multiset from* $[0,1]$ *whose neutral membership sequence is* *defined by* $(\tau_Z^1(r), \tau_Z^2(r), \cdots, \tau_Z^n(r))$ *and nmc is the crisp multiset from* $[0,1]$ *whose negative membership sequence is defined by* $(\eta_Z^1(r), \eta_Z^2(r), \cdots, \eta_Z^n(r))$*, these can be either decreasing or increasing functions satisfying* $0 \leq \sigma_Z^k(r) + \tau_Z^k(r) + \eta_Z^k(r) \leq 1\ \forall r \in Y$, $k = 1,2, \cdots, n$.

Thus, Z is represented by

$$Z_1 = \{\langle r, \sigma_{Z_1}^k(r), \tau_{Z_1}^k(r)), \eta_{Z_1}^k(r)\rangle | \, r \in Y\}$$

$k = 1,2, \cdots, n$.

Definition 2.7. *[11] Let*

$$Z_1 = \{\langle r, \sigma_{Z_1}^k(r), \tau_{Z_1}^k(r)), \eta_{Z_1}^k(r)\rangle | \, r \in Y\}$$

and

$$Z_2 = \{\langle r, \sigma_{Z_2}^k(r)), \tau_{Z_2}^k(r), \eta_{Z_2}^k(r))\rangle | \, r \in Y\}$$

be two PFMSs drawn from Y. Then,

- $Z_1 \subseteq Z_2, \Leftrightarrow (\sigma_{Z_1}^k(r) \leq \sigma_{Z_2}^k(r)), (\tau_{Z_1}^k(r) \leq \tau_{Z_2}^k(r))$ and $(\eta_{Z_1}^k(r) \geq \eta_{Z_2}^k(r))$; $k = 1,2, \cdots, n, r \in Y$.
- $Z_1 = Z_2, \Leftrightarrow Z_1 \subseteq Z_2$ and $Z_2 \subseteq Z_1$.
- $Z_1 \cup Z_2 = \{(r, (\sigma_{Z_1}^k(r) \vee \sigma_{Z_2}^k(r)), (\tau_{Z_1}^k(r) \wedge \tau_{Z_2}^k(r)), (\eta_{Z_1}^k(r) \wedge \eta_{Z_2}^k(r)))| \, r \in Y\}, k = 1,2, \cdots, n$.
- $Z_1 \cap Z_2 = \{(r, (\sigma_{Z_1}^k(r) \wedge \sigma_{Z_2}^k(r))(\tau_{Z_1}^k(r) \wedge \tau_{Z_2}^k(r)), (\eta_{Z_1}^k(r) \vee \eta_{Z_2}^k(r)))| \, r \in Y\}, k = 1,2, \cdots, n$.
- $Z_1' = \{(r, \eta_{Z_1}^k(r), \tau_{Z_1}^k(r), \sigma_{Z_1}^k(r))| \, r \in Y\}, k = 1,2, \cdots, n$.

3. Operations of picture fuzzy multirelations

This section defines operations of picture fuzzy multirelations and establishes some properties associated with the operations. It also defines the Arithmetic Mean Operator, the Geometric Mean Operator and the Harmonic Mean Operator. Examples were given to illustrate both operations and operators of PFMRs.

Definition 3.1. *Let Z be a nonempty set. Then, a picture fuzzy multirelation (PFMR) U on Z is PFMS defined by*

$$U = \{\langle (r_1, r_2), \sigma_U^k(r_1, r_2), \tau_U^k(r_1, r_2), \eta_U^k(r_1, r_2)\rangle | (r_1, r_2) \in Z_1 \times Z_2\}$$

where $k = 1,2, \cdots, \beta$ (β is the cardinality of the PFMS Z) $\sigma_Z^k(r), \tau_Z^i(r), \eta_Z^k(r): Y \to W$, and W is the set of all crisp multisets drawn from $[0,1]$.

Definition 3.2. *Let Y be a nonempty set and Z_1 and Z_2 be PFMSs in Y with positive membership $\sigma_{Z_1}^k(r)$ and $\sigma_{Z_2}^k(r)$, neutral membership $\tau_{Z_1}^k(r)$ and $\tau_{Z_2}^k(r)$ and negative membership $\eta_{Z_1}^k(r)$ and $\eta_{Z_2}^k(r)$ such that*

$$\sigma_{Z_1}^k(r), \sigma_{Z_2}^k(r), \tau_{Z_1}^k(r), \tau_{Z_2}^k(r), \eta_{Z_1}^k(r), \eta_{Z_2}^k(r): Y \to W$$

and W is the set of all crisp multisets drawn from $[0,1]$. Then, Cartesian product of Z_1 *and* Z_2, $Z_1 \times Z_2$ is the PFMS in $Y \times Y$ defined by

$$\sigma_{Z_1 \times Z_2}^k(r_1, r_2) = \wedge \{\sigma_{Z_1}^k(r_1), \sigma_{Z_2}^k(r_2)\},$$

$$\tau^k_{Z_1\times Z_2}(r_1,r_2) = \wedge \{\tau^k_{Z_1}(r_1), \tau^k_{Z_2}(r_2)\}$$

and

$$\eta^k_{Z_1\times Z_2}(r_1,r_2) = \vee \{\eta^k_{Z_1}(r_1), \eta^k_{Z_2}(r_2)\}$$

$\forall r_1, r_2 \in Y$, $k = 1,2,\cdots,\beta$ (β is the cardinality of the PFMS Z_1 and Z_2).

Definition 3.3. *Let U be a PFMS $(Y \times Y)$, $U \subseteq Z_1 \times Z_2$. Then, U is called a PFMR from Z_1 to Z_2 if for all $(r_1,r_2) \in Y \times Y$,*

$$\sigma^k_U(r_1,r_2) \le \sigma^k_{Z_1\times Z_2}(r_1,r_2), \tau^k_U(r_1,r_2) \le \tau^k_{Z_1\times Z_2}(r_1,r_2), \eta^k_U(r_1,r_2) \ge \eta^k_{Z_1\times Z_2}(r_1,r_2),$$

with $0 \le \sigma^k_U(r_1,r_2) + \tau^k_U(r_1,r_2) + \eta^k_U(r_1,r_2) \le 1$.
In particular, if $Z_1 = Z_2$, then U is called a PFMR on Z_1.

Using matrix representation for the PFMR U from Z_1 to Z_2 we have

$$U = [\sigma_{kl}, \tau_{kl}, \eta_{kl}]$$

where

$$\sigma_{kl} = \sigma_U(r_k,r_l), \tau_{kl} = \tau_U(r_k,r_l), \text{and } \eta_{kl} = \eta_U(r_k,r_l), k,l = 1,2,\cdots,n.$$

Example 3.1. *Let $Y = \{r_1,r_2,r_3\}$ be nonempty set. Let*

$$Z_1 = \begin{bmatrix} r_1 & r_2 & r_3 \\ (0.1,0.5,0.3) & (0.2,0.4,0.7) & (0.5,0.2,0.3) \\ (0.3,0.4,0.2) & (0.4,0.6,0.0) & (0.2,0.7,0.3) \\ (0.6,0.1,0.5) & (0.4,0.0,0.3) & (0.3,0.1,0.4) \end{bmatrix} \text{ and}$$

$$Z_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ (0.5,0.3,0.5) & (0.4,0.1,0.3) & (0.7,0.1,0.2) \\ (0.3,0.2,0.2) & (0.2,0.6,0.5) & (0.2,0.4,0.4) \\ (0.2,0.5,0.3) & (0.4,0.3,0.2) & (0.1,0.5,0.4) \end{bmatrix}$$

Then,

$$Z_1 \times Z_2 = \begin{bmatrix} (0.23,0.23,0.53) & (0.13,0.27,0.47) & (0.13.0.27,0.53) \\ (0.33,0.17,0.40) & (0.20,0.27,0.33) & (0.17,0.20,0.43) \\ (0.33,0.20,0.40) & (0.27,0.37,0.37) & (0.27,0.30,0.40) \end{bmatrix}$$

Let

$$U = \begin{bmatrix} (0.17,0.13,0.70) & (0.13,0.27,0.50) & (0.13,0.23,0.57) \\ (0.30,0.20,0.50) & (0.17,0.27,0.56) & (0.17,0.20,0.53) \\ (0.33,0.17,0.43) & (0.23,0.33,0.38) & (0.27,0.30,0.43) \end{bmatrix}.$$

Then, U is a relation from Z_1 to Z_2.

Definition 3.4. *Given a PFMR U on Z, U complement is a PFMR, U^c, where $\sigma^k_{U^c} = \eta^k_U, \tau^k_{U^c} = \tau^k_U$, and $\eta^k_{U^c} = \sigma^k_U$, i.e;*

$$U^c = \{((r_1,r_2), \eta^k_U(r_1,r_2), \tau^k_U(r_1,r_2), \sigma^k_U(r_1,r_2)) \mid (r_1,r_2) \in Y \times Y\}, k = 1,2,\cdots,n.$$

Definition 3.5. *Let $U \in PFMR(Z_1 \times Z_2)$. The inverse relation of U, denoted by U^{-1} from Z_2 to Z_1 is defined by*

$$\sigma^k_{U^{-1}}(r_2,r_1) = \sigma^k_U(r_1,r_2), \tau^k_{U^{-1}}(r_2,r_1) = \tau^k_U(r_1,r_2), \eta^k_{U^{-1}}(r_2,r_1) = \eta^k_U(r_1,r_2).$$

Definition 3.6. *Let $U, V \in PFMR(Z_1 \times Z_2)$. Then, $U \subseteq V$ if for every $r_1, r_2 \in Y$ $(\sigma^k_U(r_1,r_2) \leq \sigma^k_V(r_1,r_2))$, $(\tau^k_U(r_1,r_2) \leq \tau^k_V(r_1,r_2))$ and $(\eta^k_U(r_1,r_2) \geq \eta^k_V(r_1,r_2))$; $k = 1,2,\cdots,n$. If $U \subseteq V$ and $V \subseteq U$, then $U = V$.*

Example 3.2. *From Example* 3.1, *let*

$$U = \begin{bmatrix} (0.17,0.13,0.70) & (0.13,0.27,0.50) & (0.13,0.23,0.57) \\ (0.30,0.20,0.50) & (0.17,0.27,0.56) & (0.17,0.20,0.53) \\ (0.33,0.17,0.43) & (0.23,0.33,0.38) & (0.27,0.30,0.43) \end{bmatrix}$$

and

$$V = \begin{bmatrix} (0.27,0.13,0.53) & (0.33,0.37,0.30) & (0.13,0.23,0.57) \\ (0.53,0.20,0.27) & (0.27,0.27,0.40) & (0.13,0.23,0.57) \\ (0.40,0.17,0.43) & (0.33,0.33,0.27) & (0.37,0.30,0.33) \end{bmatrix}$$

be PFMRs from Z_1 to Z_2.

$$U^c = \begin{bmatrix} (0.70,0.13,0.17) & (0.50,0.27,0.13) & (0.57,0.23,0.13) \\ (0.50,0.20,0.30) & (0.56,0.27,0.17) & (0.53,0.20,0.17) \\ (0.43,0.17,0.33) & (0.38,0.33,0.23) & (0.43,0.30,0.27) \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} (0.70,0.13,0.17) & (0.30,0.20,0.50) & (0.33,0.17,0.43) \\ (0.13,0.27,0.50) & (0.17,0.27,0.56) & (0.23,0.33,0.38) \\ (0.13,0.23,0.57) & (0.17,0.20,0.53) & (0.27,0.30,0.43) \end{bmatrix}$$

and $U \subseteq V$.

Definition 3.7. *Let $U, V \in PFMR(Z_1 \times Z_2)$. Then, $U \cup V$ is a PFMR from Z_1 to Z_2 such that*

$$\sigma^k_{U\cup V}(r_1,r_2) = \vee \{\sigma^k_U(r_1,r_2), \sigma^k_V(r_1,r_2)\},$$

$$\tau^k_{U\cup V}(r_1,r_2) = \wedge \{\tau^k_U(r_1,r_2), \tau^k_V(r_1,r_2)\}$$

and

$$\eta^k_{U\cup V}(r_1,r_2) = \wedge \{\eta^k_U(r_1,r_2), \eta^k_V(r_1,r_2)\}$$

$k = 1,2,\cdots,n$.

Example 3.3. *From Example* 3.1, *let*

$$U = \begin{bmatrix} (0.17,0.13,0.70) & (0.13,0.27,0.50) & (0.13,0.23,0.57) \\ (0.30,0.20,0.50) & (0.17,0.27,0.56) & (0.17,0.20,0.53) \\ (0.33,0.17,0.43) & (0.23,0.33,0.38) & (0.27,0.30,0.43) \end{bmatrix}$$

and

$$V = \begin{bmatrix} (0.27,0.03,0.63) & (0.23,0.37,0.40) & (0.10,0.23,0.60) \\ (0.23,0.10,0.67) & (0.17,0.17,0.60) & (0.13,0.23,0.64) \\ (0.10,0.17,0.63) & (0.33,0.33,0.27) & (0.37,0.20,0.43) \end{bmatrix}$$

be PFMRs from Z_1 to Z_2. Then,

$$U \cup V = \begin{bmatrix} (0.27,0.03,0.63) & (0.23,0.27,0.40) & (0.13,0.23,0.57) \\ (0.30,0.10,0.50) & (0.17,0.17,0.56) & (0.17,0.20,0.53) \\ (0.33,0.17,0.43) & (0.30,0.33,0.27) & (0.37,0.20,0.43) \end{bmatrix}$$

Definition 3.8. *Let $U, V \in PFMR(Z_1 \times Z_2)$. Then, $U \cap V$ is a PFMR from Z_1 to Z_2 such that*

$$\sigma^k_{U\cap V}(r_1, r_2) = \wedge \{\sigma^k_U(r_1, r_2), \sigma^k_V(r_1, r_2)\},$$

$$\tau^k_{U\cap V}(r_1, r_2) = \wedge \{\tau^k_U(r_1, r_2), \tau^k_V(r_1, r_2)\}$$

and

$$\eta^k_{U\cap V}(r_1, r_2) = \vee \{\eta^k_U(r_1, r_2), \eta^k_V(r_1, r_2)\}$$

$k = 1,2, \cdots, n$.

Example 3.4. *From Example* 3.5,

$$U \cap V = \begin{bmatrix} (0.17,0.03,0.70) & (0.13,0.27,0.50) & (0.10,0.23,0.60) \\ (0.23,0.10,0.67) & (0.17,0.17,0.60) & (0.13,0.20,0.64) \\ (0.10,0.17,0.63) & (0.23,0.33,0.38) & (0.27,0.20,0.43) \end{bmatrix}$$

Proposition 3.1. *Let $U, V, W \in PFMR(Z_1 \times Z_2)$. Then,*

- $(U^{-1})^{-1} = U$
- $(U \cup V)^{-1} = U^{-1} \cup V^{-1}$
- $(U \cap V)^{-1} = U^{-1} \cap V^{-1}$
- $U \cap (V \cup W) = (U \cap V) \cup (V \cap W)$
- $U \cup (V \cap W) = (U \cup V) \cap (V \cup W)$

Proof:

• Since U^{-1} is a PFMR between Z_2 and Z_1, it implies that

$$\sigma^k_{U^{-1}}(r_2, r_1) = \sigma^k_U(r_1, r_2), \tau^k_{U^{-1}}(r_2, r_1) = \tau^k_U(r_1, r_2) \text{ and } \eta^k_{U^{-1}}(r_2, r_1) = \eta^k_U(r_1, r_2).$$

Now,

$$\sigma^k_U(r_1, r_2) = \sigma^k_{U^{-1}}(r_2, r_1) = \sigma^k_{(U^{-1})^{-1}}(r_1, r_2),$$

$$\tau^k k_U(r_1, r_2) = \tau^k_{U^{-1}}(r_2, r_1) = \tau^k_{(U^{-1})^{-1}}(r_1, r_2)$$

and

$$\eta^k_U(r_1, r_2) = \eta^k_{U^{-1}}(r_2, r_1) = \eta^k_{(U^{-1})^{-1}}(r_1, r_2)$$

• From the definition of inverse,

$$\sigma^k_{U^{-1}}(r_2, r_1) = \sigma^k_U(r_1, r_2), \tau^k_{U^{-1}}(r_2, r_1) = \tau^k_U(r_1, r_2), \eta^k_{U^{-1}}(r_2, r_1) = \eta^k_U(r_1, r_2)$$

and

$$\sigma^k_{V^{-1}}(r_2, r_1) = \sigma^k_V(r_1, r_2), \tau^k_{V^{-1}}(r_2, r_1) = \tau^k_V(r_1, r_2), \eta^k_{V^{-1}}(r_2, r_1) = \eta^k_V(r_1, r_2).$$

Thus,

$$\sigma^k_{(U\cup V)^{-1}}(r_2, r_1) = \sigma^k_{U\cup V}(r_1, r_2)$$

$$= \sigma_U^k(r_1,r_2) \vee \sigma_V^k(r_1,r_2)$$
$$= \sigma_{U^{-1}}^k(r_2,r_1) \vee \sigma_{V^{-1}}^k(r_2,r_1)$$
$$= \sigma_{U^{-1}}^k \cup \sigma_{V^{-1}}^k(r_2,r_1),$$

$$\tau_{(U\cup V)^{-1}}^k(r_2,r_1) = \tau_{U\cup V}^k(r_1,r_2)$$
$$= \tau_U^k(r_1,r_2) \wedge \tau_V^k(r_1,r_2)$$
$$= \tau_{U^{-1}}^k(r_2,r_1) \wedge \tau_{V^{-1}}^k(r_2,r_1)$$
$$= \tau_{U^{-1}}^k \cup \tau_{V^{-1}}^k(r_2,r_1)$$

and

$$\eta_{(U\cup V)^{-1}}^k(r_2,r_1) = \eta_{U\cup V}^k(r_1,r_2)$$
$$= \eta_U^k(r_1,r_2) \wedge \eta_V^k(r_1,r_2)$$
$$= \eta_{U^{-1}}^k(r_2,r_1) \wedge \eta_{V^{-1}}^k(r_2,r_1)$$
$$= \eta_{U^{-1}}^k \cup \eta_{V^{-1}}^k(r_2,r_1)$$

• From the definition of inverse,

$$\sigma_{U^{-1}}^k(r_2,r_1) = \sigma_U^k(r_1,r_2), \tau_{U^{-1}}^k(r_2,r_1) = \tau_U^k(r_1,r_2), \eta_{U^{-1}}^k(r_2,r_1) = \eta_U^k(r_1,r_2)$$

and

$$\sigma_{V^{-1}}^k(r_2,r_1) = \sigma_V^k(r_1,r_2), \tau_{V^{-1}}^k(r_2,r_1) = \tau_V^k(r_1,r_2), \eta_{V^{-1}}^k(r_2,r_1) = \eta_V^k(r_1,r_2).$$

Thus,

$$\sigma_{(U\cap V)^{-1}}^k(r_2,r_1) = \sigma_{U\cap V}^k(r_1,r_2)$$
$$= \sigma_U^k(r_1,r_2) \wedge \sigma_V^k(r_1,r_2)$$
$$= \sigma_{U^{-1}}^k(r_2,r_1) \wedge \sigma_{V^{-1}}^k(r_2,r_1)$$
$$= \sigma_{U^{-1}}^k \cap \sigma_{V^{-1}}^k(r_2,r_1),$$

$$\tau_{(U\cap V)^{-1}}^k(r_2,r_1) = \tau_{U\cap V}^k(r_1,r_2)$$
$$= \tau_U^k(r_1,r_2) \wedge \tau_V^k(r_1,r_2)$$
$$= \tau_{U^{-1}}^k(r_2,r_1) \wedge \tau_{V^{-1}}^k(r_2,r_1)$$
$$= \tau_{U^{-1}}^k \cap \tau_{V^{-1}}^k(r_2,r_1)$$

and

$$\eta_{(U\cap V)^{-1}}^k(r_2,r_1) = \eta_{U\cap V}^k(r_1,r_2)$$
$$= \eta_U^k(r_1,r_2) \vee \eta_V^k(r_1,r_2)$$
$$= \eta_{U^{-1}}^k(r_2,r_1) \vee \eta_{V^{-1}}^k(r_2,r_1)$$
$$= \eta_{U^{-1}}^k \cap \eta_{V^{-1}}^k(r_2,r_1)$$

•

$$\sigma_{U\cap(V\cup W)}^k(r_1,r_2) = \sigma_U^k(r_1,r_2) \wedge (\sigma_{V\cup W}^k(r_1,r_2))$$
$$= \sigma_U^k(r_1,r_2) \wedge (\sigma_V^k(r_1,r_2) \vee \sigma_W^k(r_1,r_2))$$
$$= (\sigma_U^k(r_1,r_2) \wedge \sigma_V^k(r_1,r_2)) \vee \sigma_U^k(r_1,r_2) \wedge \sigma_W^k(r_1,r_2)$$
$$= \sigma_{U\cap V}^k(r_1,r_2) \cup \sigma_{U\cap W}^k(r_1,r_2),$$

$$\tau_{U\cap(V\cup W)}^k(r_1,r_2) = \tau_U^k(r_1,r_2) \wedge (\tau_{V\cup W}^k(r_1,r_2))$$
$$= \tau_U^k(r_1,r_2) \wedge (\tau_V^k(r_1,r_2) \vee \tau_W^k(r_1,r_2))$$

$$= (\tau_U^k(r_1,r_2) \wedge \tau_V^k(r_1,r_2)) \vee \tau_U^k(r_1,r_2) \wedge \tau_W^k(r_1,r_2)$$
$$= \tau_{U\cap V}^k(r_1,r_2) \cup \tau_{U\cap W}^k(r_1,r_2)$$

$$\eta_{U\cap(V\cup W)}^k(r_1,r_2) = \eta_U^k(r_1,r_2) \wedge (\eta_{V\cup W}^k(r_1,r_2))$$
$$= \eta_U^k(r_1,r_2) \vee (\eta_V^k(r_1,r_2) \wedge \eta_W^k(r_1,r_2))$$
$$= (\eta_U^k(r_1,r_2) \vee \eta_V^k(r_1,r_2)) \wedge \eta_U^k(r_1,r_2) \wedge \eta_W^k(r_1,r_2)$$
$$= \eta_{U\cap V}^k(r_1,r_2) \cup \eta_{U\cap W}^k(r_1,r_2).$$

• Replace ∩ with ∪ and ∪ with ∩ in the above proof.

Definition 3.9. *Let $U, V \in PFMR(Z_1 \times Z_2)$. Then, $U@V$ is an Arithmetic Mean Operator from Z_1 to Z_2 given by*

$$U@V = (\sigma_{U@V}^k(r_1,r_2), \tau_{U@V}^k(r_1,r_2), \eta_{U@V}^k(r_1,r_2))$$

where

$$\sigma_{U@V}^k(r_1,r_2) = \frac{\sigma_U^k(r_1,r_2)+\sigma_V^k(r_1,r_2)}{2},$$

$$\tau_{U@V}^k(r_1,r_2) = \frac{\tau_U^k(r_1,r_2)+\tau_V^k(r_1,r_2)}{2}$$

and

$$\eta_{U@V}^k(r_1,r_2)) = \frac{\eta_U^k(r_1,r_2)+\eta_V^k(r_1,r_2)}{2}$$

Definition 3.10. *Let $U, V \in PFMR(Z_1 \times Z_2)$. Then, $U\wp V$ is a Geometric Mean Operator from Z_1 to Z_2 given by*

$$U\wp V = (\sigma_{U\wp V}^k(r_1,r_2), \tau_{U\wp V}^k(r_1,r_2), \eta_{U\wp V}^k(r_1,r_2))$$

where

$$\sigma_{U\wp V}^k(r_1,r_2) = \sqrt{\sigma_U^k(r_1,r_2).\,\sigma_V^k(r_1,r_2)},$$

$$\tau_{U\wp V}^k(r_1,r_2) = \sqrt{\tau_U^k(r_1,r_2).\,\tau_V^k(r_1,r_2)}$$

and

$$\eta_{U\wp V}^k(r_1,r_2)) = \sqrt{\eta_U^k(r_1,r_2).\,\eta_V^k(r_1,r_2)}$$

Definition 3.11. *Let $U, V \in PFMR(Z_1 \times Z_2)$. Then, $U©V$ is a Harmonic Mean Operator from Z_1 to Z_2 given by*

$$U©V = (\sigma_{U©V}^k(r_1,r_2), \tau_{U©V}^k(r_1,r_2), \eta_{U©V}^k(r_1,r_2))$$

where

$$\sigma_{U©V}^k(r_1,r_2) = \frac{2\sigma_U^k(r_1,r_2).\sigma_V^k(r_1,r_2)}{\sigma_U^k(r_1,r_2)+\sigma_V^k(r_1,r_2)},$$

$$\tau_{U©V}^k(r_1,r_2) = \frac{2\tau_U^k(r_1,r_2).\tau_V^k(r_1,r_2)}{\tau_U^k(r_1,r_2)+\tau_V^k(r_1,r_2)}$$

and

$$\eta^k_{U©V}(r_1,r_2) = \frac{2\eta^k_U(r_1,r_2).\eta^k_V(r_1,r_2)}{\eta^k_U(r_1,r_2)+\eta^k_V(r_1,r_2)}$$

Example 3.5. *From Example* 3.1, *let*

$$U = \begin{bmatrix} (0.17,0.13,0.70) & (0.13,0.27,0.50) & (0.13,0.23,0.57) \\ (0.30,0.20,0.50) & (0.17,0.27,0.56) & (0.17,0.20,0.53) \\ (0.33,0.17,0.43) & (0.23,0.33,0.38) & (0.27,0.30,0.43) \end{bmatrix}$$

and

$$V = \begin{bmatrix} (0.27,0.03,0.63) & (0.23,0.37,0.40) & (0.10,0.23,0.60) \\ (0.23,0.10,0.67) & (0.17,0.17,0.60) & (0.13,0.23,0.64) \\ (0.10,0.17,0.63) & (0.33,0.33,0.27) & (0.37,0.20,0.43) \end{bmatrix}$$

be PFMRs from Z_1 to Z_2. Then,

$$U@V = \begin{bmatrix} (0.22,0.08,0.67) & (0.18,0.32,0.45) & (0.12,0.23,0.59) \\ (0.27,0.15,0.59) & (0.17,0.22,0.58) & (0.15,0.22,0.59) \\ (0.22,0.17,0.53) & (0.28,0.33,0.33) & (0.32,0.25,0.43) \end{bmatrix}$$

$$U\wp V = \begin{bmatrix} (0.21,0.06,0.66) & (0.17,0.32,0.45) & (0.11,0.23,0.58) \\ (0.26,0.14,0.58) & (0.17,0.21,0.58) & (0.15,0.21,0.58) \\ (0.18,0.17,0.52) & (0.28,0.33,0.32) & (0.32,0.24,0.43) \end{bmatrix}$$

$$U©V = \begin{bmatrix} (0.21,0.05,0.66) & (0.17,0.31,0.44) & (0.11,0.23,0.58) \\ (0.26,0.13,0.57) & (0.17,0.21,0.58) & (0.15,0.21,0.58) \\ (0.15,0.17,0.51) & (0.27,0.33,0.32) & (0.31,0.24,0.43) \end{bmatrix}$$

4. Composite relation of picture fuzzy multirelations

In this section, we defined max-min-max composition of PFMRs and obtained some associated properties.

Definition 4.1. *Let* $U \in PFMR(Z_1 \times Z_2)$ *and* $V \in PFMR(Z_2 \times Z_3)$. *The composite relation* $V \circ U$ *is a PFMR between* Z_1 *and* Z_3 *defined by*

$$V \circ U = \{\langle (r_1,r_3), \sigma^k_{V\circ U}(r_1,r_3), \tau^k_{V\circ U}(r_1,r_3), \eta^k_{V\circ U}(r_1,r_3)\rangle | (r_1,r_3) \in Z_1 \times Z_3\}$$

where $\forall\ (r_1,r_3) \in Z_1 \times Z_3$ and $\forall\ r_2 \in Z_2$, its positive membership, neutral membership and negative membership functions are defined by

$$\sigma^k_{V\circ U}(r_1,r_3) = \bigvee_{r_2\in V} \{\sigma^k_U(r_1,r_2) \wedge \sigma^k_V(r_2,r_3)\},$$

$$\tau^k_{V\circ U}(r_1,r_3) = \bigwedge_{r_2\in V} \{\tau^k_U(r_1,r_2) \wedge \tau^k_V(r_2,r_3)\}$$

and

$$\eta^k_{V\circ U}(r_1,r_3) = \bigwedge_{r_2\in V} \{\eta^k_U(r_1,r_2) \vee \eta^k_V(r_2,r_3)\},$$

respectively.

Example 4.1. *Consider* U *and* V *from Example* 3.5. *Let*

$$U = \begin{bmatrix} (0.17,0.13,0.70) & (0.13,0.27,0.50) & (0.13,0.23,0.57) \\ (0.30,0.20,0.50) & (0.17,0.27,0.56) & (0.17,0.20,0.53) \\ (0.33,0.17,0.43) & (0.23,0.33,0.38) & (0.27,0.30,0.43) \end{bmatrix}$$

and

$$V = \begin{bmatrix} (0.27,0.03,0.63) & (0.23,0.37,0.40) & (0.10,0.23,0.60) \\ (0.23,0.10,0.67) & (0.17,0.17,0.60) & (0.13,0.23,0.64) \\ (0.10,0.17,0.63) & (0.33,0.33,0.27) & (0.37,0.20,0.43) \end{bmatrix}$$

be PFMRs from Z_1 to Z_2. Then,

$$V \circ U = \begin{bmatrix} (0.17,0.03,0.63) & (0.17,0.13,0.57) & (0.13,0.13,0.57) \\ (0.27,0.03,0.63) & (0.23,0.17,0.50) & (0.17,0.20,0.53) \\ (0.27,0.03,0.63) & (0.27,0.17,0.43) & (0.10,0.17,0.43) \end{bmatrix}$$

Proposition 4.1. *Let $U \in PFMR(Z_1 \times Z_2)$ and $V \in PFMR(Z_2 \times Z_3)$. The composite relation $V \circ U$ is in $PFMR(Z_1 \times Z_3)$.*

Proof: For all $(r_1, r_3) \in Z_1 \times Z_3$, let check

$$\sigma^k_{V\circ U}(r_1,r_3) + \tau^k_{V\circ U}(r_1,r_3) + \eta^k_{V\circ U}(r_1,r_3) \le 1$$

For all $\epsilon > 0$, there exists $r_2^\star \in Z_2$ such that

$$\sigma^k_{V\circ U}(r_1,r_3) < \sigma^k_U(r_1,r_2^*) \wedge \sigma^k_V(r_2^*,r_3) + \epsilon \qquad (\star)$$

It is easily seen that

$$\tau^k_{V\circ U}(r_1,r_3) \le \tau^k_U(r_1,r_2^*) \wedge \tau^k_V(r_2^*,r_3) \qquad (\star\star)$$

and

$$\eta^k_{V\circ U}(r_1,r_3) \le \eta^k_U(r_1,r_2^*) \vee \eta^k_V(r_2^*,r_3) \qquad (\star\star\star)$$

Combining $(\star)$, $(\star\star)$, $(\star\star\star)$, we have

$$\begin{aligned} &\sigma^k_{V\circ U}(r_1,r_2) + \tau^k_{V\circ U}(r_1,r_3) + \eta^k_{V\circ U}(r_1,r_3) \\ &\quad < \sigma^k_U(r_1,r_2^*) \wedge \sigma^k_V(r_2^*,r_3) + \tau^k_U(r_1,r_2^*) \wedge \tau^k_V(r_2^*,r_3) \\ &\quad + \eta^k_U(r_1,r_2^*) \vee \eta^k_V(r_2^*,r_3) + \epsilon \end{aligned}$$

Here, two things to check,

Firstly: If

$$\eta^k_U(r_1,r_2^*) \vee \eta^k_V(r_2^*,r_3) = \eta^k_U(r_1,r_2^*).$$

Then,

$$\begin{aligned} &\sigma^k_U(r_1,r_2^*) \wedge \sigma^k_V(r_2^*,r_3) + \tau^k_U(r_1,r_2^*) \wedge \tau^k_V(r_2^*,r_3) + \eta^k_U(r_1,r_2^*) \vee \eta^k_V(r_2^*,r_3) + \epsilon \\ &= \sigma^k_U(r_1,r_2^*) \wedge \sigma^k_V(r_2^*,r_3) + \tau^k_U(r_1,r_2^*) \wedge \tau^k_V(r_2^*,r_3) + \eta^k_U(r_1,r_2^*) + \epsilon \\ &\le \sigma^k_U(r_1,r_2^*) + \tau^k_U(r_1,r_2^*) + \eta^k_U(r_1,r_2^*) + \epsilon \\ &\le 1 + \epsilon \end{aligned}$$

Secondly: If

$$\eta^k_U(r_1,r_2^*) \vee \eta^k_V(r_2^*,r_3) = \eta^k_V(r_2^*,r_3).$$

Then,

$$\begin{aligned} &\sigma^k_U(r_1,r_2^*) \wedge \sigma^k_V(r_2^*,r_3) + \tau^k_U(r_1,r_2^*) \wedge \tau^k_V(r_2^*,r_3) + \eta^k_U(r_1,r_2^*) \vee \eta^k_V(r_2^*,r_3) + \epsilon \\ &= \sigma^k_U(r_1,r_2^*) \wedge \sigma^k_V(r_2^*,r_3) + \tau^k_U(r_1,r_2^*) \wedge \tau^k_V(r_2^*,r_3) + \eta^k_U(r_1,r_2^*) + \epsilon \\ &\le \sigma^k_U(r_1,r_2^*) + \tau^k_U(r_1,r_2^*) + \eta^k_V(r_2^*,r_3) + \epsilon \\ &\le 1 + \epsilon \end{aligned}$$

Thus,

$$\sigma^k_{V\circ U}(r_1,r_3)+\tau^k_{V\circ U}(r_1,r_3)+\eta^k_{V\circ U}(r_1,r_2)<1+\epsilon \text{ for all } \epsilon>0$$

Therefore,

$$\sigma^k_{V\circ U}(r_1,r_3)+\tau^k_{V\circ U}(r_1,r_3)+\eta^k_{V\circ U}(r_1,r_3)\le 1$$

Proposition 4.2. *Let* $U\in PFMR(Z_1\times Z_2)$ *and* $V\in PFMR(Z_2\times Z_3)$. *Then* $(V\circ U)^{-1}=U^{-1}\circ V^{-1}$

Proof:

$$\begin{aligned}
\sigma^k_{(V\circ U)^{-1}}(r_3,r_1) &= \sigma^k_{V\circ U}(r_1,r_3)\\
&= \bigvee_{r_2\in Z_2}\{\sigma^k_U(r_1,r_2)\wedge\sigma^k_V(r_2,r_3)\}\\
&= \bigvee_{r_2\in Z_2}\{\sigma^k_{U^{-1}}(r_2,r_1)\wedge\sigma^k_{V^{-1}}(r_3,r_2)\}\\
&= \bigvee_{r_2\in Z_2}\{\sigma^k_{V^{-1}}(r_3,r_2)\wedge\sigma^k_{U^{-1}}(r_2,r_1)\}\\
&= \sigma^k_{U^{-1}\circ V^{-1}}(r_3,r_1)
\end{aligned}$$

$$\begin{aligned}
\tau^k_{(V\circ U)^{-1}}(r_3,r_1) &= \tau^k_{V\circ U}(r_1,r_3)\\
&= \bigvee_{r_2\in Z_2}\{\tau^k_U(r_1,r_2)\wedge\tau^k_V(r_2,r_3)\}\\
&= \bigvee_{r_2\in Z_2}\{\tau^k_{U^{-1}}(r_2,r_1)\wedge\tau^k_{V^{-1}}(r_3,r_2)\}\\
&= \bigvee_{r_2\in Z_2}\{\tau^k_{V^{-1}}(r_3,r_2)\wedge\tau^k_{U^{-1}}(r_2,r_1)\}\\
&= \tau^k_{U^{-1}\circ V^{-1}}(r_3,r_1)
\end{aligned}$$

$$\begin{aligned}
\eta^k_{(V\circ U)^{-1}}(r_3,r_1) &= \eta^k_{V\circ U}(r_1,r_3)\\
&= \bigwedge_{r_2\in Z_2}\{\eta^k_U(r_1,r_2)\vee\eta^k_V(r_2,r_3)\}\\
&= \bigwedge_{r_2\in Z_2}\{\eta^k_{U^{-1}}(r_2,r_1)\vee\eta^k_{V^{-1}}(r_3,r_2)\}\\
&= \bigwedge_{r_2\in Z_2}\{\eta^k_{V^{-1}}(r_3,r_2)\vee\eta^k_{U^{-1}}(r_2,r_1)\}\\
&= \eta^k_{U^{-1}\circ V^{-1}}(r_3,r_1)
\end{aligned}$$

Proposition 4.3. *Let* $U,V\in PFMR(Z_2\times Z_3)$ *and* $W\in PFMR(Z_1\times Z_2)$. *Then,*

- $(V\wedge U)\circ W=(V\circ W)\wedge(U\circ W)$
- $(V\vee U)\circ W=(V\circ W)\vee(U\circ W)$

Proof:

$$\begin{aligned}
\sigma^k_{(V\wedge U)\circ W}(r_1,r_3) &= \bigvee_{r_2}\{\sigma^k_W(r_1,r_2)\wedge(\sigma^k_V(r_2,r_3)\wedge\sigma^k_U(r_2,r_3))\}\\
&= \bigvee_{r_2}\{(\sigma^k_W(r_1,r_2)\wedge\sigma^k_V(r_2,r_3))\wedge(\sigma^k_W(r_1,r_2)\wedge\sigma^k_U(r_2,r_3))\}\\
&= \bigvee_{r_2}\{\sigma^k_W(r_1,r_2)\wedge\sigma^k_V(r_2,r_3)\}\wedge\bigvee_{r_2}\{\sigma^k_W(r_1,r_2)\wedge\sigma^k_U(r_2,r_3)\}\\
&= \sigma^k_{V\circ W}(r_1,r_3)\wedge\sigma^k_{U\circ V}(r_1,r_3)
\end{aligned}$$

$$\tau^k_{(V\wedge U)\circ W}(r_1,r_3) = \bigwedge_{r_2}\{\tau^k_W(r_1,r_2)\wedge(\tau^k_V(r_2,r_3)\wedge\tau^k_U(r_2,r_3))\}$$

$$=\bigwedge_{r_2}\{(\tau_W^k(r_1,r_2)\wedge\tau_V^k(r_2,r_3))\wedge(\tau_W^k(r_1,r_2)\wedge\tau_U^k(r_2,r_3))\}$$
$$=\bigwedge_{r_2}\{\tau_W^k(r_1,r_2)\wedge\tau_V^k(r_2,r_3)\}\wedge\bigwedge_{r_2}\{\tau_W^k(r_1,r_2)\wedge\tau_U^k(r_2,r_3)\}$$
$$=\tau_{V\circ W}^k(r_1,r_3)\wedge\tau_{U\circ W}^k(r_1,r_3)$$

$$\eta_{(V\wedge U)\circ W}^k(r_1,r_3)=\bigwedge_{r_2}\{\eta_W^k(r_1,r_2)\vee(\eta_V^k(r_2,r_3)\wedge\eta_U^k(r_2,r_3))\}$$
$$=\bigwedge_{r_2}\{(\eta_W^k(r_1,r_2)\vee\eta_V^k(r_2,r_3))\wedge(\eta_W^k(r_1,r_2)\vee\eta_U^k(r_2,r_3))\}$$
$$=\bigwedge_{r_2}\{\eta_W^k(r_1,r_2)\vee\eta_V^k(r_2,r_3)\}\wedge\bigwedge_{r_2}\{\eta_W^k(r_1,r_2)\vee\eta_U^k(r_2,r_3)\}$$
$$=\eta_{V\circ W}^k(r_1,r_3)\wedge\eta_{U\circ W}^k(r_1,r_3)$$

-

$$\sigma_{(V\vee U)\circ W}^k(r_1,r_3)=\bigvee_{r_2}\{\sigma_W^k(r_1,r_2)\wedge(\sigma_V^k(r_2,r_3)\vee\sigma_U^k(r_2,r_3))\}$$
$$=\bigvee_{r_2}\{(\sigma_W^k(r_1,r_2)\wedge\sigma_V^k(r_2,r_3))\vee(\sigma_W^k(r_1,r_2)\wedge\sigma_U^k(r_2,r_3))\}$$
$$=\bigvee_{r_2}\{\sigma_W^k(r_1,r_2)\wedge\sigma_V^k(r_2,r_3)\}\vee\bigvee_{r_2}\{\sigma_W^k(r_1,r_2)\wedge\sigma_U^k(r_2,r_3)\}$$
$$=\sigma_{V\circ W}^k(r_1,r_3)\vee\sigma_{U\circ W}^k(r_1,r_3)$$

$$\tau_{(V\vee U)\circ W}^k(r_1,r_3)=\bigwedge_{r_2}\{\tau_W^k(r_1,r_2)\wedge(\tau_V^k(r_2,r_3)\vee\tau_U^k(r_2,r_3))\}$$
$$=\bigwedge_{r_2}\{(\tau_W^k(r_1,r_2)\wedge\tau_V^k(r_2,r_3))\vee(\tau_W^k(r_1,r_2)\wedge\tau_U^k(r_2,r_3))\}$$
$$=\bigwedge_{r_2}\{\tau_W^k(r_1,r_2)\wedge\tau_V^k(r_2,r_3)\}\vee\bigwedge_{r_2}\{\tau_W^k(r_1,r_2)\wedge\tau_U^k(r_2,r_3)\}$$
$$=\tau_{V\circ W}^k(r_1,r_3)\vee\tau_{U\circ W}^k(r_1,r_3)$$

$$\eta_{(V\vee U)\circ W}^k(r_1,r_3)=\bigwedge_{r_2}\{\eta_W^k(r_1,r_2)\vee(\eta_V^k(r_2,r_3)\vee\eta_U^k(r_2,r_3))\}$$
$$=\bigwedge_{r_2}\{(\eta_W^k(r_1,r_2)\vee\eta_V^k(r_2,r_3))\vee(\eta_W^k(r_1,r_2)\wedge\eta_U^k(r_2,r_3))\}$$
$$=\bigwedge_{r_2}\{\eta_W^k(r_1,r_2)\vee\eta_V^k(r_2,r_3)\}\vee\bigwedge_{r_2}\{\eta_W^k(r_1,r_2)\vee\eta_U^k(r_2,r_3)\}$$
$$=\eta_{V\circ W}^k(r_1,r_3)\vee\eta_{U\circ W}^k(r_1,r_3)$$

Proposition 4.4. *Let $V\in PFMR(Z_1\times Z_2), U\in PFMR(Z_2\times Z_3)$ and $W\in PFMR(Z_3\times Z_4)$. Then, $(W\circ U)\circ V=W\circ(U\circ V)$.*

Proof:

$$\sigma_{(W\circ U)\circ V}^k(r_1,r_4)=\bigvee_{r_2}\{\sigma_V^k(r_1,r_2)\wedge(\sigma_{W\circ U}^k)(r_2,r_3)\}$$
$$=\bigvee_{r_2}\{\sigma_V^k(r_1,r_2)\wedge\{\bigvee_{r_3}\{\sigma_U^k(r_2,r_3)\wedge\sigma_W^k(r_3,r_4)\}\}\}$$
$$=\bigvee_{r_2}\{\bigvee_{r_3}\{\sigma_V^k(r_1,r_2)\wedge\{\sigma_U^k(r_2,r_3)\wedge\sigma_W^k(r_3,r_4)\}\}\}$$
$$=\bigvee_{r_2}\{\bigvee_{r_3}\{\sigma_V^k(r_1,r_2)\wedge\sigma_U^k(r_2,r_3)\}\wedge\sigma_W^k(r_3,r_4)\}$$
$$=\bigvee_{r_3}\{\sigma_{U\circ V}^k(r_1,r_3)\wedge\sigma_W^k(r_3,r_4)$$
$$=\sigma_{W\circ(U\circ V)}^k(r_1,r_4)$$

$$\begin{aligned}\tau^k_{(W\circ U)\circ V}(r_1,r_4) &= \bigwedge_{r_2}\{\tau^k_V(r_1,r_2)\wedge(\tau^k_{W\circ U})(r_2,r_3)\}\\ &= \bigwedge_{r_2}\{\tau^k_V(r_1,r_2)\wedge\{\bigwedge_{r_3}\{\tau^k_U(r_2,r_3)\wedge\tau^k_W(r_3,r_4)\}\}\}\\ &= \bigwedge_{r_2}\{\bigwedge_{r_3}\{\tau^k_V(r_1,r_2)\wedge\{\tau^k_U(r_2,r_3)\wedge\tau^k_W(r_3,r_4)\}\}\}\\ &= \bigwedge_{r_2}\{\bigwedge_{r_3}\{\tau^k_V(r_1,r_2)\wedge\tau^k_U(r_2,r_3)\}\wedge\tau^k_W(r_3,r_4)\}\\ &= \bigwedge_{r_3}\{\tau^k_{U\circ V}(r_1,r_3)\wedge\tau^k_W(r_3,r_4)\\ &= \tau^k_{W\circ(U\circ V)}(r_1,r_4)\end{aligned}$$

$$\begin{aligned}\eta^k_{(W\circ U)\circ V}(r_1,r_4) &= \bigwedge_{r_2}\{\eta^k_V(r_1,r_2)\wedge(\eta^k_{W\circ U})(r_2,r_3)\}\\ &= \bigwedge_{r_2}\{\eta^k_V(r_1,r_2)\wedge\{\bigwedge_{r_3}\{\eta^k_U(r_2,r_3)\wedge\eta^k_W(r_3,r_4)\}\}\}\\ &= \bigwedge_{r_2}\{\bigwedge_{r_3}\{\eta^k_V(r_1,r_2)\wedge\{\eta^k_U(r_2,r_3)\wedge\eta^k_W(r_3,r_4)\}\}\}\\ &= \bigwedge_{r_2}\{\bigwedge_{r_3}\{\eta^k_V(r_1,r_2)\wedge\eta^k_U(r_2,r_3)\}\wedge\eta^k_W(r_3,r_4)\}\\ &= \bigwedge_{r_3}\{\eta^k_{U\circ V}(r_1,r_3)\wedge\eta^k_W(r_3,r_4)\\ &= \eta^k_{W\circ(U\circ V)}(r_1,r_4).\end{aligned}$$

5. Conclusion

In this paper, it has been established that the Picture Fuzzy MultiRelation (PFMR) is an extension of the Picture Fuzzy Relation (PFR). Some operations (union, intersection and complement) and some operators (Arithmetic mean operator, Geometric mean operator and Harmonic mean operator) have been studied with examples. Finally, the composition of PFMRs was introduced, and some of its properties were obtained. For future work, some applications of PFMR in decision-making, medical diagnosis, electoral systems, appointment procedures and pattern recognition will be explored.

Acknowledgement. The author would like to express sincere appreciation to the esteemed reviewers for their valuable comments and suggestions, which significantly contributed to the quality of the paper.

Conflict of interest. The author declares no conflicts of interest.

Authors' Contributions. This is the sole work of the author.

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