

Certain Amenability Results of Banach Algebras Including Their Basic Considerable Properties: A Review

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Abstract. Banach Algebra is a fundamental and practically applicable concept, and its importance seems indispensable. Suppose we have a Banach B-bimodule S. Then, B is said to be amenable provided that every continuous derivation D from B into its dual space S* is inner. In this work, we review fundamental results on basic properties and characterisations of amenability in Banach algebras.

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1. Introduction

The notion and concepts of amenability can be traced back to the early twentieth century. At that time, the form of amenable group classes was known, introduced by John Von Neumann. He also conducted the first study of the concept. Hence, according to John Von Neumann, any given group G which is known to be locally compact is going to be amenable if it is possible that there exists a given left translation which is invariant mean on G: This is to say, if there exists a linear functional which is positive $\mu \in L^\infty(G)^\infty$, with $\mu(1) = 1$, $\mu(f) \geq 0$, for $f \geq 0$, and is left-translation invariant. And so, we have as follows, the conditions given by:

- (i) $\langle 1, \mu \rangle = 1$
- (ii) $\langle \delta_x * f, \mu \rangle = \langle f, \mu \rangle \quad (x \in G, f \in L^\infty(G)).$

We know that B. E. Johnson, in 1972, showed that the amenability of a locally compact group G can be characterised in certain ways. These particular characterizations, presented in terms of the Hochschild cohomology for the given group algebra $L^1(G)$. From here, we verify the essential theoretical facts about the amenability characterisations and properties of the Banach algebras, as expected [1]. At this point, we invite readers to consult some of our other references for a more detailed understanding. For instance, our readers can also check [2].

Johnson, in [1], has actually made us understand that a Banach algebra given by B is actually going to fulfil the amenability conditions provided that each and every derivation map which is given by the expression $V: B \rightarrow X^*$, which is continuous, is

actually inner, and this is actually possible for every given Banach B -bimodule X . Here, our notation given as X^* actually gives a particular representation to the dual of X . Now, suppose that X^* here is replaced with X ; in this particular given definition, then we are going to say that this particular B is actually referred to as a given Banach algebra which is actually called contractible. In [1] as well as in [3], the eminent Johnson was really able to give the proof as well as establish at least certain parts of the intrinsic characterisations, conditions, as well as the required hereditary properties of such Banach algebras B , which are ascertained to be amenable in these properties. Hence, the given notions and the concepts of amenability have been a versatile and, in a way, fruitful area of ongoing research, with further activity in mathematical analysis and in the study of Banach algebras. In fact, these notions seem to extend to other fruitful and enriching areas, such as operator algebras, and even to aspects of abstract harmonic analysis.

2. Closed Ideals of Banach Algebras X^* actually gives a particular representation of

2.1. Preliminaries and definitions

Definition 2.1.1. Now, suppose that we have the algebra B . Let v be any given ideal for the algebra B , including any particular given subset of B in which case, the following are feasible

- a. v being a vector subspace for B ;
- b. $Bv \subseteq v$ as well as $vB \subseteq v$.

Clearly, if it is known that B satisfies the conditions of being a Banach space and v also, is also known as a closed ideal of B , then, the given quotient space B/v is a actually, a Banach space. This is for the quotient norm which could be given by: $\|a+v\| = \inf_{x \in v} \|a+x\|$, for every $a \in B$.

So, suppose that the required product could be defined such that:

$(x+v)(y+v) = xy+v$ on B/v for each of the x as well as y , been elements of B . So, as a result of this particular given case, B/v should at this particular point in time, now going to be a Banach algebra as stated and required.

Definition 2.1.2 Given for B to be any a Banach algebra, then, we are going to have and also itemized as follows:

- i. Any particular given ideal v in B is said to be proper, provided that v is not equal to B .
- ii. Any ideal v is said to be an ideal which is maximal as an ideal of B , and also, provided v is a proper ideal in which v is also particularly known not to be contained in any of the strictly as well as larger proper ideals for the given algebra B .

Definition 2.1.3. Given that the following X as well as Y are given as Banach algebras. Define a homomorphism from X to Y as a linear map $t: X \rightarrow Y$. This is multiplicative, and we have as follows: $t(xy) = t(x)t(y)$, $(x, y \in X)$.

We are going to have the kernel of t to be given by:

$$\ker t = \{x \in X \mid t(x) = 0\}.$$

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Now let these two X as well as Y be the Banach algebras with identities. Then, the homomorphism $t: X \rightarrow Y$ is with identity, provided that $t(1_X) = 1_Y$. t would be bijection provided that $t: X \rightarrow Y$ is going to be an *isomorphism*. Observe that: For every $x_1, x_2 \in X$, $t(x_1) = t(x_2) \Rightarrow x_1 = x_2$. Then, t is an homomorphism, well defined, injective, (one – to – one, mono). Also, for every $x \in X$, there exists $y \in Y$ of which $t(x) = y \in Y$. By this it is very clear that t , definitely, is an isomorphism. Now, suppose that the isomorphism is existing. Then, these given Banach algebras X as well as Y are going to have an *isomorphic property as possed*. Hence, the kernel of the given t is said to be a proper ideal of t and is not going to be identical or as much equal to 0. Also, this given quotient map, directly expressed as $h: X \rightarrow X/\nu$ is a surjection and we have that: $\ker h = \nu$. Then, h is well defined since if $x \in X$, then, $h(x) = x\nu$ (multiplicatively). Now, if $t: X \rightarrow Y$ will be a homomorphism of the Banach algebras such that: $\text{Ker } t = \nu$. We have an embedding $k: X/\nu \rightarrow Y$.

In which case, there is an existence of the homomorphism $k: X/\nu \rightarrow Y$, such that: $t = k \circ h$. So, having the homomorphisms:

$h: X \rightarrow X/\nu$, $t: X \rightarrow Y$ and $k: X/\nu \rightarrow Y$, considering the diagram below:

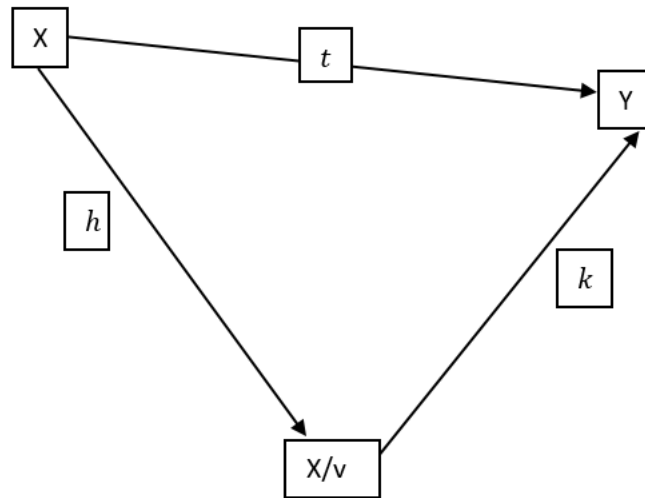


Figure 1:

So, we are going to have it that: $t = k \circ h$. And the diagram commutes

Theorem 2.1.4. (Please, see [4]) Suppose that ν is a ideal which is closed for the given Banach algebra X . Hence, We are going to have that: X / ν is a Banach algebra. Now, given that X is abelian, then so is the quotient X / ν .

Proof. Clearly, the product $(x + \nu)(y + \nu) = xy + \nu$ is well defined, and this is for each of the $x, y \in X$. Now, X/ν is a Banach space. We claim that:

$\|(x + \nu)(y + \nu)\| \leq \|x + \nu\| \|y + \nu\|$. Here, for each of the x , and $y \in X$,

$$\|(x + \nu)(y + \nu)\| = \|xy + \nu\| \quad (*)$$

By taking the infimum, $\left(\inf_{a \in v}\right)$ of the RHS of (*), we are going to have that:

$$\begin{aligned} \inf_{a \in v} \|xy + v\| &\leq \inf_{a, b \in v} \|xy + xb + ay + ab\| = \inf_{a, b \in v} \|(a + x)(b + y)\| \\ &\leq \inf_{a, b \in v} \|a + x\| \|b + y\| = \|x + v\| \|y + v\| \end{aligned}$$

We have that, X/v is now, going to be a Banach algebra. Hence, suppose that this particular given X will be possessing an abelian property, in which case, its commutativity property could then be easily ascertained as much as possible as desired. And so, we are going to have that for each of the $x, y \in X$,

$$(x + v)(y + v) = xy + v = yx + v = (y + v)(x + v).$$

This actually shows that, X / v is abelian. ■

Lemma 2.1.5. [4] Given that B is any Banach algebra having an identity element. So, suppose that v is an ideal of B , then v could be said to be a proper ideal iff $v \cap \text{Inv}(B) = \emptyset$.

Theorem 2.1.6. [4] Let B be given as is a Banach algebra having an identity element. Then, we are going to have as follows:

- i. Given that v is given as a proper ideal for B . In this case, the closure of v is a proper ideal of B .
- ii. For any given maximal ideal v for B , v is said to now be closed.

3. Amenability in Banach algebras

At this point, the real intention is to provide the definitions for some essential as well as very important notions for amenable Banach algebras, and this is considered generally having the reviewed results.

Definition 3.1.1. Suppose that B is given as any Banach algebra, and suppose that A is a Banach B -bimodule. We are going to have as formally defined that any given and defined derivation, setting from B into A is going to be a bounded linear map rightly given $D: B \rightarrow A$ in such a way that:

$$D(xy) = xD(y) + D(x)y, \text{ for all } x, y \in B$$

Now, consider the following map given by: $\delta_a: B \rightarrow A$. And allow this to be determined as follows:

$$\delta_a(x) = xa - ax, \text{ for each of the } x, y \in B.$$

Here, it should be well noted that:

$$\delta_a(xy) = xya - axy = xya - xay + xay - axy = x(ya - ay) + (xa - ax)y = x\delta_a(y) + \delta_a(x)y.$$

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From here, we have that the δ_a could definitely be seen indeed as a derivation. Hence, any derivations following this particular role could be referred simply to as the inner derivations. Now, if $P^1(B, A)$ represents the given and represented space for each of the given continuous derivations which are right from the B into the A . So let $Q^1(B, A)$ be for the space of all the inner derivations from B into A that are indeed continuous.

We are going to have, the 1st kind *Cohomology group* for B which are actually with the coefficients in A . and is the even the quotient vector space. This is actually given by: $C^1(B, A) := P^1(B, A)/Q^1(B, A)$. From here, it is obvious that if every continuous derivation is linear, then, there are no non zero inner derivations and $Q^1(B, A) = \{0\}$. If every derivation is inner, then, we have exactly that $P^1(B, A) = Q^1(B, A)$ or equivalently, $Q^1(B, A) = 0$. This means that $Q^1(B, A) = 0$ iff for every continuous derivation $D: B \rightarrow A$ is inner

Implication: Any derivation from B to A is inner, provided it is continuous.

Definition 3.1.2. Given that B is a Banach algebra. This implies that B is amenable, and this is also going to happen if and only if $Q^1(B, A^*) = \{0\}$, for every Banach B -bimodule A . So, as a result of this, every form of derivations which is said to be continuous, i.e the map specially given by: $D: B \rightarrow A^*$ is going to be an inner map as required.

Definition 3.1.3. Given that B is a Banach algebra. It's going to give each of the following sets of given implications as now itemized:

1. B also could now be said to be very contractible provided for each given Banach B – bimodule A , each of the given derivations which is known to be continuous, that is, the map represented by: $D: B \rightarrow A$ is going to be as such, inner;
2. B can be amenable provided that, for every Banach B -bimodule A , every derivation which is continuous $D: B \rightarrow A^*$ is indeed also, inner;
3. B is could be weakly amenable provided that every derivation $D: B \rightarrow B^*$ which is continuous is actually inner;
4. Supposed that v is a closed ideal for B . Then, B is a v -weakly amenable if every derivation $D: B \rightarrow v^*$ (the dual of v) which is continuous is inner. Then, B is ideally amenable provide that B is a v -weakly amenable for each of those such closed ideal v of B ;
5. B is actually and ideally amenable provide that B is a v -weakly amenable for each of the closed ideal v of B ;
6. B could be approximately having a contractible property provided that, for every Banach B - bimodule A , every derivation $D: B \rightarrow A$ which is actually continuous is also seen to be approximately inner;
7. B could actually be is approximately amenable provided that for every Banach B -bimodule A , each of the derivations $D: B \rightarrow A^*$ is which is continuous is actually approximately

inner;

8. B is actually approximately weakly amenable provide that each of the derivations $D: B \rightarrow B^*$ which is continuous is approximately inner;
9. B could actually be boundedly approximately contractible provided that, for each of the Banach B -bimodule A and each of the derivation given by: $D: B \rightarrow A$ which is continuous, we have that $S > 0$ and also a net (a_α) from A^* in which case, $\|x \cdot a_\alpha - a_\alpha \cdot x\| \leq S\|x\|$ for each of the $x \in B$ as well as for every x with $\text{Supp}_\alpha \|a_{\alpha_{\{a_\alpha\}}}\| < \infty$, finite (bounded):

$$D(x) = \lim_\alpha (x \cdot a_\alpha - a_\alpha \cdot x), (x \in B);$$
10. B could also, be said to be boundedly as well as approximately amenable provided that, given for each of the Banach B -bimodule A as well as every derivation $D: B \rightarrow A$ which actually is continuous, we are going to have that $S > 0$ with the given net (a_α) for A^* in which case, $\|x \cdot a_\alpha - a_\alpha \cdot x\| \leq S\|x\|$ given for each of the $x \in B$ and $D(x) = \lim_\alpha (x \cdot a_\alpha - a_\alpha \cdot x), (x \in B).$

Example 3.1.4. Every given set of the complex numbers ably represented and denoted by $\mathbb{C}$, is actually a Banach algebra which is amenable. This is given as actually with the usual product and the given norm in whichever cases given.

Proof: Given that $\{e\}$, a trivial group and that $B := \ell^1(\{e\})$ is given as an amenable Banach algebra. We define $\varphi: B \rightarrow \mathbb{C}$ in such a way as $\varphi(f) = f(\{e\})$. We are going to have that, φ to be an isomorphism. Also, for each of the $f \in B$, we would have that: $|\varphi(f)| = |f(\{e\})| = \|f\|_1$.

This actually would imply that $B \cong \mathbb{C}$. Therefore, $\mathbb{C}$ is also amenable just simply because B is amenable. The implication here is that the isometric algebra isomorphism transfers amenability to the complex $\mathbb{C}$ ■

Hence, the following:

Theorem 3.1.1. $\ell^1(G)$ is amenable if and only if G is amenable

Corollary 3.1.1. The trivial group is amenable. For other useful as well as illustrative ideas on Banach algebras, we implore our prospective readers to also consult other parts of our suggested references ([5] to [19])

Scope of further studies:

Future work could include complete hereditary characterisations and properties in Banach algebras, such as amenability, and how these behaviours pass to quotients, closed ideals, subalgebras, direct sums, tensor products, etc., as much as possible.

4. Conclusion

So far, we have briefly reviewed certain results involving amenability properties in Banach algebras. It has been ascertained, as proposed by John Von Neumann, that a known group G , which is said to be locally compact, is amenable provided there is a translation-invariant (left) mean on the group G . It was authenticated and verified that the concepts are essential theoretical facts of the amenability characterisations, as well as the remarkable properties for the Banach algebras, as expected. This notion in particular, as well as the concept of

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amenability, has thus been a versatile and, in a way, fruitful area of ongoing research. Both in mathematical analysis and in the study of entire Banach algebras. This includes operator algebras and related areas such as abstract harmonic analysis.

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