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Short Communication

An Upper Bound for the Domination Number of a Connected Graph

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Abstract. Let $G = (V, E)$ be a graph. A subset D of the vertex set V of G is called a dominating set if every vertex in G is either in the set D or is adjacent to at least one vertex in D. The domination number of G is the minimum size of a dominating set in G. In this note, we present an upper bound for the domination number of a connected graph.

Keywords: the domination number; the upper bound

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph $G = (V, E)$, we use n and e to denote its order $|V(G)|$ and size $|E(G)|$, respectively. The neighborhood of a vertex u in G is denoted by $N_G(u)$. For a subset S of V, we use $G[S]$ to denote the subgraph of G which is induced by S. We define $N(S)$ as $\cup_{u \in S} N_G(u)$. If a vertex $u \in V - S$, where S is a subset of the vertex set V, we define $d_{G[S]}(u)$ as $|N_G(u) \cap S|$. For disjoint vertex subsets S_1 and S_2 of V, we define $E(S_1, S_2)$ as $\{e: e = pq \in E, p \in S_1, q \in S_2\}$. A subset D of the vertex set V of G is called a dominating set if every vertex in G is either in the set D or is adjacent to at least one vertex in D. The domination number of G, denoted $\gamma(G) := \gamma$, is the minimum size of a dominating set in G. In this note, we present an upper bound for the domination number of a connected graph. We need to define a family of graphs to state our result. The family of the graphs is denoted by $\mathcal{H}(n, \gamma(G))$ and it is defined as $\{G = (V, E): V = X \cup A \cup B$, where X, A, B are pairwise disjoint, $X = \{x_1, x_2, \dots, x_{\gamma(G)}\}$, $A = \{a_1, a_2, \dots, a_{\gamma(G)}\}$, $N(a_t) \cap X = \{x_t\}$ for each t with $1 \le t \le \gamma(G)$, $B = \{b_1, b_2, \dots, b_{n-2\gamma(G)}\}$, $x_p b_q \in E$ for each p with $1 \le p \le \gamma(G)$ and for each q with $1 \le$

$q \leq n - 2\gamma(G)$, both $G[X]$ and $G[V - X]$ are complete}. The main result of this note is as follows.

Theorem 1. Let G be a connected graph of order $n \geq 2$ and size e. Then

$$\gamma(G) \leq \frac{1 + \sqrt{1 + 2(n^2 - n - 2e)}}{2}$$

with equality if and only if $G \in \mathcal{H}(n, \gamma(G))$.

2. Proofs

To prove Theorem 1, we need the following result, which is Lemma 2 on Page 392 in [2].

Lemma. Let G be a connected graph with domination number $\gamma(G) = \gamma$. Then there exists a dominating set X of G with $|X| = \gamma(G) = \gamma$ such that for each vertex $x \in X$, $|N(v, X)| \geq 1$, where $N(v, X) = \{u \in V(G) - X : N(u) \cap X = \{v\}\}$.

Proof of Theorem 1. The ideas in the proof of Theorem 8 in [2] are used here. From the above lemma, there exists a dominating set $X := \{x_1, x_2, \dots, x_\gamma\}$ in G such that $|N(v, X)| \geq 1$ for each $v \in X$. Thus, we have that

$$e = |E(G[X])| + |E(X, V - X)| + |E(G[V - X])|$$
$$\leq \frac{1}{2}\gamma(\gamma - 1) + |E(X, V - X)| + \frac{1}{2}(n - \gamma)(n - \gamma - 1)$$
$$\leq \frac{1}{2}\gamma(\gamma - 1) + \sum_{v \in X} |N(v, X)| + |X|\left(n - |X| - \sum_{v \in X} |N(v, X)|\right) + \frac{1}{2}(n - \gamma)(n - \gamma - 1)$$
$$= \frac{1}{2}\gamma(\gamma - 1) + \gamma(n - \gamma) - (\gamma - 1)\sum_{v \in X} |N(v, X)| + \frac{1}{2}(n - \gamma)(n - \gamma - 1)$$
$$\leq \frac{1}{2}\gamma(\gamma - 1) + \gamma(n - \gamma) - \gamma(\gamma - 1) + \frac{1}{2}(n - \gamma)(n - \gamma - 1).$$

Solving the inequality, we have that

$$\gamma \leq \frac{1 + \sqrt{1 + 2(n^2 - n - 2e)}}{2}.$$

If $\gamma = \frac{1 + \sqrt{1 + 2(n^2 - n - 2e)}}{2}$, we, from the above proofs, have that

$$|E(X, V - X)|$$
$$= \sum_{v \in X} |N(v, X)| + |X|\left(n - |X| - \sum_{v \in X} |N(v, X)|\right)$$
$$= \gamma(n - \gamma) - (\gamma - 1)\sum_{v \in X} |N(v, X)|$$
$$= \gamma(n - \gamma) - \gamma(\gamma - 1).$$

Thus $\sum_{v \in X} |N(v, X)| = \gamma$. Since $|N(v, X)| \geq 1$ for each $v \in X$, we have that $|N(v, X)| = 1$ for each $v \in X$. Set $A := \cup_{v \in X} N(v, X)$. Then $|A| = |X| = \gamma$. Define $A := \{a_1, a_2, \dots, a_\gamma\}$.

Then a_i has a unique neighbor in X for each i with $1 \leq i \leq \gamma$. Without loss of generality, we can assume $N(a_t) \cap X = \{x_t\}$ for each t with $1 \leq t \leq \gamma$. Set $B := V - (X \cup A) := \{b_1, b_2, \dots, b_{n-2\gamma}\}$. We also have that $|E(X, V-(X \cup A))| = |E(X,B)| = |X|(n - |X| - \sum_{v \in X}|N(v,X)|) = |X||B|$. Thus $x_p b_q \in E$ for each p with $1 \leq p \leq \gamma$ and for each q with $1 \leq q \leq n - 2\gamma$. We, from the above proofs again, have that $|E(G[X])| = \frac{1}{2}\gamma(\gamma-1)$ and $|E(G[V-X])| = \frac{1}{2}(n-\gamma)(n-\gamma-1)$. Thus both $G[X]$ and $G[V-X]$ are complete. Hence $G \in \mathcal{H}(n, \gamma(G))$.

If $G \in \mathcal{H}(n, \gamma(G))$, then $e = \frac{1}{2}\gamma(\gamma-1) + \gamma + \gamma(n-2\gamma) + \frac{1}{2}(n-\gamma)(n-\gamma-1)$. Thus $2e = n^2 - n - 2\gamma^2 + 2\gamma$. Therefore

$$\frac{1 + \sqrt{1 + 2(n^2 - n - 2e)}}{2} = \gamma.$$

This completes the proof of Theorem 1.

3. Conclusions

In this note, we present an upper bound for the domination number of a connected graph. We also characterized the graphs achieving the upper bound.

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