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Short Communication

On Diophantine Equation $3^x73^y + 37^t = z^2$

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Abstract. In this paper, we investigated and solved the Diophantine equation $3^x73^y + 37^t = z^2$ for non-negative x, y, t and z and found that it has three non-negative solutions (x, y, t, z) = (1, 0, 0, 2), (1, 1, 1, 16) and (3, 0, 1, 8). We showed that for y = 0 and t = 0, the equation has a solution (1, 0, 0, 2); for x = z = 1, it has a solution (1, 1, 1, 16); and for x = 3, y = 0, it has a solution (3, 0, 1, 8).

Keywords: Diophantine equations, Integral solutions.

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1. Introduction

Many authors have investigated exponential Diophantine equations. In 2002, Sandor studied equations $3^x + 3^y = 6^z$ [6]and $4^x + 18^y = 22^z$ [7]. In 2013, Sroysang [8] was asked for the set of all solutions (x, y, z, w)∈N^4 for the Diophantine equation $7^x + 8^y = z^2$ and in 2014, Jerico and Rabago [2] found all the solutions of the equation $3^x + 5^y + 7^z = w^2$ as (0,0,1,3), (1,1,0,3) and (3,1,2,9). Also in the year 2014, Suvarnamani [10] solved $p^x + q^y = z^2$ where p is an odd prime number for which q-p =2 and x and y are non-negative integers. In 2017, Sadani [4] solved the Diophantine equation $2^x+2 + 7(3^y) + 11 = z^2$, and in 2018, Suganda et al. [9] solved the nonlinear Diophantine equation $11^x + 13^y = z^2$ and found that this system of equations has no solutions. Subsequently, Rao [3] found the solution of the Diophantine equation $3^x + 7^y = z^2$ as (1,0,2) and (2,1,4). In 2020, Tahiliani ([13,14]) found the solution of the Exponential Diophantine equation $2^x + 41^y = z^2$ as (3,0,1) and (7,1,13) and in 2021, same author investigated the solution of the Diophantine equations $2(3^x) + 5(7^x) + 11 = z^2$ and $2^{x+3} + 11(3^y) - 1 = z^2$ and many more Diophantine equations using factorials. Recently, Sadani [5], solved the Diophantine equation $2^x71^t + 3^y = z^2$ and proved that the equation has one integral solution. Most recently, Tadee solved four new Diophantine equations $n^x + 5^y = z^2$, $9^x - 3^y = z^2$, $13^x - 7^y = z^2$, $6^x + 4^y = z^2$ and $24^x + 4^y = z^2$

([1,11,12]). Based on the study of the above researchers, the author was motivated to write a research paper on the Diophantine equation $3^x73^y + 37^t = z^2$.

In this research paper, the author solved the Diophantine equation $3^x73^y + 37^t = z^2$ and found that it has three solutions (x, y, t, z) = (1, 0, 0, 2), (1, 1, 1,16) and (3, 0, 1, 8).

2. Main results

We now present our results.

Theorem 1. The Diophantine equation $3^x73^y + 37^t = z^2$ has three solutions (x, y, t, z) = (1, 0, 0, 2), (1, 1, 1, 16) and (3, 0, 1, 8).

Proof: **Case 1**. For y=0 and t=0, equation reduces to $3^x + 1 = z^2$,which has a unique solution (1,2) ([3], Lemma 2.1).

Hence, the equation $3^x73^y + 37^t = z^2$ has a solution (x, y, t, z) = (1, 0, 0, 2).

Case 2.: For x =0, the equation $73^y + 37^t = z^2$ has no solution.

Proof: (i).For y with the even value and z with the odd value, if y has an even value, its obtained $73^y \equiv 1 \pmod 4$ and then if z has the odd value, then $z^2 \equiv 1 \pmod 4$. So, $73^y + 37^t = z^2 \leftrightarrow 1 \pmod 4 + 1 \pmod 4 \equiv 2 \pmod 4$.Also, if z has the odd value, then $z^2 \equiv 1 \pmod 4$, which is a contradiction. Consequently, for this case, the Diophantine equation $73^y + 37^t = z^2$ has no solution.

(ii). For y with the even value and z with even value, if y has the even value, then it is obtained $73^y \equiv 1 \pmod 4$, and if z has even value, then it is obtained that $z^2 \equiv 0 \pmod 4$.We have $73^y + 37^t = = z^2 \leftrightarrow 1 \pmod 4 + 1 \pmod 4 \equiv 2 \pmod 4$. But, as z has an even value, so $z^2 \equiv 0 \pmod 4$, which is a contradiction. Hence, the Diophantine equation $73^y + 37^t = z^2$ has no solution.

(iii) For y with the odd value and z with odd value, if y has the odd value, then it is obtained $73^y \equiv 1 \pmod 4$ and if z has odd value, then it is obtained that $z^2 \equiv 1 \pmod 4$.We have $73^y + 37^t = = z^2 \leftrightarrow 1 \pmod 4 + 1 \pmod 4 \equiv 2 \pmod 4$. But, as z has an even value, so $z^2 \equiv 1 \pmod 4$, which is a contradiction. therefore, the Diophantine equation $73^y + 37^t = z^2$ has no solution.

(iv). For y to have an odd value and z with an even value, the case is similar to (ii).

Based on these four explanations, it can be concluded that the Diophantine equation $73^y + 37^t = z^2$ has no solution.

Case 3. For x, z = 1, we have $3 \cdot 73^y + 37 = t^2$. Hence we have $3.73^y + 1 = t^2 - 36$. Now L.H.S is even, so putting t = 2z gives $3.73^y + 1 = 4z^2 - 36 = 4(z - 3)(z+3)$. As 3 is a triangular number so (y,z) =(1,8) is a solution of the equation $3.73^y + 1 = 4z^2 - 36 = 4(z-3)(z+3)$ and hence (1,16) is the solution of the equation $3 \cdot 73^y + 37 = t^2$.

Hence, the equation $3^x73^y + 37^t = z^2$ has a solution (x, y, t, z) = (1, 1, 1, 16).

For x=0, t=0, the equation reduces to $73^y + 1 = z^2$. Similarly for x=0, y=0, equation reduces to $37^t + 1 = z^2$ which again has no solution ([9], Theorem 2).

Case 4. For y=0, the equation reduces to $3^x + 37^t = z^2$.

For x, t $\geq$ 1, the equation has no solution.

Proof: Let x,t $\geq$ 1.

Let us analyze the equation modulo 3.

$3^x \equiv 0 \pmod 3$ for $x \geq 1$ and $37 \equiv 1 \pmod 3$,

so $3^x + 37^t \equiv 0 + 1^y \equiv 1 \pmod 3$,

This means that $z^2 \equiv 1 \pmod 3$, so $z \equiv 0 \pmod 3$ is not possible.

Let us analyze the equation for modulo 4.

$3^x + 37^t = z^2$ becomes $(-1)^x + 1^t \equiv z^2 \pmod 4$, since $3 \equiv (-1) \bmod 4$ and $37 \equiv 1 \pmod 4$. So this means $(-1)^x + 1 \equiv z^2 \pmod 4$.

If x is even, then $2 = z^2 \pmod 4$.

Since $z^2 \equiv 2 \pmod 4$ is impossible, there is no solution when x is even or positive.

If x is odd, say x= 2k +1, then $-1 + 1 = 0$.So $z^2 \equiv 0 \pmod 4$. This is possible when z is even.

For t = 1, x must be odd.

Consider modulo 5

$3^x + 2^t = z^2 \pmod 5$ because $37 \equiv 2 \pmod 5$.

The possible values of $z^2 \pmod 5$ are 0,1, and 4.

Let us check the exponents. The powers of 3(mod 5) cycle are 3,4,2,1, whereas for the 2 (mod 5) cycle are 2, 4, 3, 1.

Let us consider small values of x.

If x=1, z^2 =40, so no solution.

If x=3, 3^3 +37 =64= z^2, so z=8. Thus (x, t, z) =(3,1, 8) is the solution.

Hence the equation $3^x73^y + 37^t = z^2$ has a solution (x,y, t, z) = (3,0,1,8).

If x=5, then z^2 =280, no solution

If x=7, then z^2 =2224, no solution.

Modulo 37

If t≥1, then $3^x + 37^t = z^2$.

So, $3^x = z^2 \pmod{37}$.

This means 3^x must be quadratic modulo 37. We can check the quadratic residue modulo 37 using the Legendre symbol $\frac{3}{37}$.

Using quadratic reciprocity,

$$\left(\frac{3}{37}\right) = \left(\frac{37}{3}\right) -1^{\frac{(3-1)(37-1)}{4}} = \left(\frac{1}{3}\right)(-1)^{18} = (1)(1)=1$$

This implies that 3 is a quadratic residue modulo 37.

Since $3^x \equiv z^2 \pmod{37}$, x must be even for 3^x to be a quadratic residue. However, we have already shown that x must be odd. This is a contradiction.

Hence, we proved that Diophantine equation Diophantine equation $3^x73^y + 37^t = z^2$ has three solutions (x, y, t, z) = (1, 0, 0, 2), (1, 1, 1,16) and (3, 0, 1, 8).

3. Conclusion

It is found that the Diophantine equation $3^x73^y + 37^t = z^2$ has three solutions (x, y, t, z) = (1, 0, 0, 2), (1, 1, 1,16) and (3, 0, 1, 8). Indeed, there is much scope for further work based on these types of Diophantine equations.

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Conflict of interest. A single author writes the paper, so there is no conflict of interest.

Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

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