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Generalized Intuitionistic Fuzzy BE-Algebras

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Abstract. In this paper, we introduce the notion of $(\in,\in\vee q_k)$-intuitionistic fuzzy subalgebrasas a generalization of an $(\in,\in\vee q)$-intuitionistic fuzzy subalgebrasin BE-algebras. Characterizations $(\in,\in\vee q_k)$-intuitionistic fuzzy subalgebrasare established. Finally, we show that the Cartesian product of $(\in,\in\vee q_k)$-intuitionistic fuzzy subalgebras is an $(\in,\in\vee q_k)$-intuitionistic fuzzy subalgebras.

Keywords: BE-algebra, fuzzy point, $(\in,\in\vee q_k)$-intuitionistic fuzzy subalgebra

1. Introduction

The theory of fuzzy set was introduced by Zadehin 1965 [16]. The generalization of the crisp set to fuzzy sets relied on spreading positive information that point {1} into the interval [0,1]. Since then the notion of fuzzy sets is actively applied to many branches in Mathematics andit is known that several generalization of a fuzzy set were extensively investigated by many researchers and properties have been considered systematically. Fuzzy sets and its extensions have provided successful results dealing with uncertainty in different in many real word problems.

The fuzzification of algebraic structures was initiated by Rosenfeld's [14] and he introduced the notion of fuzzy subgroups. The idea of "intuitionistic fuzzy set" was first introduced by Atanassov [2,3], as a generalization of the notion of fuzzy set.These kind of fuzzy sets have gained a wide recognition as a useful tool in the modeling of some uncertain phenomena. Bhakat and Das gave the concepts of (α,β)-fuzzy subgroup [4,5,6] by using the "belong to" relation $(\in)$ and "quasi-coincident" relation (q) between a fuzzy point x_λ and a fuzzy set μ. Then theyintroduced a new type of fuzzy subgroup, that is, the $(\in,\in\vee q)$-fuzzy subgroups and $(\in,\in\vee q)$-fuzzy subrings and obtained some fundamental results pertaining to these notions. J. Zhan and et al. introduced the notion of $(\in,\in\vee q)$-fuzzy *p*-ideals, $(\in,\in\vee q)$-fuzzy *q*-ideals and $(\in,\in\vee q)$-fuzzy *a*-ideals in BCI-algebras and investigate some of their properties [17]. Jun et al. introduced the notion of $(\in,\in\vee q_k)$-fuzzy subgroupssuch as an important generalization of Rosenfeld's fuzzy subgroup.Then many authors have studied about it for other algebraic structures.

BE-algebras have been defined in [8] as ageneralization of BCK-algebras.Ahn et al. [1] fuzzified the concept of BE-algebras and investigated some of their Properties.Song et al. was defined the notion of fuzzy ideals in BE-algebras and investigated related properties [15]. Rezaei et al. investigated the relationship between Hilbert algebras and BE-algebras. In fact we showed that a commutative implicative BE-algebra is equivalent to the commutative self distributive BE-algebra and prove that Hilbert algebras and commutative self distributive BE-algebras are equivalent [11]. Rezaei and BorumandSaeid [12] introduced the notions of ε-generalized fuzzy filters(ideals) in BE-algebras. Recently, we introduced the notion of hesitant fuzzy (implicative) filters and got some results on BE-algebras and show that every hesitant fuzzy implicative filter is a hesitant fuzzy filter but not the converse [13].

The aim of this paper is to introduce the notion of $(\in,\in\vee q_k)$-intuitionistic fuzzy subalgebras, which is an extended notion of an$(\in,\in\vee q)$-intuitionistic fuzzy subalgebras in BE-algebras. Then we prove some theorems and review some basic properties on this algebra by using our definitions.The Cartesian product of $(\in,\in\vee q_k)$-intuitionistic fuzzy subalgebrashas been discussed.

2. Preliminaries

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

Definition 2.1. [8] By a BE-algebra we shall mean an algebra $(X;*,1)$ of type (2,0) satisfying the followingaxioms:

(BE1) $x*x=1$,

(BE2) $x*1=1$,

(BE3) $1*x=x$,

(BE4) $x*(y*z)=y*(x*z)$, forall $x,y,z\in X$.

The binary relation"$\leq$" on X is defined by $x\leq y$ if and only if $x*y=1$, for all $x,y\in X$.

Definition 2.2. [8] A BE-algebra X is said to be self distributive if

$$x*(y*z)=(x*y)*(x*z), \text{ for all } x,y,z\in X.$$

A fuzzy set μ in $X$is a function $\mu: X\to[0,1]$and the complement of μ, denoted by $\bar{\mu}$ is the fuzzy set in X given by $\bar{\mu}(x)=1-\mu(x)$ for all $x\in X$. For $\alpha\in[0,1]$ the set

$$U(\mu,\alpha)=\{x\in X:\mu(x)\geq\alpha\} \text{ (resp. } L(\mu,\alpha)=\{x\in X:\mu(x)\leq\alpha\})$$

is called an upper (resp. lower) level set of μ.

Definition 2.3. [10] Fuzzy set μ is called a fuzzy subalgebra of X if

$$\mu(x*y)\geq min(\mu(x),\mu(y)), \text{ for all } x,y\in X.$$

Note that if μ is a fuzzy subalgebra of X then $\mu(1)\geq\mu(x)$for all $x\in X$.

A fuzzy subset μ of a set X of the form

$$\mu(x)=\begin{cases} t\in(0,1] & if \quad y=x \\ 0 & if \quad y\neq x \end{cases}$$

issaid to be a fuzzy point with support x and value t and it is denoted by x_t.

Let $f: X \to Y$ be a function and δ be a fuzzy set of Y.Then the fuzzy set $f^{-1}(\delta)$ of X is defined by:

$$f^{-1}(\delta)(x) = \delta(f(x)),\text{for all } x \in X.$$

Let $f: X \to Y$ be a function and μ be a fuzzy set of X.Then the fuzzy set $f(\mu)(x)$ of Y is defined by:

$$f(\mu)(x) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & if \quad f^{-1}(y) \neq \phi \\ 0 & if \quad f^{-1}(y) = \phi \end{cases}$$

An intuitionistic fuzzy set (briefly, IFS) A in a nonempty set Xisan object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ where the function $\mu_A: X \to [0,1]$ and $\gamma_A: X \to [0,1]$denoted the degree of membership and the degree of non membership, respectively, where $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

An intuitionistic fuzzy set$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ in X can be identified with an ordered pair (μ_A, γ_A) in $I^X \times I^X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$. We define $0(x) = 0$ and $1(x) = 1$, for all $x \in X$.

Pu and Liu [9] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. We say that x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set μ. To say that ($x_t \in \vee q$ (resp. $x_t \in \wedge q$)), we mean $x_t \in \mu$ or $x_t q \mu$ (resp. $x_t \in \mu$ and $x_t q \mu$). By the symbol $x_t q_k \mu$ we mean $\mu_A(x) + t + k > 1$. where $k \in (0,1)$. and ($x_t \in \vee q_k$ (resp. $x_t \in \wedge q_k$)), we mean $x_t \in \mu$ or $x_t q_k \mu$ (resp. $x_t \in \mu$ and $x_t q_k \mu$). The symbol $x_t \overline{\alpha\mu}$ means that $x_t \alpha \mu$ does not hold for $\alpha \in \{\in, q, \in \vee q, \in \vee q_k, \in \wedge q\}$.

3. On $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebras of BE-algebras

From now on, X is a BE-algebra unless otherwise specified.

Definition 3.1. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$of the form

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & if \quad y = x \\ (0,1) & if \quad y \neq x \end{cases}$$

is said to be an intuitionistic fuzzy point with support x and value (α, β) and is denoted by $x_{(\alpha,\beta)}$ A fuzzy point $x_{(\alpha,\beta)}$ is said to intuitionistic belongs to (resp. intuitionistic qusi-coincident) with intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$written $x_{(\alpha,\beta)} \in A$ (resp. $x_{(\alpha,\beta)} q A$) if $\mu_A(x) \geq \alpha$ and $\gamma_A(x) \leq \beta$ (resp. $\mu_A(x) + \alpha > 1$ and $\gamma_A(x) + \beta < 1$). By the symbol $x_{(\alpha,\beta)} q_k A$ we mean $\mu_A(x) + \alpha + k > 1and\gamma_A(x) + \beta + k < 1$where $k \in (0,1)$.

We used the symbol $x_t \in \mu_A$ implies $\mu_A(x) \geq t$ and $\frac{t}{x}[\in]\gamma_A$ implies $\gamma_A(x) \leq t$, in the whole paper. To say that $x_{(\alpha,\beta)} \in \vee q A$ (resp. $x_{(\alpha,\beta)} \in \wedge q A$), we mean $x_{(\alpha,\beta)} \in A$ or $x_{(\alpha,\beta)} q A$ (resp. $x_{(\alpha,\beta)} \in A$ and $x_{(\alpha,\beta)} q A$).

By the symbol $x_{(\alpha,\beta)} \in \vee q_k A$ (resp. $x_{(\alpha,\beta)} \in \wedge q_k A$) we mean $x_{(\alpha,\beta)} \in A$ or $x_{(\alpha,\beta)} q_k A$ (resp. $x_{(\alpha,\beta)} \in A$ and $x_{(\alpha,\beta)} q_k A$) where $k \in (0,1)$.

Definition 3.2. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ of X is said to be $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X if

$$x_{(t_1,t_2)} \in A,\ y_{(t_3,t_4)} \in A \text{ imply } (x*y)_{(t_1\wedge t_2, t_3\vee t_4)} \in \vee q_k A,$$

for all $x, y \in X, t_1, t_2, t_3, t_4, k \in (0,1)$.

Or equivalently we have

An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ of X is said to be $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X if

- $x_{(t_1)} \in \mu_A,\ y_{(t_2)} \in \mu_A \Rightarrow (x*y)_{t_1\wedge t_2} \in \vee q_k \mu_A,\ \forall x, y \in X, t_1, t_2, k \in (0,1],$
- $\frac{t_3}{x}[\in]\gamma_A,\ \frac{t_4}{y}[\in]\gamma_A \Rightarrow \frac{t_3 \vee t_4}{x*y}[\in] \wedge [q_k]\gamma_A,\ \forall x, y \in X, t_3, t_4, k \in [0,1).$

Example 3.3. Let $X = \{1, a, b, c\}$ be a set with the following table:

*	1	a	b
1	1	a	b
a	1	1	a
b	1	1	1

Then $(X;*,1)$ is a BE-algebra. Define the intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ of X as:

X	1	a	b
$\mu_A(x)$	0.4	0.3	0.6
$\gamma_A(x)$	0.2	0.2	0.3

and

t_1	0.3
t_2	0.2
t_3	0.4
t_4	0.5
k	0.1

Then $A = (\mu_A, \gamma_A)$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X.

Theorem 3.4. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ of X is said to be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X if and only if

$$\mu_A(x*y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}, \text{ and } \gamma_A(x*y) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\}.$$

Proof: Let $A = (\mu_A, \gamma_A)$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X. By contrary assume that there exist some $t \in (0,1]$ and $r \in [0,1)$ such that

$$\mu_A(x * y) < t < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$$

and

$$\gamma_A(x * y) > r > \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\}.$$

This implies $\mu_A(x) \geq t, \mu_A(y) \geq t, \mu_A(x * y) < t$. Then we have

$$\mu_A(x * y) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1.$$

Hence $(x * y)_t \overline{\in \vee q_k} A$. Which is a contraction. Therefore

$$\mu_A(x * y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}.$$

Also, we have $\gamma_A(x) \leq r, \gamma_A(y) \leq r, \gamma_A(x * y) > r$. Then we have

$$\gamma_A(x * y) + r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1.$$

Hence $\frac{r}{x * y} [\in] \wedge [q_k] A$. Which is a contraction. Therefore

$$\gamma_A(x * y) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\}.$$

Conversely, assume that

$\mu_A(x * y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}$, and $\gamma_A(x * y) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\}$.

Let $(x)_{t_1} \in \mu_A$ and $(y)_{t_2} \in \mu_A$ for all $x, y \in X, t_1, t_2, k \in (0,1]$ and $\frac{t_3}{x} \in \gamma_A$, $\frac{t_4}{y} \in \gamma_A$ for all $x, y \in X, t_3, t_4, k \in [0,1)$.

This implies that $\mu_A(x) \geq t_1, \mu_A(y) \geq t_2$ and $\gamma_A(x) \leq t_3, \gamma_A(y) \leq t_4$.

Consider $\mu_A(x * y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \geq \left\{t_1, t_2, \frac{1-k}{2}\right\}$. If $t_1 \wedge t_2 > \frac{1-k}{2}$, then $\mu_A(x * y) \geq \frac{1-k}{2}$ and so $\mu_A(x * y) + (t_1 \wedge t_2) + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Which implies that $(x * y)_{t_1 \wedge t_2} q_k \mu_A$. Also, $\gamma_A(x * y) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \leq \left\{t_3, t_4, \frac{1-k}{2}\right\}$.

If $t_1 \vee t_2 < \frac{1-k}{2}$, then $\gamma_A(x * y) \leq \frac{1-k}{2}$ and so $\gamma_A(x * y) + (t_1 \vee t_2) + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$.

Which implies that $\frac{t_1 \vee t_2}{x * y} [\in] \vee [q_A] \gamma_A$. Therefore $A = (\mu_A, \gamma_A)$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X.

Corollary 3.5. Let $A = (\mu_A, \gamma_A)$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X.Then$\mu_A(1) \geq min\left\{\mu_A(x), \frac{1-k}{2}\right\}$ and $\gamma_A(1) \leq max\left\{\gamma_A(x), \frac{1-k}{2}\right\}$ for all $x \in X$.

Theorem 3.6. Let $A = (\mu_A, \gamma_A)$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X, for some $k \in (0,1)$. If $\mu_A(1) < \frac{1-k}{2}$, and$\gamma_A(1) > \frac{1-k}{2}$,then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subalgebra of X.

Proof: Let$A = (\mu_A, \gamma_A)$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X, for some $k \in (0,1)$, and $\mu(1) < \frac{1-k}{2}$. Then by Corollary 3.5, we have

$min\left\{\mu_A(x), \frac{1-k}{2}\right\} \leq \mu_A(1) < \frac{1-k}{2}$and so $\mu_A(x) \leq \frac{1-k}{2}$, for all $x \in X$. Thus

$$\mu_A(x*y) \geq min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} = min\{\mu_A(x), \mu_A(y)\}.$$

Also, by a similar way we have

$$\gamma_A(x*y) \leq max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} = max\{\gamma_A(x), \gamma_A(y)\}.$$

Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subalgebra of X.

Theorem 3.7. If $A_i = (\mu_{A_i}, \gamma_{A_i})$ for all$i \in I$, is a family of $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X, for some $k \in (0,1)$.Then $\bigcap_{i\in I} A_i$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X,where $\bigcap_{i\in I} A_i = (\vee\mu_i, \wedge\gamma_i)$.

Proof: For all $i \in I$and $x, y \in X$, we have

$$\vee\mu_i(x*y) \geq \vee\left(min\left(\mu_i(x), \mu_i(y), \frac{1-k}{2}\right)\right) = min\left(\vee\mu_i(x), \vee\mu_i(y), \frac{1-k}{2}\right)$$

and

$$\wedge\gamma_i(x*y) \leq \wedge\left(max\left(\gamma_i(x), \gamma_i(y), \frac{1-k}{2}\right)\right) = max\left(\wedge\gamma_i(x), \wedge\gamma_i(y), \frac{1-k}{2}\right)$$

Hence by Theorem 3.4, we have $\bigcap_{i\in I} A_i = (\vee\mu_i, \wedge\gamma_i)$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X.

Theorem 3.8. Let A be a subalgebra of X and $k \in (0,1)$. Then $\bar{I} = (\mathcal{X}_A, \bar{\mathcal{X}}_A)$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X.

Proof: Let $k \in (0,1]$. If $x, y \in A$, then $x*y \in A$ and $\mathcal{X}_A(x) = 1$, $\bar{\mathcal{X}}_A(x) = 0$ and $\mathcal{X}_A(y) = 1$, $\bar{\mathcal{X}}_A(y) = 0$ Hence

$$\mathcal{X}_A(x*y) = 1 \geq min\left\{\mathcal{X}_A(x), \mathcal{X}_A(y), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Also,

$$\bar{\mathcal{X}}_A(x*y) = 1 - \mathcal{X}_A(x*y) = 0 \leq max\left\{\bar{\mathcal{X}}_A(x), \bar{\mathcal{X}}_A(y), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

If $x \in A$ and $y \notin A$, (or, $x \notin A$ and $y \in A$), then $\mathcal{X}_A(x) = 1$, $\bar{\mathcal{X}}_A(x) = 0$and $\mathcal{X}_A(y) = 0$, $\bar{\mathcal{X}}_A(y) = 1$ or($\mathcal{X}_A(x) = 0$, $\bar{\mathcal{X}}_A(x) = 1$ and $\mathcal{X}_A(y) = 1$, $\bar{\mathcal{X}}_A(y) = 0$).

Thus we have

$$\mathcal{X}_A(x*y) \geq min\left\{\mathcal{X}_A(x), \mathcal{X}_A(y), \frac{1-k}{2}\right\} = 0$$

and

$$\bar{\mathcal{X}}_A(x*y) \leq max\left\{\bar{\mathcal{X}}_A(x), \bar{\mathcal{X}}_A(y), \frac{1-k}{2}\right\} = 1.$$

Definition 3.9. Let $A = (\mu_A, \gamma_A)$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X. Define the intuitionistic level set as

$$A_{(\alpha,\beta)} = \left\{x \in X : \mu_A(x) \geq \alpha, \gamma_A(x) \leq \beta, where, \alpha \in (0, \frac{1-k}{2}], \beta \in [\frac{1-k}{2}, 1)\right\}.$$

Example 3.10. From Example 3.3, we have $A_{(0.3,0.5)} = \{1, a, b\}$ and $A_{(0.4,0.5)} = \{1, b\}$.

Theorem 3.11. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ of X is said to be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X if and only if $A_{(\alpha,\beta)} \neq \emptyset$ is a subalgebra of X.

Proof: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subalgebra of X and $x, y \in A_{(\alpha,\beta)}$. Then $\mu_A(x) \geq \alpha$, $\mu_A(y) \geq \alpha$ and $\gamma_A(x) \leq \beta, \gamma_A(y) \leq \beta$. Since $A = (\mu_A, \gamma_A)$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X, we have

$$\mu_A(x*y) \geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \geq \min\left\{\alpha, \alpha, \frac{1-k}{2}\right\} \geq \min\left\{\alpha, \frac{1-k}{2}\right\} = \alpha$$

and

$$\gamma_A(x*y) \leq \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\} \leq \max\left\{\beta, \beta, \frac{1-k}{2}\right\} \leq \max\left\{\beta, \frac{1-k}{2}\right\} = \beta.$$

Hence $x*y \in A_{(\alpha,\beta)}$. Therefore $A_{(\alpha,\beta)}$ is a subalgebra of X.

Conversely, assume that $A_{(\alpha,\beta)} \neq \emptyset$ is a subalgebra of X. By contrary assume that there exist some $x, y \in X$ such that

$$\mu_A(x*y) < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \text{and } \gamma_A(x*y) > \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\}.$$

Choose $\alpha \in (0, \frac{1-k}{2}], \beta \in [\frac{1-k}{2}, 1)$ such that

$$\mu_A(x*y) < \alpha < \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\} \text{and } \gamma_A(x*y) > \beta > \max\left\{\gamma_A(x), \gamma_A(y), \frac{1-k}{2}\right\}.$$

Hence $x*y \notin A_{(\alpha,\beta)}$, which is acontradiction to the hypothesis. Therefore $A = (\mu_A, \gamma_A)$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X.

Definition 3.12. Let X be a BE-algebra and $A \subseteq X$. An $(\in, \in \vee q_k)$-intuitionistic characteristic function $\mathcal{X}_A = \{(x, \mu_{\mathcal{X}_A}, \gamma_{\mathcal{X}_A}) : x \in X\}$ where $\mu_{\mathcal{X}_A}$ and $\gamma_{\mathcal{X}_A}$ are fuzzy sets respectively, defined as:

$$\mu_{\chi_A} : X \to [0,1], x \to \mu_{\chi_A}(x) := \begin{cases} \frac{1-k}{2} & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}$$

and

$$\gamma_{\chi_A} : X \to [0,1], x \to \gamma_{\chi_A}(x) := \begin{cases} 1 & if \quad x \in A \\ \frac{1-k}{2} & if \quad x \notin A \end{cases}$$

Proposition 3.13. Let X be a BE-algebra and $\emptyset \neq A \subseteq X$. A is a subalgebra of X if and only if the characteristic intuitionistic set $\mathcal{X}_A = \{(x, \mu_{\mathcal{X}_A}, \gamma_{\mathcal{X}_A}) : x \in X\}$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X.

Proof: Let $k \in (0,1)$. If $x, y \in A$, then $x * y \in A$ and $\mu_{\mathcal{X}_A}(x) = \frac{1-k}{2}$, $\gamma_{\mathcal{X}_A}(x) = 1$ and $\mu_{\mathcal{X}_A}(y) = \frac{1-k}{2}$, $\gamma_{\mathcal{X}_A}(y) = 1$ Hence

$$\mu_{\mathcal{X}_A}(x * y) = \frac{1-k}{2} \geq min\left\{\mu_{\mathcal{X}_A}(x), \mu_{\mathcal{X}_A}(y), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Also,

$$\gamma_{\mathcal{X}_A}(x * y) = 1 \leq max\left\{\gamma_{\mathcal{X}_A}(x), \gamma_{\mathcal{X}_A}(y), \frac{1-k}{2}\right\} = 1.$$

If $x \in A$ and $y \notin A$, (or, $x \notin A$ and $y \in A$), then $\mu_{\mathcal{X}_A}(x) = \frac{1-k}{2}$, $\mu_{\mathcal{X}_A}(y) = 0$ and $\gamma_{\mathcal{X}_A}(x) = 1$, $\gamma_{\mathcal{X}_A}(y) = \frac{1-k}{2}$ or ($\mu_{\mathcal{X}_A}(y) = \frac{1-k}{2}$, $\mu_{\mathcal{X}_A}(x) = 0$ and $\gamma_{\mathcal{X}_A}(y) = 1$, $\gamma_{\mathcal{X}_A}(x) = \frac{1-k}{2}$).

Thus we have

$$\mu_{\mathcal{X}_A}(x * y) \geq min\left\{\mu_{\mathcal{X}_A}(x), \mu_{\mathcal{X}_A}(y), \frac{1-k}{2}\right\} = 0.$$

and

$$\gamma_{\mathcal{X}_A}(x * y) \leq max\left\{\gamma_{\mathcal{X}_A}(x), \gamma_{\mathcal{X}_A}(y), \frac{1-k}{2}\right\} = 1.$$

Theorem 3.14. Let $f: X \to Y$ be a homomorphism of BE-algebras and $B = (\delta_B, \vartheta_B)$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of Y, for some $k \in (0,1)$. Then $f^{-1}(B) = (f^{-1}(\delta_B), f^{-1}(\vartheta_B))$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X.

Proof: Let $B = (\delta_B, \vartheta_B)$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of Y, for some $k \in (0,1)$ and $x, y \in X$. Then we have

$$\begin{aligned} f^{-1}(\delta_B)(x * y) &= \delta_B(f(x * y)) = \delta_B(f(x) * f(y)) \\ &\geq min\left\{\delta_B(f(x)), \delta_B(f(y)), \frac{1-k}{2}\right\} \\ &= min\left\{f^{-1}(\delta_B(x)), f^{-1}(\delta_B(y)), \frac{1-k}{2}\right\}. \end{aligned}$$

And

$$
\begin{aligned}
f^{-1}(\vartheta_B)(x*y) &= \vartheta_B\big(f(x*y)\big) = \vartheta_B\big(f(x)*f(y)\big) \\
&\leq max\left\{\vartheta_B\big(f(x)\big), \vartheta_B\big(f(y)\big), \frac{1-k}{2}\right\} \\
&= max\left\{f^{-1}\big(\vartheta_B(x)\big), f^{-1}\big(\vartheta_B(y)\big), \frac{1-k}{2}\right\}.
\end{aligned}
$$

Definition 3.15. Let $f: X \to Y$ be a function. An $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra $A = (\mu_A, \gamma_A)$ of X is said to be f-invariant, if $f(x) = f(y)$ implies that $\mu_A(x) = \mu_A(y)$and $\gamma_A(x) = \gamma_A(y)$, for all $x, y \in X$.

Theorem 3.16. Let $f: X \to Y$ be a homomorphism and $A = (\mu_A, \gamma_A)$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of X. If A if f-invariant, then $f(A) = \big(f(\mu_A), f(\gamma_A)\big)$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebra of Y, where,

$$
f(\mu_A)(x) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & if \quad f^{-1}(y) \neq \phi \\ 0 & if \quad f^{-1}(y) = \phi \end{cases}
$$

and

$$
f(\gamma_A)(x) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \gamma_A(x) & if \quad f^{-1}(y) \neq \phi \\ 0 & if \quad f^{-1}(y) = \phi \end{cases}.
$$

Proof: Let $y_1, y_2 \in Y$. If $f^{-1}(y_1) = \emptyset$or $f^{-1}(y_2) = \emptyset$, then the proof is obvious. Otherwise, let $f^{-1}(y_1) \neq \emptyset$ or $f^{-1}(y_2) \neq \emptyset$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$and $f(x_2) = y_2$. Thus we have

$$
\begin{aligned}
f(\mu_A)(y_1 * y_2) &= \sup_{x \in f^{-1}(y_1 * y_2)} \mu_A(x) = \sup_{x \in f^{-1}\left(f(x_1) * f(x_2)\right)} \mu_A(x) = \sup_{x \in f^{-1}\left(f(x_1 * x_2)\right)} \mu_A(x) \\
&= \mu_A(x_1 * x_2) \geq min\left\{\mu_A(x_1), \mu_A(x_2), \frac{1-k}{2}\right\} \\
&= min\left\{\sup_{x \in f^{-1}\left(f(x_1)\right)} \mu_A(x), \sup_{x \in f^{-1}\left(f(x_2)\right)} \mu_A(x), \frac{1-k}{2}\right\} \\
&= min\left\{\sup_{x \in f^{-1}(y_1)} \mu_A(x), \sup_{x \in f^{-1}(y_2)} \mu_A(x), \frac{1-k}{2}\right\} \\
&= min\left\{f(\mu_A)(y_1), f(\mu_A)(y_2), \frac{1-k}{2}\right\}
\end{aligned}
$$

and

$$f(\gamma_A)(y_1 * y_2) = \inf_{x\in f^{-1}(y_1 * y_2)} \gamma_A(x) = \inf_{x\in f^{-1}\left(f(x_1)*f(x_2)\right)} \gamma_A(x) = \inf_{x\in f^{-1}\left(f(x_1*x_2)\right)} \gamma_A(x)$$
$$= \gamma_A(x_1 * x_2) \le max\left\{\gamma_A(x_1), \gamma_A(x_2), \frac{1-k}{2}\right\}$$
$$= max\left\{\inf_{x\in f^{-1}\left(f(x_1)\right)} \gamma_A(x), \inf_{x\in f^{-1}\left(f(x_2)\right)} \gamma_A(x), \frac{1-k}{2}\right\}$$
$$= max\left\{\inf_{x\in f^{-1}(y_1)} \gamma_A(x), \inf_{x\in f^{-1}(y_2)} \gamma_A(x), \frac{1-k}{2}\right\}$$
$$= max\left\{f(\gamma_A)(y_1), f(\gamma_A)(y_2), \frac{1-k}{2}\right\}.$$

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebras in X, for some $k \in (0,1)$. The Cartesian product of A and B is defined by
$A \times B = (\mu_A \times \mu_B, \gamma_A \times \gamma_B)$, where

$$(\mu_A \times \mu_B)(x, y) = min(\mu_A(x), \mu_B(y)) \text{ and } (\gamma_A \times \gamma_B)(x, y) = min(\gamma_A(x), \gamma_B(y)).$$

Theorem 3.17. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebras in X, for some $k \in (0,1)$. Then $A \times B = (\mu_A \times \mu_B, \gamma_A \times \gamma_B)$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebras in $X \times X$.

Proof: For any $(x_1, y_1), (x_2, y_2) \in X \times X$, we have

$$(\mu_A \times \mu_B)\left((x_1, y_1) * (x_2, y_2)\right) = (\mu_A \times \mu_B)(x_1 * x_2, y_1 * y_2)$$
$$= min\left(\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)\right)$$
$$\ge min\left\{min\left\{\mu_A(x_1), \mu_A(x_2), \frac{1-k}{2}\right\}, min\left\{\mu_B(y_1), \mu_B(y_2), \frac{1-k}{2}\right\}\right\}$$
$$= min\left\{min\{\mu_A(x_1), \mu_B(y_1)\}, min\{\mu_A(x_2), \mu_B(y_2)\}, \frac{1-k}{2}\right\}$$
$$= min\left\{(\mu_A \times \mu_B)(x_1, y_1), (\mu_A \times \mu_B)(x_2, y_2), \frac{1-k}{2}\right\}$$

and

$$(\gamma_A \times \gamma_B)\left((x_1, y_1) * (x_2, y_2)\right) = (\gamma_A \times \gamma_B)(x_1 * x_2, y_1 * y_2)$$
$$= min\left(\gamma_A(x_1 * x_2), \gamma_B(y_1 * y_2)\right)$$
$$\le min\left\{max\left\{\gamma_A(x_1), \gamma_A(x_2), \frac{1-k}{2}\right\}, max\left\{\gamma_B(y_1), \gamma_B(y_2), \frac{1-k}{2}\right\}\right\}$$
$$= min\left\{max\{\gamma_A(x_1), \gamma_B(y_1)\}, max\{\gamma_A(x_2), \gamma_B(y_2)\}, \frac{1-k}{2}\right\}$$
$$= min\left\{(\gamma \times \gamma_B)(x_1, y_1), (\gamma_A \times \gamma_B)(x_2, y_2), \frac{1-k}{2}\right\}$$

Proposition 3.18. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebras in X, for some $k \in (0,1)$. Then $(A \times B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \times B_{(\alpha,\beta)}$, where $\alpha \in (0, \frac{1-k}{2}]$ and $\beta \in \left[\frac{1-k}{2}, 1\right)$.

8. Conclusion

In this paper, we extended $(\in, \in \vee q)$-intuitionistic fuzzy subalgebras to $(\in, \in \vee q_k)$-intuitionistic fuzzy subalgebras for purpose of an extension of fuzzy sets in BE-algebras. Since filter theory is a very interesting and important area of research in the theory of algebraic structures in mathematics we wish these concepts can be further generalized to $(\in, \in \vee q)$-intuitionistic fuzzy filters (ideals) and congruence relations related to this structures. Also, in the future work we will introduce the notion of an interval-valued $(\in, \in \vee q_k)$-fuzzy BE-algebra and investigated relation between $(\in, \in \vee q_k)$-fuzzy point. These concepts can further be generalized and we hope its have been some applications in the real world.

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