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Analysis of Delayed Prey-Predator System with Ratio-Dependent III Functional Response

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Abstract. In this paper, we proposed and analyzed a prey-predator system with discrete time delay incorporating ratio-dependent III functional response. The equilibrium of proposed system are determined and behavior of the system is investigated around equilibrium. In the presence of delay, the condition for boundedness of the system is established. Choosing delay as a bifurcation parameter, the existence of Hopf bifurcation of the system has been investigated. We also show that increasing delay may cause bifurcations into periodic solutions. Some numerical simulation has been performed to substantiate our analytical findings. .

***Keywords*:** Prey-Predator system, discrete time delay, Ratio dependent III functional response, Hopf bifurcation.

1. Introduction

The predator-prey model forms the building block of ecosystem. The dynamical behavior of predator and its prey exhibits a variety of pattern. It paves interest for many biologists and mathematicians to develop significant models for challenging situations. A system of differential or difference equation is used to formulate a predator-prey model mathematically. The key component in the predator-prey interaction is the functional response. Generally, functional response depends on prey density. If the functional response relies on the predator-prey ratio, then such a functional response is called ratio-dependent functional response.

Decades ago, predator-prey models having ratio-dependent functional response have been proposed.(see[1,2,3]) and reference sited there in. it is pointed out qualitative analysis of food chain and multispecies models based on ratio-dependent approach exists in Kesh [11], Gakkar [6], Baek [5]. The study of Hsu [8] and Xiao [17] portrays that ratio-dependent model produce richer dynamics. A recent findings of Jost et al [9] shows that prey dependent and ratio dependent models can fit well with time series generated by eacch other. Jost and Ellner [10] proposed and analyzed a two species model with ratio dependent III functional response.

2. Mathematical model

Mathematical model considered is based on the predator-prey system with Ratio dependent III functional response.

$$\frac{dX}{dT} = rX\left(1-(\frac{X}{K})\right) - \frac{\alpha X^2 Y}{mY^2 + X^2}$$

$$\frac{dY}{dT} = \frac{\beta X^2 Y}{mY^2 + X^2} - \gamma Y \qquad (1)$$

where r is the intrinsic growth rate of prey, K is the environmental carrying capacity of the prey, α is the maxima relative increase of predation, m is the half-saturation constant, β is the conversion factor. γ is the death rate of predator.

In order to minimize the number of parameters involved with the model system, it is extremely useful to write the system in non-dimensionalized form. For this purpose we introduce the variables X, Y and T as follows.

$$x \to \frac{X}{K}, y \to \frac{\sqrt{a}Y}{K} \quad \text{and} \quad t \to Tr$$

In terms of the non-dimensionalized variables the model system (1) become

$$\frac{dx}{dt} = x(1-(x)) - \frac{cx^2 y}{y^2 + x^2}$$

$$\frac{dy}{dt} = \frac{dx^2 y}{y^2 + x^2} - ey \qquad (2)$$

where

$$c = \frac{\alpha}{r\sqrt{m}}, d = \frac{\beta}{r}, e = \frac{\gamma}{r}$$

The non-negative initial conditions are associated with system (2)

$$x \geq 0, y \geq 0 \qquad (3)$$

The objective of this paper is to perform a qualitative analysis on this ratio dependent III functional response in the system with discrete time delay. . The paper is organized as follows: In section 3 we present some positive invariance and boundedness results. In section 4 we obtain the existence of the equilibrium points of model (2) .In section 5, we investigate local behavior of the equilibrium points in absence of delay. In section 6 analysis of the model in presence of discrete delay is discussed. In section 7 Numerical simulations are used to illustrate some of our result.

3. Positive invariance and boundedness

Preliminaries

Let x and y represent the prey and predator population respectively. We have formed positive invariance and boundedness for the system (2). The positive x and y ensures that population ever survives. Since the resources are limited, the boundedness may depict a natural suspension to growth.

Positive invariance

Theorem 1. The positive quadrant int(R_+^2) is invariant for system(2)

Proof: We prove that for all $t \in [0, Q[, x(t) > 0, y(t) > 0$ .we show this by method of contradiction

Suppose, it is not true, there must exists one , t_q , $0 < t_q < Q$ such that $\forall\ t \in [0, t_q[, x(t) > 0, y(t) > 0$

And minimum of $x(t_q), y(t_q)$ vanish.

From the system (2), we have

$$x(t) = x(0)\exp\left(\int_0^t G_1(x,y)dt\right)$$

$$y(t) = y(0)\exp\left(\int_0^t G_2(x,y)dt\right)$$

where

$$G_1(x,y) = (1 - x - \frac{cxy}{x^2+y^2})$$

$$G_2(x,y) = (\frac{dx^2}{x^2+y^2} - e)$$

Since (x, y) are defined and continuous on $[0, t_q]$ there exists a $L \geq 0$ such that $\forall t \in [0, t_q[$

$$x(t) = x(0)\exp\left(\int_0^t G_1(x,y)dt\right) \geq x(0)\exp(-t_q L)$$

$$y(t) = y(0)\exp\left(\int_0^t G_2(x,y)dt\right) \geq y(0)\exp(-t_q L)$$

It is clear that if limit $t \rightarrow t_q$ we obtain

$$x(t_q) \geq x(0)\exp(-t_q L)$$

$$y(t_q) \geq y(0)\exp(-t_q L)$$

which contradicts the fact minimum of one $x(t_q), y(t_q)$ vanish

There fore $\forall\ t \in [0, Q[, x(t) > 0, y(t) > 0$

This completes the proof.

Theorem 2. All the solution of the system (2) with initial condition (3) that initiate in R_+^2 are uniformly bounded.

4. Existence of equilibrium points

In this section we first determine the existence of fixed points of the differential equations (2) and then we investigate their stability by calculating the eigenvalues for the variational matrix (2) at each fixed point. To determine the fixed points, the equilibrium is the solution of pair of equation given below

$$x(1-x-\frac{cxy}{y^2+x^2})=0 \tag{4}$$
$$y(\frac{dx^2}{y^2+x^2}-e)=0$$

By simple computation of the above algebraic system, it was found that there are two non-negative fixed points

i) $E_1(1,0)$ is axial fixed point is always exists, as the prey population grows to the carrying capacity in the absence of predation

ii) $E_2(x^*,y^*)$ is the positive equilibrium point exists in the interior of the first quadrant

where $x^*=[\frac{d-c\sqrt{e(d-e)}}{d}]$ and $y^*=\sqrt{\frac{d-e}{e}}x^*$

5. Local Stability analysis in absence of delay

In order to check the stability of the model (2), the variational matrix corresponding to each equilibrium point is calculated.

The variational matrix of equilibrium point at $E_1(1,0)$ is

$$E_1=\begin{pmatrix}-1 & -c\\ 0 & d-e\end{pmatrix}$$

The eigenvalues of E_1 are -1 and $d-e$ .Therefore the model system (2) is stable around E_1 for $d<e$ for which, $x-y$ plane is the stable. On the other hand the system is always unstable around E_1 if $d>e$ which is, infact , a saddle point and whose stable manifold in x-direction and unstable in y direction for $d>e$ Hence we state the following theorem.

Theorem 3. The equilibrium point E_1 is stable if $d<e$.

6. Analysis of delayed model

Ideally, a real system is modeled with time-delay using differential equation. Time -delay occurs in any manmade or natural phenomenon. In general, delay differential equation exhibit much more complicated dynamics than ordinary differential equation since a time delay can cause stable equalibrium to become unstable and then population fluctuate. Ignorance of time delay is ignorance of reality. The significance and application of time-delay in realistic models is elaborated in books of Gopalsamy[7], Kuang[12].more realistic and importance models of population ecology should be taken into account with the time delay and the stability of an ecological systems with time delays has been studied by many authors[13-16,18-19]

In this section we analyze the model system (2) with delay τ (discrete time delay in predator response function).then the model system (2) takes the following form

$$\frac{dx}{dt} = x\big(1-(x)\big) - \frac{cx^2y}{y^2+x^2}$$

$$\frac{dy}{dt} = \frac{dx^2(t-\tau)y(t-\tau)}{y^2((t-\tau)+x^2(t-\tau)} - e \tag{5}$$

With the initial densities

$$x_0(\theta) = \phi_1(\theta) > 0, y_0(\theta) = \phi_2(\theta) > 0, \phi_r \in C([-\tau,0] \to R_+), \theta \in (-\tau,0), \tau > 0 \tag{6}$$

Boundedness of the system with τ>0

Theorem 4. All solution of the system (5) are uniformly bounded with an ultimate bound.

Proof: Define the function

$$\omega(t) \quad = \quad x(t-\tau) + \frac{c}{d}y(t)$$

which on differentiation with respect to t

$$\frac{d\omega}{dt} \quad = \quad \frac{dx}{dt}(t-\tau) + \frac{c}{d}\frac{dy}{dt}$$

$$= x(t-\tau)(1-x(t-\tau) - \frac{(x^2(t-\tau)y(t-\tau)}{x^2(t-\tau)+y^2(t-\tau)} + \frac{c}{d}\left[\frac{dx^2(t-\tau)y(t-\tau)}{x^2(t-\tau)+y^2(t-\tau)} - ey(t)\right]$$

$$= \quad -\left(x(t-\tau)+\frac{c}{d}y(t)\right) + x(t-\tau)(2-x(t-\tau)) + cy(t)\left(\frac{1-e}{d}\right)$$

$$\leq \quad -w + \left(1+\frac{(1-e)^2}{4d}\right)$$

which yields

$$\lim_{t\to\infty}\sup w(t) \quad \leq \quad 1+\frac{(1-e)^2}{4d}$$

$$\equiv \quad M$$

Then there exists positive constant M>0 such that ω(t) < M for large t.

Local stability analysis in presence of delay

Now we direct our attention due to discuss of stability of system at E_1.

The variational matrix of the system at E_1 takes the form

$$E_1 \quad = \quad \begin{bmatrix} -1 & -c \\ 0 & de^{-\lambda\tau} - e \end{bmatrix}$$

The characteristic equation of E_1 is of the form

$$(\lambda+1)(\lambda+e-de^{-\lambda\tau})=0$$

Here λ=-1 is a negative eigenvalue we now consider the equation

$$\lambda \quad = \quad de^{-\lambda\tau} - e \tag{7}$$

If τ=0, and d<e, the equilibrium E_1 is locally asymptotically stable. If substitute λ=iμ in (7) and equating real and imaginary parts, we obtain,

$$\mu \quad = \quad -d\sin\mu\tau$$

$$e \quad = \quad d\cos\mu\tau \tag{8}$$

Eliminating τ from (8) we obtain

$$\mu^2 \quad = \quad d^2 - e^2 \tag{9}$$

We know that (9) has positive root μ_+ is d>e. Hence there is positive constant τ_+ such that $\tau>\tau_+$, E_1 becomes unstable.

The main purpose of this section to study the stability behavior of $E_2(x^*, y^*)$ in the presence of discrete delay ($\tau \neq 0$). Now to prove the stability behavior of $E_2(x^*, y^*)$ for the system (5), First we linearize the system (5) by using following transformation

$$u'(t) = a_{11}u(t) + a_{12}v(t)$$

$$v'(t) = c_{21}u(t-\tau) + c_{22}v(t-\tau)$$

where

$$a_{11} = -x^* + \frac{cx^* y^* (x^{*^2} - y^{*^2})}{(x^{*^2} + y^{*^2})^2}, a_{12} = \frac{-cx^{*^2}(x^{*^2} - y^{*^2})}{(x^{*^2} + y^{*^2})^2}$$

$$c_{21} = \frac{2dx^* y^{*^3}}{(x^{*^2} + y^{*^2})^2}, c_{22} = \frac{-2dx^{*^2} y^{*^2}}{(x^{*^2} + y^{*^2})^2}, a_{22} = -e$$

The characteristic equation is $\Delta(\lambda,\tau) = (\lambda^2 + l_1\lambda + l_2) + (l_3\lambda + l_4)e^{-\lambda\tau} = 0$ (10)

where

$$l_1 = -a_{11} - a_{22}, l_2 = a_{11}a_{22}, l_3 = -c_{22}, l_4 = c_{22}a_{11} - a_{12}c_{21}$$

If $\tau = 0$ in (10) the characteristic equation becomes

$$\lambda^2 + (l_1 + l_3)\lambda + (l_2 + l_4) = 0 \tag{11}$$

The root of (11) is

$$\lambda = \frac{-(l_1 + l_3) \pm \sqrt{(l_1 + l_3)^2 - 4(l_2 + l_4)}}{2} \tag{12}$$

From (12), we have λ has negative real parts if and only if

$$l_1 + l_3 > 0, l_2 + l_4 > 0 \tag{13}$$

Now for $\tau \neq 0$,Put $\lambda = iw$ in equation in (10), we get

$$(-w^2 + il_1 w + l_2) + (il_3 w + l_4)(\cos w\tau - i\sin w\tau) = 0$$

Equating real, imaginary parts we get

$$w^2 - l_2 = l_4 \cos w\tau + l_3 w \sin w\tau$$

$$-l_1 w = l_3 w \cos w\tau - l_4 \sin w\tau$$

Squaring and adding we get (14)

$$w^4 + w^2(l_1^{\,2} - l_3^{\,2} - 2l_2) + (l_2^{\,2} - l_4^{\,2}) = 0$$

We get the roots of (14) is

$$w^2 = \frac{-(l_1^{\,2} - l_3^{\,2} - 2l_2) + \sqrt{(l_1^{\,2} - l_3^{\,2} - 2l_2)^2 - 4(l_2^{\,2} - l_4^{\,2})}}{2} \quad (15)$$

It follows

$(l_1^{\,2} - l_3^{\,2} - 2l_2) > 0$ and $(l_2^{\,2} - l_4^{\,2}) > 0$ (16)

are satisfied .Hence the equation does not have any positive solutions.

We conclude the following theorem

Theorem 4. If the conditions (13) and (16) are satisfied ,then all the roots of the equation (10) have negative real parts for all $\tau \ge 0$ .Then the equilibrium $E_2(x^*, y^*)$ is stable for $\tau \ge 0$.

Put $w^2 = \delta$ then the equation (14) becomes

$$\delta^2 + \delta(l_1^{\,2} - l_3^{\,2} - 2l_2) + (l_2^{\,2} - l_4^{\,2}) = 0$$

If $(l_2^{\,2} - l_4^{\,2}) < 0$ holds then (14) has unique positive root δ_0 then

$$\delta_0 = \sqrt{\frac{-(l_1^{\,2} - l_3^{\,2} - 2l_2) + \sqrt{(l_1^{\,2} - l_3^{\,2} - 2l_2)^2 - 4(l_2^{\,2} - l_4^{\,2})}}{2}}$$

The corresponding time delay is

$$\tau_0 = \frac{1}{w_0}\cos^{-1}\left(\frac{(l_4 - l_1 l_3)w_0^{\,2} - l_2 l_4}{l_3^{\,2} w_0^{\,2} + l_4^{\,2}}\right) + \frac{2k\pi}{w_0}, k = 0,1,2......$$

Differentiate (10) with respect to τ

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + l_1}{\lambda(l_3\lambda + l_4)e^{-\lambda\tau}} + \frac{l_3}{\lambda\left(l_3\lambda + l_4\right)} - \frac{\tau}{\lambda}$$

$$= \frac{\lambda^2 - l_2 - (l_3\lambda + l_4)e^{-\lambda\tau}}{\lambda^2(l_3\lambda + l_4)e^{-\lambda\tau}} + \frac{l_3\lambda}{\lambda^2\left(l_3\lambda + l_4\right)} - \frac{\tau}{\lambda}$$

$$= \frac{(\lambda^2 - l_2)}{\lambda^2(l_3\lambda + l_4)e^{-\lambda\tau}} - \frac{1}{\lambda^2} + \frac{l_3\lambda}{\lambda^2\left(l_3\lambda + l_4\right)} - \frac{\tau}{\lambda}$$

$$= \frac{(\lambda^2 - l_2)}{-\lambda^2(\lambda^2 + l_1\lambda + l_2)} - \frac{l_4}{\lambda^2\left(l_3\lambda + l_4\right)} - \frac{\tau}{\lambda}$$

Taking $\lambda = iw_0$ in above equation, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda = iw_0} = \frac{(iw_0)^2 - l_2}{-(iw_0)^2\left(\left(iw_0\right)^2 + l_1(iw_0) + l_2\right)} + \frac{-l_4}{(i\omega_0)^2\left(l_3\left(iw_0\right) + l_4\right)} + \frac{\tau i}{w_0}$$

$$= \left[\frac{(w_0^2 + l_2)}{w_0{}^2[\left(w_0^2 - l_2\right) - i\left(l_1 w_0\right)} \cdot \frac{(w_0^2 - l_2) + i(l_1 w_0)}{(w_0^2 - l_2) + i(l_1 w_0)}\right] +$$

$$\left[\frac{l_4}{w_0^2\left((l_4 + il_3 w_0\right)} \cdot \frac{(l_4 - il_3 w_0)}{(l_4 - il_3 w_0)}\right] + \frac{\tau i}{w_0}$$

$$\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda = iw_0} = \left[\frac{\left(w_0^4 - l_2{}^2\right)}{w_0^2\left[\left(w_0^2 - l_2\right)^2 + \left(l_1 w_0\right)^2\right]}\right] + \frac{(l_4)^2}{w_0^2[l_4{}^2 + l_3^2 w_0^2]}$$

Thus we obtain $\quad Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda = iw_0} > 0$

Therefore transversality condition holds and hence hopf bifurcation occurs at $\tau = \tau_0$ This signifies that there exits at least or equal value with positive real part for $\tau > \tau_0$

Theorem 5. If E_2 exists with the condition (14) and $\delta = \omega_0{}^2$ be positive root of (14), then there exists a $\tau = \tau_0{}^*$ such that

(i) E_2 is locally asymptotically stable for $0 \le \tau < \tau_0{}^*$

(ii) E_2 is unstable for $\tau > \tau_0{}^*$

(iii) The system (5) undergoes a Hopf –bifurcation around E_2 at $\tau = \tau_0{}^*$

$$\tau_0{}^* = \min\ h(\omega_0)$$

where

$$h(\omega_0) = \frac{1}{w_0}\cos^{-1}\left(\frac{(l_4 - l_1 l_3)w_0{}^2 - l_2 l_4}{l_3{}^2 w_0{}^2 + l_4{}^2}\right) + \frac{2k\pi}{w_0}, k = 0,1,2......$$

and the minimum taken over all positive ω_0 such that $\delta = \omega_0{}^2$ is a solution of (14)

Analysis of Global Stability

In this section, we study the global asymptotic stability of co-existence equilibrium point E_2 of the system (2) by Bendixson- Dulac criterion, in absence of delay.

Theorem 6. The positive equilibrium point E_2 of non-delayed model system (2) is globally asymptotically stable.

Proof: Let us consider the function

$$H(x, y) = \frac{1}{xy}$$

Clearly H (x, y) is positive for both x > 0, y > 0

$$G_1(x,y) = x(1 - x - \frac{cxy}{x^2 + y^2})$$

$$G_2(x,y) = y(\frac{dx^2}{x^2 + y^2} - e)$$

Now

$$\Delta(x,y) = \frac{\partial}{\partial x}(G_1 H) + \frac{\partial}{\partial y}(G_2 \ H)$$

$$= \frac{-1}{y} - \frac{c}{x^2 + y^2} + \frac{2c\,x^2}{\left(x^2 + y^2\right)^2} - \frac{2dxy}{\left(x^2 + y^2\right)^2}$$

$$= \frac{-1}{y} - c\left[\frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\right] - \frac{2dxy}{\left(x^2 + y^2\right)^2}$$

$$= \frac{-1}{y} - \left[\frac{cy^2 - cx^2 + 2dxy}{\left(x^2 + y^2\right)^2}\right]$$

$$< 0$$

From the above equation, we note that Δ(x, y) does not change sign and is not identically zero in the interior of the positive quadrant we show that of x-y plane. In the following theorem E_2 is globally asymptotically stable.

7. Numerical simulations

In this section, we present some numerical simulation result to validate our analytical findings. Let us consider the parameter of the system (2) as in appropriate units. In Fig.1(a) and Fig1(b),we take $d = 0.85, e = 1.32, c = 0.83$,the Eigen values of E_1 are $\lambda_1 = -1, \lambda_2 = -0.47$ .In this case E_1 is locally asymptotically stable. From Fig 1(a) we observe that the predator is driven to extinction, where as the prey approaches the carrying capacity.Fig1 (b) phase portrait is also tends to the boundary equilibrium point (1,0).In Fig 2(a) and Fig 2(b),we take $d = 1.55, e = 1.32, c = 2.3$.

In this case it satisfy the condition of theorem $l_1 + l_3 = 1.3029, l_1^{\,2} - l_3^{\,2} - 2l_2 = 0.90146 > 0, l_2^{\,2} - l_4^{\,2} = 0.1634 > 0$ .We conclude that $E_2(x^*, y^*)$ is locally asymptotically stable. We observe that in Fig 2(a) both prey and predator converges to the positive equilibrium. A phase portrait also shows solution tends to the positive equilibrium.From Fig3(a) and Fig3(b) if we take

$d = 1.55, e = 1.32, c = 0.6$ the eigen values of delayed system is $\lambda_1 = -1, \lambda_2 = 0.23$.In this case E_1 is unstable and the value of $\tau = 0.613$.In this case time series and phase portrait of the system in Fig 3(a) and 3(b).If we take d=1.55,e=1.32,c=2.5,in this case prey and predator population shows periodic solutions see Fig 4(a) and phase portrait as shown in Fig 4(b).

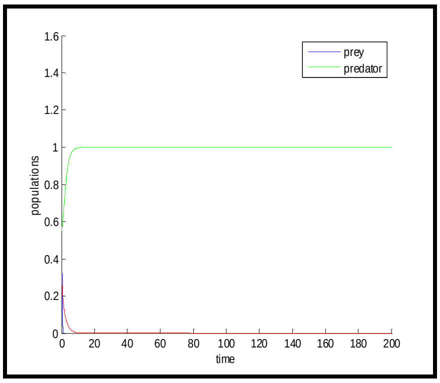

Figure 1(a): Time series

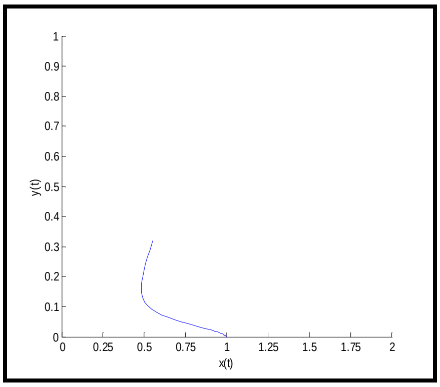

Figure 1(b): Phase Portrait

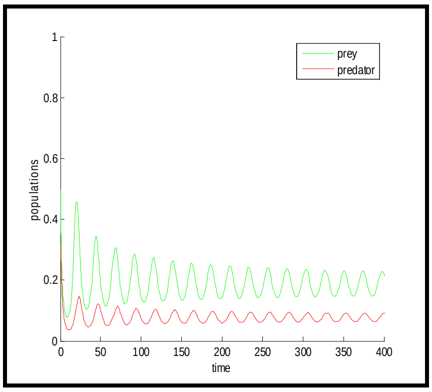

Figure 2(a): Time series

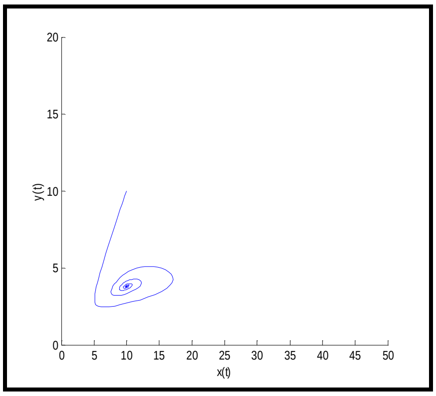

Figure 2(b): Phase Portrait

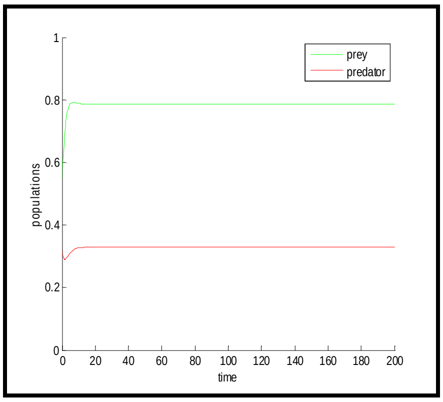

Figure 3(a): Time series

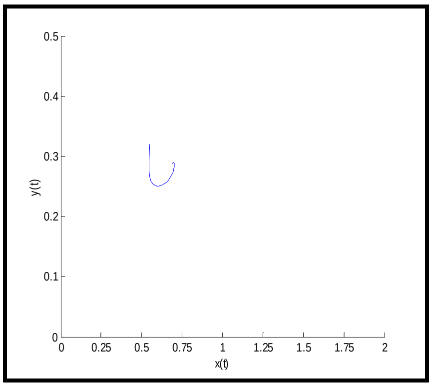

Figure 3(b): Phase Portrait

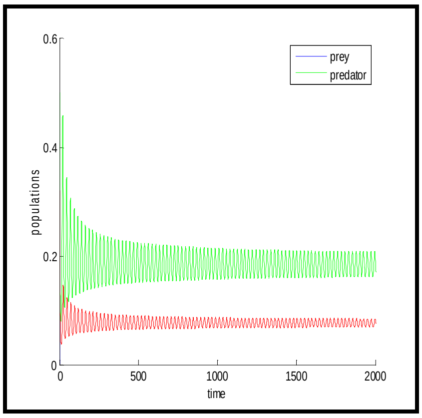

Figure 4(a): Time series

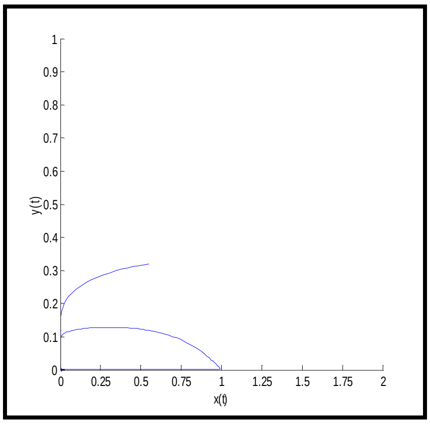

Figure 4(b): Phase Portrait

8. Conclusion

In this work, the local stability condition for various equilibrium points and the boundedness of the system with ratio-dependent III functional response is investigated in the absence of gestational delay. And the stability condition for the interior equilibrium points is analyzed in the presence of discrete time delay. It has been evident that the system undergoes Hopf bifurcation while choosing delay as bifurcation parameter. Also increasing the delay has caused bifurcation into periodic solution. Finally, numerical simulations have substantiated the analytical findings.

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