Journal of Mathematics and Informatics
Vol. 29, 2025, 23-26
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 9 December 2025
www.researchmathsci.org
DOI:http://dx.doi.org/10.22457/jmi.v29a03262

Journal of **Mathematics and Informatics**

Short Communication

On the Diophantine Equation $p^x + (p-1)^y = z^2$ where p is a Prime Number with $p \equiv 3 \pmod 4$

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Received 28 October 2025; accepted 8 December 2025

Abstract. In this paper, we prove that the Diophantine equation $p^x + (p-1)^y = z^2$, where p is a prime number with $p \equiv 3 \pmod 4$ and x, y, z are non-negative integers, has exactly three solutions, which are (p,x,y,z) ϵ {(3,0,3,3), (3,1,0,2), (3,2,4,5)}.

Keywords: Diophantine equation; Congruence; Legendre symbol

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

There are lots of studies about the exponential Diophantine equation of type $a^x + b^y = z^2$, where a, b are positive integers and x, y, z are non-negative integers (see [1, 2, 3]). For instance, Sroysang [4] proved in 2013 that $(0,1,2)$, $(3,0,3)$ and $(4,2,5)$ are all non-negative integer solutions (x, y, z) of the Diophantine equation $2^x + 3^y = z^2$. In 2022, Gayo and Bacani [5] presented all non-negative integer solutions of the Diophantine equation $M^x + (M-1)^y = z^2$, where M is a Mersenne prime. In 2024, Gayo and Siong [6] proved that the Diophantine equation $11^x + 10^y = z^2$ and $17^x + 16^y = z^2$ are unsolvable in non-negative integers. In 2025, Gayo et al. [7] showed that the Diophantine equation $19^x + 18^y = z^2$ has no non-negative integer solution. Meanwhile, Assiry [8] showed that the Diophantine equation $23^x + 22^y = z^2$ has no non-negative integer solution. Moreover, Gayo [9] also proved that the Diophantine equation $29^x + 28^y = z^2$ has no non-negative integer solution.

Inspired by the above research studies, we will solve the Diophantine equation $p^x + (p-1)^y = z^2$, where p is a prime number with $p \equiv 3 (\bmod 4)$ and x, y, z are non-negative integers.

2. Preliminaries

In this section, we present some helpful Definitions and Theorems.

Definition 2.1. Let p be an odd prime and a be an integer such that $\gcd(a, p) = 1$. If the congruence $z^2 \equiv a \pmod{p}$ has an integer solution, then a is said to be a quadratic residue of p. Otherwise, a is called a quadratic non-residue of p.

Definition 2.2. Let p be an odd prime and a be an integer such that $\gcd(a, p) = 1$. The Legendre symbol, $\left(\frac{a}{p}\right)$, is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

Theorem 2.1. [10] Let p be an odd prime and a, b be integers with $\gcd(a, p) = 1$ and $\gcd(b, p) = 1$.

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right).$$

Theorem 2.2. [10] Let p be an odd prime.

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Theorem 2.3. [4] The triple $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three non-negative integer solutions (x, y, z) of the Diophantine equation $2^x + 3^y = z^2$.

Theorem 2.4. [6] The triple $(3, 3, 3)$ is the unique non-negative integer solution (p, y, z) of the Diophantine equation $1 + (p-1)^y = z^2$, where p is a prime number.

3. Main results

In this section, we present our results.

Theorem 3.1. Let p be a prime number with $p \equiv 3 \pmod{4}$ and x, y, z be non-negative integers. Then all solutions of the Diophantine equation $p^x + (p-1)^y = z^2$ are

$$(p, x, y, z) \in \{(3, 0, 3, 3), (3, 1, 0, 2), (3, 2, 4, 5)\}.$$

Proof: Since x is a non-negative integer, we will separate it into two cases:

Case 1. $x = 0$. Then $1 + (p-1)^y = z^2$. By Theorem 2.4, we get $(p, x, y, z) = (3, 0, 3, 3)$.

Case 2. $x \geq 1$. Then $p^x \equiv 0 \pmod p$. It implies that $p^x + (p-1)^y \equiv (-1)^y \pmod p$. Thus $z^2 \equiv (-1)^y \pmod p$ and so $\left(\frac{(-1)^y}{p}\right) = 1$. By Theorem 2.1, we get $\left(\frac{-1}{p}\right)^y = 1$. Since p is a prime number with $p \equiv 3 \pmod 4$ and Theorem 2.2, we have $(-1)^y = 1$. It follows that y is even. There exists a non-negative integer k such that $y = 2k$. It implies that $p^x + (p-1)^{2k} = z^2$. Then $\left[z-(p-1)^k\right]\left[z+(p-1)^k\right] = p^x$. Since p is a prime number, we have $z-(p-1)^k = p^u$ and $z+(p-1)^k = p^{x-u}$ for some non-negative integer u. Then $2(p-1)^k = p^u\left(p^{x-u}-1\right)$. Therefore $u = 0$ and so $2(p-1)^k = p^x - 1$. Since $2(p-1)^k \equiv 2(-1)^k \equiv \pm 2 \pmod p$ and $p^x - 1 \equiv -1 \pmod p$, we get $\pm 2 \equiv -1 \pmod p$. Since p is a prime number, we have $p = 3$. By Theorem 2.3, we obtain that $(p,x,y,z) \in \{(3,1,0,2),(3,2,4,5)\}$.

From both cases, we can conclude that $(3,0,3,3),(3,1,0,2)$ and $(3,2,4,5)$ are all non-negative integer solutions (p,x,y,z) of the equation $p^x + (p-1)^y = z^2$.

By Theorem 3.1, we can easily show that some previous researches are true.

Corollary 3.2. [5] The Diophantine equation $7^x + 6^y = z^2$ has no non-negative integer solution.

Corollary 3.3. [6] The Diophantine equation $11^x + 10^y = z^2$ has no non-negative integer solution.

Corollary 3.4. [7] The Diophantine equation $19^x + 18^y = z^2$ has no non-negative integer solution.

Corollary 3.5. [8] The Diophantine equation $23^x + 22^y = z^2$ has no non-negative integer solution.

4. Conclusion

By using a modular arithmetic method, the Diophantine equation $p^x + (p-1)^y = z^2$, where p is a prime number with $p \equiv 3 \pmod 4$ and x, y, z are non-negative integers, has exactly three solutions, which are $(p,x,y,z) \in \{(3,0,3,3),(3,1,0,2),(3,2,4,5)\}$.

Acknowledgements. The author would like to thank reviewers for careful reading of this manuscript and the useful comments. This work was supported by Research and

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Development Institute and Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

Conflict of interest. The paper is written by a single author so there is no conflict of interest.

Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

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