Journal of Mathematics and Informatics
Vol. 31, 2026, 63-66
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 19 August 2026
www.researchmathsci.org
DOI: http://doi.org/10.22457/jmi.v31a01272

Journal of
Mathematics and
Informatics

The Domination Number and Some Hamiltonian Properties of Graphs

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Received 22 July 2026; accepted 18 August 2026

Abstract. Let G = (V(G), E(G)) be a graph. A dominating set of G is a subset of V(G) such that every vertex in G is either in the set or adjacent to a vertex in the set. The domination number is the size of a minimum dominating set in G. In this note, we present sufficient conditions involving the domination number, minimum degree, and maximum degree for some Hamiltonian properties of graphs.

Keywords: the domination number, Hamiltonian graph, traceable graph

AMS Mathematics Subject Classification (2010): 05C69, 05C45

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph G = (V(G), E(G)), we use n and e to denote its order |V(G)| and size |E(G)|, respectively. The minimum degree and maximum degree of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A subset S of the vertex set V(G) of G is independent if no two vertices in S are adjacent in G. A maximum independent set in a graph G is an independent set of the largest possible size. The independence number, denoted $\alpha = \alpha(G)$, of a graph G is the cardinality of a maximum independent set in G. A subset T of the vertex set V(G) is called a dominating set of G if every vertex in G is either in T or adjacent to a vertex in T. A minimum domination set in a graph G is a domination set of the smallest possible size. The domination number of G, denoted $\gamma = \gamma(G)$, is the size of a minimum dominating set in G. We use $K_{s,\,t}$ to denote a complete bipartite graph with two partition sets Y and Z such that |Y| = s and |Z| = t. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. In this note, we present sufficient conditions involving the domination number, minimum degree, and maximum degree for Hamiltonian graphs and traceable graphs. The main results are as follows.

Theorem 1.1. Let G be a k-connected ($k \geq 2$) graph of order $n \geq 3$ and size e. If

$$\gamma \leq n - \frac{(n-k-1)(\delta+\Delta)^2}{4\delta},$$

then G is Hamiltonian.

Theorem 1.2. Let G be a k-connected ($k \geq 1$) graph of order $n \geq 2$ and size e. If

$$\gamma \leq n - \frac{(n-k-2)(\delta+\Delta)^2}{4\delta},$$

then G is traceable.

2. Lemma

We need the following results as lemmas when we prove our theorems. Lemma 2. and Lemma 2.2 below are from [2].

Lemma 2.1. Let G be a k-connected graph of order $n \geq 3$. If $\alpha \leq k$, then G is Hamiltonian.

Lemma 2.2. Let G be a k-connected graph of order n. If $\alpha \leq k + 1$, then G is traceable.

Lemma 2.3 follows from Theorem 2.22 on Page 55 in [4].

Lemma 2.3. If G is connected graph of order $n \geq 2$, then $n - e \leq \gamma$ with equality if and only if G is $K_{1, n-1}$.

Lemma 2.4 is Corollary 4 on Page 67 in [3]. See also [5].

Lemma 2.4. Let the real numbers γ_k ($k = 1, 2, \ldots, n$) satisfy $0 < m \leq \gamma_k \leq M$. Then

$$\sum_{k=1}^{n} \frac{1}{\gamma_k} \sum_{k=1}^{n} \gamma_k \leq \frac{(m+M)^2}{4mM} n^2.$$

If $M > m$, then the equality sign holds in above inequality if and only if n is an even number; while, at the same time, for n/2 values of k one has $\gamma_k = m$ and for the remaining n/2 values of k one has $\gamma_k = M$. If $M = m$, the equality always holds in the above inequality.

3. Proofs

Proof of Theorem 1.1. Let G be a k-connected graph ($k \geq 2$) with $n \geq 3$ vertices, e edges, and the maximum degree Δ satisfying the conditions in Theorem 1.1. Suppose G is not Hamiltonian. Then Lemma 2.1 implies that $\alpha \geq k + 1$. Clearly, $n \geq \alpha + 1 \geq k + 2$ and therefore $n - k - 1 \geq 1$. Let $I_1 := \{u_1, u_2, ..., u_\alpha\}$ be a maximum independent set in G. Then $I := \{u_1, u_2, ..., u_{k+1}\}$ is an independent set in G.

Clearly, $\sum_{y \in I} d(y) = |E(I, V - I)| \leq \sum_{z \in V - I} d(z)$. Since $\sum_{y \in I} d(y) + \sum_{z \in V - I} d(z) = 2e$, we have that $\sum_{y \in I} d(y) \leq e \leq \sum_{z \in V - I} d(z)$. From Lemma 2.3, we have $n - \gamma \leq e \leq \sum_{z \in V - I} d(z)$. Note that $\delta \leq d(z) \leq \Delta$ for each $z \in V - I$. From Lemma 2.4, we have that

$$\frac{n-k-1}{\Delta} \sum_{z \in V - I} d(z) \leq \sum_{z \in V - I} \frac{1}{d(z)} \sum_{z \in V - I} d(z) \leq \frac{(\delta + \Delta)^2 (n-k-1)^2}{4\delta\Delta}.$$

Thus

$$\sum_{z \in V - I} d(z) \leq \frac{(n-k-1)(\delta+\Delta)^2}{4\delta}.$$

Therefore

$$\frac{(n-k-1)(\delta+\Delta)^2}{4\delta}$$
$$\leq n-\gamma \leq e \leq \sum_{z \in V - I} d(z) \leq$$

$$\frac{(n-k-1)(\delta+\Delta)^2}{4\delta}.$$

Hence $n - \gamma = e$. By Lemma 2.3, we have G is $K_{1,\, n-1}$, contradicting the assumption that G is k-connected, where $k \geq 2$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let G be a k-connected graph ($k \geq 1$) with $n \geq 2$ vertices, e edges, and the maximum degree Δ satisfying the conditions in Theorem 1.2. Suppose G is not traceable. Then Lemma 2.2 implies that $\alpha \geq k + 2$. Clearly, $n \geq \alpha + 1 \geq k + 3$ and therefore $n - k - 2 \geq 1$. Let $I_1 := \{u_1, u_2, ..., u_\alpha\}$ be a maximum independent set in G. Then $I := \{u_1, u_2, ..., u_{k+2}\}$ is an independent set in G. Using the proofs which are similar to ones in the above Proof of Thereon 1.1, we have that $n - \gamma \leq e \leq \sum_{z \in V - I} d(z)$ and

$$\frac{n-k-2}{\Delta} \sum_{z \in V - I} d(z) \leq \sum_{z \in V - I} \frac{1}{d(z)} \sum_{z \in V - I} d(z) \leq \frac{(\delta+\Delta)^2 (n-k-2)^2}{4\delta\Delta}.$$

Thus

$$\sum_{z \in V - I} d(z) \leq \frac{(n-k-2)(\delta+\Delta)^2}{4\delta}.$$

Therefore

$$\frac{(n-k-2)(\delta+\Delta)^2}{4\delta}$$
$$\leq n-\gamma \leq e \leq \sum_{z \in V - I} d(z) \leq$$

$$\frac{(n-k-2)(\delta+\Delta)^2}{4\delta}.$$

Hence

$$n-\gamma = e = \sum_{z \in V - I} d(z) = \frac{(n-k-2)(\delta+\Delta)^2}{4\delta}.$$

From Lemma 2.3, we have that G is $K_{1,\, n-1}$. We also have that $e = \sum_{z \in V - I} d(z)$ which implies that $e = \sum_{y \in I} d(y)$ and therefore G is a bipartite graph with partition sets of I and V - I. From

$$\sum_{z \in V - I} d(z) = \frac{(n-k-2)(\delta+\Delta)^2}{4\delta},$$

We have that

$$\frac{n-k-2}{\Delta} \sum_{z \in V - I} d(z) = \sum_{z \in V - I} \frac{1}{d(z)} \sum_{z \in V - I} d(z) = \frac{(\delta+\Delta)^2 (n-k-2)^2}{4\delta\Delta}.$$

Thus all the vertices in V - I must have the same degree of Δ. Therefore $|V - I| = 1$ and $d(z) = (n - 1)$, where z is the only vertex in V - I. Consequently,

$$(n-1) = \Delta = \sum_{z \in V - I} d(z) = \frac{(n-k-2)(\delta+\Delta)^2}{4\delta}$$
$$= \frac{(n-k-2)(1+n-1)^2}{4*1} = \frac{|V-I|\, n^2}{4} = \frac{n^2}{4}.$$

So, n = 2, contradicting the assumption that $n \geq 4$. This completes the proof of Theorem 1.2.

4. Conclusion

Using Schweitzer's inequality, we present sufficient conditions involving the domination number, minimum degree, and maximum degree for Hamiltonian and traceable graphs.

Acknowledgements. The authors thank the referee for their comments and suggestions, which led to improvements to the initial manuscript.

Conflicts of Interest: The author confirms that there are no conflicts of interest associated with the research, authorship, or publication of this manuscript.

Author Contributions: The author solely conceptualized the study, conducted the research, prepared the manuscript, and reviewed and approved the final version for publication.

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