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Fuzzy Translations of Fuzzy Subalgebras in BG-algebras

***Monoranjan Bhowmik*[a] *and Tapan Senapati*[b]**

[a]Department of Mathematics, V.T.T. College, Midnapore 721101, India
E-mail: mbvttc@gmail.com
[b]Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102 India
E-mail: math.tapan@gmail.com

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Abstract. In this paper, the concepts of fuzzy translation to fuzzy subalgebras in BG-algebras are introduced. The notion of fuzzy extensions and fuzzy multiplications of fuzzy subalgebras are introduced and several related properties are investigated. In this paper, the relationships between fuzzy translations and fuzzy extensions of fuzzy subalgebras are investigated.

Keywords: BG-algebra, fuzzy subalgebra, fuzzy translation, fuzzy extension, fuzzy multiplication

1. Introduction

The study of BCK/BCI-algebras [6,7] was initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Neggers and Kim [14] introduced a new notion, called a B-algebras which is related to several classes of algebras of interest such as BCI/BCK-algebras. Kim and Kim [13**Error! Reference source not found.**] introduced the notion of BG-algebras, which is a generalization of B-algebras. Ahn and Lee [1**Error! Reference source not found.**] fuzzified BG-algebras. Senapati et al. done lot of works on B-algebras. The authors [5,15-33] presented the concept and basic properties of intuitionistic fuzzy subalgebras, intuitionistic L-fuzzy ideals, interval-valued intuitionistic fuzzy subalgebras, interval-valued intuitionistic fuzzy closed ideals of BG-algebras. Also, Senapati et al. introduced L-fuzzy G-subalgebras of G-algebras which is related to B-algebras.

In this paper, fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy subalgebras in BG-algebras are discussed. Relations among fuzzy translations and fuzzy extensions of fuzzy subalgebras in BG-algebras is also investigated.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included. A *BG*-algebra is an important class of logical algebras introduced by Kim and Kim and was extensively investigated by several researchers. This algebra is defined as follows.

A non-empty set X with a constant 0 and a binary operation $*$ is called a *BG*-algebra if it satisfies the following axioms

1. $x*x=0$

2. $x*0=x$
3. $(x*y) * (0*y) = x$, for all $x,y\in X$.

A non-empty subset S of a *BG*-algebra X is called a subalgebra of X if $x*y\in S$ for any $x,y\in S$.

Let X be the collection of objects denoted generally by x, then a fuzzy set μ in X is defined as $A=\{<x, \mu(x)>::x\in X\}$, where $\mu(x)$ is called the membership value of x in μ and $0\leq \mu(x)\leq 1$.

A fuzzy set μ in a *BG*-algebra X is called a fuzzy subalgebra of X if $\mu (x*y)\geq \min\{ \mu(x), \mu(y)\}$ for all $x,y\in X$.

3. Fuzzy translations of fuzzy subalgebras

Throughout this paper, we take $\top=1-\sup\{ \mu(x)|x\in X\}$ for any fuzzy set μ of X.

Definition 3.1. Let μ be a fuzzy subset of X and let $\alpha\in [0,\top]$. A mapping $\mu_\alpha^T:X\rightarrow[0,1]$ is called a fuzzy α-translation of μ if it satisfies $\mu_\alpha^T(x)= \mu(x)+ \alpha$ for all $x\in X$.

Theorem 3.2. Let μ be a fuzzy subalgebras of X and let $\alpha\in [0,\top]$. Then the fuzzy α -translation μ_α^T of μ is a fuzzy subalgebras of X.

Proof: Let $x,y\in X$. Then $\mu_\alpha^T(x*y)= \mu(x*y)+ \alpha \geq \min\{\mu(x), \mu (y)\}+ \alpha= \min\{\mu(x)+ \alpha, \mu(y)+\alpha\}=\min\{\mu_\alpha^T (x), \mu_\alpha^T (y)\}$. Hence, the fuzzy α-translation μ_α^T of μ is a fuzzy subalgebras of X.

Theorem 3.3. Let μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy subalgebras of X for some $\alpha\in [0,\top]$. Then μ is a fuzzy subalgebra of X.

Proof: Assume that μ_α^T is a fuzzy subalgebras of X for some $\alpha\in [0,\top]$. Let $x,y\in X$, we have $\mu(x*y)+ \alpha= \mu_\alpha^T(x*y) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}= \min\{\mu(x)+ \alpha, \mu(y)+\alpha\}= \min\{\mu(x), \mu (y)\}+ \alpha$ which implies that $\mu(x*y) \geq \min\{\mu(x), \mu (y)\}$ for all $x,y\in X$. Hence, μ is a fuzzy subalgebras of X.

Definition 3.4. Let μ and ν be fuzzy subsets of X. If $\mu(x)\leq \nu(x)$ for all $x\in X$, then we say that B is a fuzzy extension of μ.

Definition 3.5. Let μ and ν be fuzzy subsets of X. Then ν is called a fuzzy S-extension of μ if the following assertions are valid:

(i) ν is a fuzzy extension of μ.

(ii) If μ is a fuzzy subalgebra of X, then ν is a fuzzy subalgebra of X.

From the definition of fuzzy α-translation, we get $\mu_\alpha^T (x)=\mu(x)+\alpha$ for all $x\in X$. Therefore, we have the following theorem.

Theorem 3.6. Let μ be a fuzzy subset of X and $\alpha\in [0,\top]$. Then the fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of μ.

The converse of the Theorem 3.6 is not true in general as seen in the following example.

Example 3.7. Let X={0,1,2,3,4,5} be a BG-algebra with the following Cayley table:

*	0	1	2	3	4	5
0	0	5	4	3	2	1
1	1	0	5	4	3	2
2	2	1	0	5	4	3
3	3	2	1	0	5	4
4	4	3	2	1	0	5
5	5	4	3	2	1	0

Let μ be a fuzzy subset of X defined by

X	0	1	2	3	4	5
μ	0.7	0.4	0.7	0.4	0.7	0.4

Then μ is a fuzzy subalgebra of X. Let ν be a fuzzy subset of X defined by

X	0	1	2	3	4	5
N	0.73	0.51	0.73	0.51	0.73	0.51

Then ν is a fuzzy S-extension of μ. But it is not the fuzzy α-translation μ_α^T of μ for all $\alpha \in [0,\top]$.

Clearly, the intersection of fuzzy S-extensions of a fuzzy subalgebra μ of X is a fuzzy S-extension of μ. But the union of fuzzy S-extensions of a fuzzy subalgebra μ of X is not a fuzzy S-extension of μ as seen in the following example.

Example 3.8. Let X={0,1,2,3,4,5} be a BG-algebra with the following Cayley table:

*	0	1	2	3	4	5
0	0	2	1	3	4	4
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Let μ be a fuzzy subset of X defined by

X	0	1	2	3	4	5
μ	0.6	0.2	0.2	0.2	0.2	0.2

Then μ is a fuzzy subalgebra of X. Let ν and δ be fuzzy subsets of X defined by

X	0	1	2	3	4	5
N	0.7	0.4	0.4	0.7	0.4	0.4

X	0	1	2	3	4	5
Δ	0.8	0.5	0.5	0.5	0.8	0.5

respectively.

Then ν and δ are fuzzy S-extensions of μ. Obviously, the union $\nu\cup\delta$ is a fuzzy extension of μ, but it is not a fuzzy S-extension of μ since $\nu\cup\delta(4*3)=\nu\cup\delta(1)=0.5\ngeq 0.7=\min\{0.8,0.7\}=\min\{\nu\cup\delta(4), \nu\cup\delta(3)\}$.

For a fuzzy subset μ of X, $\alpha\in[0,\top]$ and $t,s\in[0,1]$ with $t\geq\alpha$, let $U_\alpha(\mu:t)=\{x\in X \mid \mu(x)\geq t-\alpha\}$. If μ is a fuzzy subalgebra of X, then it is clear that $U_\alpha(\mu:t)$ is a subalgebra of X for all $t\in Im(\mu)$ with $t\geq\alpha$. But, if we do not give a condition that μ is a fuzzy subalgebra of X, then $U_\alpha(\mu:t)$ is not a subalgebra of X as seen in the following example.

Example 3.9. Let X={0,1,2,3,4,5} be a BG-algebra in Example and μ be a fuzzy subset of X defined by

X	0	1	2	3	4	5
μ	0.73	0.51	0.51	0.68	0.68	0.68

Since $\mu(3*5)=0.51\ngeq 0.68=\min\{\mu(3), \mu(5)\}$, therefore, μ is not a fuzzy subalgebra of X.

For $\alpha=0.15$ and $t=0.75$, we obtain $U_\alpha(\mu:t)=\{0,3,4,5\}$ which is not a subalgebra of X since $3*4=2$ does not belong to $U_\alpha(\mu:t)$.

Theorem 3.10. For $\alpha\in[0,\top]$, let μ_α^T be the fuzzy α-translation of μ.Then the following assertions are equivalent:

(i) μ_α^T is a fuzzy subalgebra of X.

(ii) $U_\alpha(\mu:t)$ is a subalgebra of X for $t\in Im(\mu)$ with $t\geq\alpha$.

Proof: Assume that μ_α^T is a fuzzy subalgebra of X. Then μ_α^T is a fuzzy subalgebra of X. Let $x,y\in X$ such that $x,y\in U_\alpha(\mu:t)$ and $t\in Im(\mu)$ with $t\geq\alpha$. Then $\mu(x)\geq t-\alpha$ and $\mu(y)\geq t-\alpha$ i.e., $\mu_\alpha^T(x)=\mu(x)+\alpha\geq t$ and $\mu_\alpha^T(y)=\mu(y)+\alpha\geq t$. Since μ_α^T is a fuzzy subalgebra of X, therefore, we have

$\mu(x*y)+\alpha=\mu_\alpha^T(x*y)\geq\min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}\geq t$

that is, $\mu(x*y)\geq t-\alpha$ so that $x*y\in U_\alpha(\mu:t)$. Therefore, $U_\alpha(\mu:t)$ is a subalgebra of X.

Conversely, suppose that $U_\alpha(\mu:t)$ is a subalgebra of X for $t\in Im(\mu)$ with $t\geq\alpha$. If there exists $a,b\in X$ such that $\mu_\alpha^T(a*b)\leq\beta\leq\min\{\mu_\alpha^T(a), \mu_\alpha^T(b)\}$, then $\mu(a)\geq\beta-\alpha$ and $\mu(b)\geq\beta-\alpha$ but $\mu(a*b)<\beta-\alpha$. This shows that $a\in U_\alpha(\mu:t)$ and $b\in U_\alpha(\mu:t)$ but $a*b$ does not belong to $U_\alpha(\mu:t)$. This is a contradiction, and therefore, $\mu_\alpha^T(x*y)\geq\min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$ for all $x,y\in X$. Consequently, μ_α^T is a fuzzy subalgebra of X.

Theorem 3.11. Let μ be a fuzzy subalgebra of X and let $\alpha, \beta \in [0, Ʈ]$. If $\alpha \geq \beta$, then the fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of the fuzzy β-translation μ_β^T of μ.

Proof: Straightforward.

For every fuzzy subalgebra μ of X and $\beta \in [0, Ʈ]$, the fuzzy β-translation μ_β^T of μ is a fuzzy subalgebra of X. If ν is a fuzzy S-extension of A_β^T, then there exists $\alpha \in [0, Ʈ]$ such that $\alpha \geq \beta$ and $B \geq A_\alpha^T$ that is $\mu_\beta(x) \geq \mu_\alpha^T(x)$ for all $x \in X$. Hence, we have the following theorem.

Theorem 3.12. Let μ be a fuzzy subalgebra of X and let $\beta \in [0, Ʈ]$. For every fuzzy S-extension ν of the fuzzy β-translation μ_β^T of μ, there exists $\alpha \in [0, Ʈ]$ such that $\alpha \geq \beta$ and B is a fuzzy S-extension of the fuzzy α-translation μ_α^T of μ.

Let us illustrate the Theorem 3.12 using the following example.

Example 3.13. Let X={0,1,2,3,4,5} be a BG-algebra and μ be a fuzzy subset of X defined in Example . Then T=0.3. If we take β=0.11, then the fuzzy β-translation μ_β^T of μ is given by

X	0	1	2	3	4	5
μ_β^T	0.81	0.51	0.81	0.51	0.81	0.51

Let ν be a fuzzy subset of X defined by

X	0	1	2	3	4	5
ν	0.89	0.57	0.89	0.57	0.89	0.57

Then ν is clearly a fuzzy subalgebra of X which is a fuzzy S-extension of the fuzzy β-translation μ_β^T of μ. But ν is not a fuzzy α-translation of μ for all $\alpha \in [0, Ʈ]$. If we take α =0.15 then α=0.15>0.11=β and the fuzzy α-translation μ_α^T of μ is given as follows:

X	0	1	2	3	4	5
μ_α^T	0.85	0.55	0.85	0.55	0.85	0.55

Note that $\nu(x) \geq \mu_\alpha^T(x)$for all $x \in X$, and hence, ν is a fuzzy S-extension of the fuzzy α -translation μ_α^T of μ.

Definition 3.14. Let μ be a fuzzy subset of X and $\gamma \in [0,1]$. A fuzzy γ-multiplication of μ, denoted by μ_γ^m and is defined by $\mu_\gamma^m(x) = \mu(x).\ \gamma$ for all $x \in X$.

For any fuzzy subset μ of X, a fuzzy 0-multiplication μ_0^m of $\mu$is a fuzzy subalgebra of X.

Theorem 3.15. If μ is a fuzzy subalgebra of X, then the fuzzy γ -multiplication of μ is a fuzzy subalgebra of X for all $\gamma \in [0,1]$.

Proof: Straightforward.

Theorem 3.16. If μ is any fuzzy subset of X, then the following assertions are equivalent:

(i) μ is a fuzzy subalgebra of X.

(ii) for all $\gamma \in (0,1]$, μ_γ^m is a fuzzy subalgebra of X.

Proof: Necessity follows from Theorem 3.15. For sufficient part let $\gamma \in (0,1]$ be such that μ_γ^m is a fuzzy subalgebra of X. Then for all $x,y \in X$, we have $\mu(x*y).\ \gamma = \mu_\gamma^m(x*y) \geq \min\{\mu_\gamma^m(x), \mu_\gamma^m(y)\} = \min\{\mu(x).\ \gamma, \mu(y).\gamma\} = \min\{\mu(x), \mu(y)\}\ .\ \gamma$. Therefore, $\mu(x*y) \geq \min\{\mu(x),\mu(y)\}$ for all $x,y \in X$ since $\gamma \neq 0$. Hence, μ is a fuzzy subalgebra of X.

Theorem 3.17. Let μ be a fuzzy subset of X, $\alpha \in [0,T]$ and $\gamma \in (0,1]$. Then every fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of the fuzzy γ-multiplication μ_γ^m of μ.

Proof: For any $x \in X$, we have $\mu_\alpha^T(x) = \mu(x) + \alpha \geq \mu(x) \geq \mu(x)\ .\ \gamma = \mu_\gamma^m(x)$, and so μ_α^T is a fuzzy extension of μ_γ^m. Assume that μ_γ^m is a fuzzy subalgebra of X. Then μ is a fuzzy subalgebra of X by Theorem 3.16. It follows from Theorem 3.2 that μ_α^T is a fuzzy subalgebra of X for all $\alpha \in [0,T]$. Hence, every fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of the fuzzy γ-multiplication μ_γ^m.

The following example illustrates Theorem 3.17.

Example 3.18. Let X={0,1,2,3,4,5} be a BG-algebra which is given in Example , and consider a fuzzy subalgebra μ of X that is defined in Example 3.7. If we take γ=0.2, then the fuzzy γ -multiplication μ_γ^m of μ is given by

X	0	1	2	3	4	5
$\mu_{0.2}^m$	0.14	0.08	0.14	0.08	0.14	0.08

Therefore, $\mu_{0.2}^m$ is a fuzzy subalgebra of X. Also, for any $\alpha \in [0,0.3]$, the fuzzy α-translation μ_α^T of μ is given by

X	0	1	2	3	4	5
μ_α^T	0.7+α	0.4+α	0.7+α	0.4+α	0.7+α	0.4+α

Then μ_α^T is a fuzzy extension of $\mu_{0.2}^m$ and μ_α^T is always a fuzzy subalgebra of X for all $\alpha \in [0,0.3]$. Hence, μ_α^T is a fuzzy S-extension of $\mu_{0.2}^m$ for all $\alpha \in [0,0.3]$.

8. Conclusions and future work

In this paper, fuzzy translation of fuzzy subalgebras in BG-algebra is introduced and investigated some of their useful properties. The relationships between fuzzy translations and fuzzy extensions of fuzzy subalgebras has been constructed. It is our hope that this work would other foundations for further study of the theory of BG-algebras.

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