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Fuzzy Dot Structure of Bounded Lattices

***S.V. Tchoffo Foka*[1], *Marcel Tonga*[1] *and Tapan Senapati*[2]**

[1]Department of Mathematics, University of Yaoundé 1,
P.O. Box 812, Yaoundé, Cameroon.
E-mail: tchoffofoka88@yahoo.fr;tongamarcel@yahoo.fr
[2]Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, India.
E-mail: math.tapan@gmail.com

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Abstract. In this paper, we introduce the notion of fuzzy dot bounded sublattices of bounded lattices and investigate some of their properties.

Keywords: Lattice, bounded sublattice, fuzzy subset, fuzzy point, fuzzy bounded sublattice, fuzzy dot bounded sublattice, fuzzy dot bounded sublattice generated.

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1. Introduction

In 1965, Zadeh (see, [9]) introduced the notion of fuzzy sets. A few years later, many researchers working on fuzzification of algebraic structures (see, [6]). In 2001, Hong (see, [5]) introduced the notion of fuzzy dot subalgebras of BCH-algebras as a generalization of the notion of fuzzy subalgebras of BCH-algebras.

In 1990, Yuan and Wu (see, [8]) introduced the notion of fuzzy sublattices of lattices and in 1994, Ajmal and Thomas (see, [1]) defined a fuzzy sublattice as a fuzzy algebra and characterized fuzzy sublattices. In this paper, the notion of fuzzy dot bounded sublattices of bounded lattices is introduced as a generalization of the notion of fuzzy bounded sublattices of bounded lattices, and some of their properties are investigated. The binary operations $\curlywedge$, $\curlyvee$, ^{0} and ^{1} are introduced. The fuzzy dot bounded sublattices of bounded lattices are characterized in terms of $\curlywedge$, $\curlyvee$, ^{0} and ^{1}. The image and the preimage of fuzzy dot bounded sublattices of bounded lattices are studied. Some fuzzy dot bounded sublattices generated by fuzzy subsets are described.

2. Preliminaries

In this section, we give some definitions and basic results which will be used for the development of the paper.

Definition 2.1. (see, [2]) A nonempty set L together with two binary operations $\wedge$ and $\vee$ (read ''meet'' and ''join'' respectively) on L, denoted by (L; $\wedge$, $\vee$), is called a lattice if it satisfies the following identities:

i. $(x\wedge y)\wedge z = x\wedge(y\wedge z)$ and $(x\vee y)\vee z = x\vee(y\vee z)$ (associative laws),
ii. $x\wedge y = y\wedge x$ and $x\vee y = y\vee x$ (commutative laws),
iii. $x\wedge x = x$ and $x\vee x = x$ (idempotent laws),
iv. $x\wedge(x\vee y) = x$ and $x\vee(x\wedge y) = x$ (absorption laws).

Remark 2.1. (see, [2]) The relation ≤ on L, defined by x ≤ y if and only if $x \wedge y = x$, is a partial order on L.

Definition 2.2. (see, [2]) A poset (P, ≤) is complete if the meet and the join of every subset of P exist in P.

Definition 2.3. (see, [2]) An algebra $(L; \wedge, \vee; 0, 1)$ with two binary operations $\wedge, \vee$ and two nullary operations 0,1 is called a bounded lattice if it satisfies the following conditions:

i. $(L; \wedge, \vee)$ is a lattice,
ii. $x \wedge 0 = 0$ and $x \vee 1 = 1$.

Definition 2.4. (see, [2]) Let $(L; \wedge, \vee; 0, 1)$ be a bounded lattice. A nonempty subset S of L is called a bounded sublattice of $(L; \wedge, \vee; 0, 1)$ if it satisfies the following conditions:

i. $x \in S$ and $y \in S$ imply $x \wedge y \in S$ and $x \vee y \in S$,
ii. $0 \in S$ and $1 \in S$.

Notation 2.1. (see, [2]) Sub(L) denotes the set of all bounded sublattices of a bounded lattice $(L; \wedge, \vee; 0, 1)$.

Notation 2.2. (see, [2]) Let $(L; \wedge, \vee; 0, 1)$ be a bounded lattice. For any subset X of L, Sg(X) denotes the bounded sublattice of $(L; \wedge, \vee; 0, 1)$ generated by X; i.e., the smallest bounded sublattice of $(L; \wedge, \vee; 0, 1)$ containing X.

Remark 2.2. (see, [2]) For any bounded lattice $(L; \wedge, \vee; 0, 1)$, Sub(L) with the usual ordering ⊆ is a complete lattice.

Definition 2.5. (see, [2]) A map $f: L_1 \rightarrow L_2$ from the underlying set L_1 of a bounded lattice to the underlying set L_2 of another bounded lattice is called a homomorphism of bounded lattices if it satisfies the following conditions:

i. $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L_1$,
ii. $f(0) = 0$ and $f(1) = 1$.

We now review some fuzzy logic concepts.

Definition 2.6. A fuzzy subset of a nonempty set L is a function $\mu: L \rightarrow [0, 1]$ from L to the real unit interval [0, 1].

Notation 2.3. F(L) denotes the set of all fuzzy subsets of a nonempty set L.
We define on F(L) the partial order ≤, called order of fuzzy sets, by: $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in L$. We also define on F(L) the binary operations $\wedge$ and $\vee$ respectively by: $(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}$ and $(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in L$.

Notation 2.4. For any family $\{\mu_i\}_{i \in I}$ of fuzzy subsets of a nonempty set L, $\inf_{i \in I}\mu_i$ and $\sup_{i \in I}\mu_i$ denote the fuzzy subsets of L respectively defined by: $(\inf_{i \in I}\mu_i)(x) = \inf_{i \in I}\mu_i(x)$ and $(\sup_{i \in I}\mu_i)(x) = \sup_{i \in I}\mu_i(x)$ for all $x \in L$. We will sometimes use the notations $\inf\{\mu_i: i \in I\}$ and $\sup\{\mu_i: i \in I\}$ to mean $\inf_{i \in I}\mu_i$ and $\sup_{i \in I}\mu_i$ respectively.

Remark 2.3. F(L) with the order ≤ is a complete lattice.

Definition 2.7. A fuzzy relation on a nonempty set L is a fuzzy subset of LxL.

3. Main results

We confuse now a bounded lattice (L; ∧, ∨; 0, 1) with it underlying set L. In the following, unless otherwise specified, L denotes a bounded lattice.

Definition 3.1. Let μ be a fuzzy subset of L. μ is called a fuzzy bounded sublattice of L if it satisfies the following condition:

$$\min\{\mu(x\wedge y),\mu(x\vee y),\mu(0),\mu(1)\} \geq \min\{\mu(x),\mu(y)\} \text{ for all } x,y \in L.$$

Definition 3.2. Let μ be a fuzzy subset of L. μ is called a fuzzy dot bounded sublattice of L if it satisfies the following condition:

$$\min\{\mu(x\wedge y),\mu(x\vee y),\mu(0),\mu(1)\} \geq \mu(x)\cdot\mu(y) \text{ for all } x,y \in L.$$

Example 3.1. Let L={0, a, b, c, 1} such that 0<a,b<c<1 and, a and b are incomparable. The fuzzy subset ξ of L, defined by $\xi(x) = \begin{cases} 0{,}5 \text{ if } x \in \{0,b,1\}, \\ 0{,}4 \text{ if } x = a, \\ 0{,}3 \text{ if } x = c. \end{cases}$ for all x ϵ L, is a fuzzy dot bounded sublattice of L.

Note that every fuzzy bounded sublattice is a fuzzy dot bounded sublattice, but the converse is not necessary true. In fact, the fuzzy dot bounded sublattice ξ of the example 3.1 is not a fuzzy bounded sublattice, because

$$\xi(a\vee b) = \xi(c) = 0{,}3 \ngeq 0{,}4 = \min\{\xi(a),\xi(b)\}.$$

Proposition 3.1. Let μ be a fuzzy subset of L. If μ is a fuzzy dot bounded sublattice of L, then $\min\{\mu(0),\mu(1)\} \geq \mu(x)^2$ for all x ϵ L.

Proof. If μ is a fuzzy dot bounded sublattice of L, then $\min\{\mu(0),\mu(1)\} \geq \min\{\mu(x\wedge x),\mu(x\vee x),\mu(0),\mu(1)\} \geq \mu(x)\cdot\mu(x)$ for all $x \in L$; thus, $\min\{\mu(0),\mu(1)\} \geq \mu(x)^2$ for all x ϵ L.

Corollary 3.1. Let μ be a fuzzy dot bounded sublattice of L. If there exists a sequence $\{x_n\}$ in L such that $\lim_{n\to\infty} \mu(x_n)^2 = 1$, then μ (0) = μ(1) = 1.

Proof. Assume that there exists a sequence $\{x_n\}$ in L such that $\lim_{n\to\infty} \mu(x_n)^2 = 1$. By proposition 3.1, $\min\{\mu(0),\mu(1)\} \geq \mu(x_n)^2$ for every positive integer n. Consider, $1 \geq \min\{\mu(0),\mu(1)\} \geq \lim_{n\to\infty} \mu(x_n)^2 = 1$.
Hence, min{μ(0),μ(1)}=1; i.e., μ(0) = μ(1) = 1.

Theorem 3.1. Let $\{\mu_i\}_{i\in I}$ be a family of fuzzy dot bounded sublattices of L. Then $\inf_{i\in I}\mu_i$ is a fuzzy dot bounded sublattice of L.

Proof. For any x,y ϵ L,

$$\begin{aligned}
&\min\{(\inf_{i\in I}\mu_i)(x\wedge y),(\inf_{i\in I}\mu_i)(x\vee y),(\inf_{i\in I}\mu_i)(0),(\inf_{i\in I}\mu_i)(1)\} = \\
&= \min\{\inf_{i\in I}\mu_i(x\wedge y),\inf_{i\in I}\mu_i(x\vee y),\inf_{i\in I}\mu_i(0),\inf_{i\in I}\mu_i(1)\} \\
&= \min\{\inf_{i\in I}\mu_i(x)\cdot\mu_i(y),\inf_{i\in I}\mu_i(x)\cdot\mu_i(y),\inf_{i\in I}\mu_i(x)\cdot\mu_i(y),\inf_{i\in I}\mu_i(x)\cdot\mu_i(y)\} \\
&= \inf_{i\in I}\mu_i(x)\cdot\mu_i(y) \\
&\geq \inf_{i\in I}(\inf_{i\in I}\mu_i(x))\cdot(\inf_{i\in I}\mu_i(y)) \\
&= (\inf_{i\in I}\mu_i(x))\cdot(\inf_{i\in I}\mu_i(y)) \\
&= (\inf_{i\in I}\mu_i)(x)\cdot(\inf_{i\in I}\mu_i)(y).
\end{aligned}$$

Hence, $\inf_{i\in I}\mu_i$ is a fuzzy dot bounded sublattice of L.

Notation 3.1. Fs(L) denotes the set of all fuzzy dot bounded sublattices of L.

Remark 3.1. Fs(L) with the order ≤ of fuzzy sets is a complete lattice.

Definition 3.3. Given a fuzzy subset μ of L, the smallest fuzzy dot bounded sublattice of L containing μ is called the fuzzy dot bounded sublattice of L generated by μ, and denoted by Fsg(μ).

Notation 3.2. For any subset B of L and α, β ϵ [0, 1], B_β^α and $[B_\beta^\alpha]$ denote the fuzzy subsets of L respectively defined by:

$$B_\beta^\alpha(x) = \begin{cases} \alpha \text{ if } x \in B, \\ \beta \text{ otherwise.} \end{cases} \text{ and } [B_\beta^\alpha](x) = \begin{cases} \alpha \text{ if } x \in Sg(B), \\ \beta \text{ otherwise.} \end{cases} \text{ for all } x \in L.$$

Lemma 3. Let B be a subset of L and α, β ϵ [0, 1] such that β ≤ α. Then $[B_\beta^\alpha]$ is a fuzzy dot bounded sublattice of L.

Proof. Let + ϵ {∧,∨} and x,y ϵ L.

If x+ y ∉ Sg(B), then x ∉ Sg(B) or y ∉ Sg(B); thus, $[B_\beta^\alpha]$(x) = β or $[B_\beta^\alpha]$(y) = β; thus, $[B_\beta^\alpha]$ (x+y) = β ≥ max{α·β,β^2} ≥ $[B_\beta^\alpha]$(x)·$[B_\beta^\alpha]$(y).

If x+y ϵ Sg(B), then $[B_\beta^\alpha]$(x+y) = α ≥ max{α^2,α·β,β^2} ≥ $[B_\beta^\alpha]$(x)·$[B_\beta^\alpha]$(y).

Thus, $[B_\beta^\alpha]$(x+y) ≥$[B_\beta^\alpha]$(x)·$[B_\beta^\alpha]$(y) for all + ϵ {∧, ∨} and x,y ϵ L.

$[B_\beta^\alpha]$(e) = α ≥ max{α^2,α·β,β^2} ≥ $[B_\beta^\alpha]$(x)·$[B_\beta^\alpha]$(y) for all e ϵ {0,1} and x,y ϵ L. So, min{$[B_\beta^\alpha]$(x∧y),$[B_\beta^\alpha]$(x∨y),$[B_\beta^\alpha]$(0),$[B_\beta^\alpha]$(1)} ≥ $[B_\beta^\alpha]$(x)·$[B_\beta^\alpha]$(y) for all x,y ϵ L; i.e., $[B_\beta^\alpha]$ is a fuzzy dot bounded sublattice of L.

If B is a bounded sublattice of L, then B_β^α is a fuzzy dot bounded sublattice of L; but the converse is not necessary true. In fact, consider the bounded lattice L of the example 3.1; $\{a,b,c,1\}_{0,4}^{0,5}$ is a fuzzy dot bounded sublattice of L but {a,b,c,1} is not a bounded sublattice of L, because 0 ∉ {a,b,c,1}.

Theorem 3.2. Let B be a subset of L and $\beta \in [0,1]$. Then $[B_\beta^1]$ is the fuzzy dot bounded sublattice of L generated by B_β^1; i.e., Fsg(B_β^1) $= [B_\beta^1]$.

Proof. Since $\beta \leq 1$, $[B_\beta^1]$ is a fuzzy dot bounded sublattice of L by the lemma 3.. It suffices now to show that $[B_\beta^1]$ is the smallest fuzzy dot bounded sublattice of L containing B_β^1. So, since $B_\beta^1(x)$ ≤ 1 =$[B_\beta^1]$ (x) for all x ϵ Sg(B) and $B_\beta^1(x)$ = β = $[B_\beta^1]$(x) for all x ∉ Sg(B), it follows that $B_\beta^1(x)$ ≤ $[B_\beta^1]$ (x) for all x ϵ L; i.e., $[B_\beta^1]$ contains B_β^1. Let ν be a fuzzy dot bounded sublattice of L containing B_β^1. For any x ∉ Sg(B), we have $[B_\beta^1]$(x) = β =$B_\beta^1(x)$ ≤ ν(x). For any

$x \in$ Sg(B), there is an n-ary lattice term t and $x_{1,\dots,}x_n$ ϵ B such that x=$t^L(x_{1,\dots,}x_n)$; thus, there is a positive integer k such that ν(x)≥ $(\nu(x_1)\cdots\nu(x_2))^k$; thus, ν(x) ≥ $(B_\beta^1(x_1)\cdots B_\beta^1(x_2))^k$ = $(1\cdots1)^k$ = 1^k = 1= $[B_\beta^1]$(x) . Consequently,

ν(x) ≥ $[B_\beta^1]$(x) for all x ϵ L; i.e., ν contains $[B_\beta^1]$. Hence, $[B_\beta^1]$ is the smallest fuzzy dot bounded sublattice of L containing B_β^1; i.e., $[B_\beta^1]$ is the fuzzy dot bounded sublattice of L generated by B_β^1.

Corollary 3.2. Let B be a subset of L and $\beta \in [0,1]$. Then B_β^1 is a fuzzy dot bounded sublattice of L if and only B is a bounded sublattice of L.
Proof. B_β^1 is a fuzzy dot bounded sublattice of L if and only if Fsg(B_β^1) $= B_\beta^1$; i.e., $[B_\beta^1] = B_\beta^1$; i.e., Sg(B) = B; i.e., B is a bounded sublattice of L.

Notation 3.3. χ_B denotes the characteristic function of a subset B of L.

Corollary 3.3. Let B be a nonempty subset of L. Then χ_B is a fuzzy dot bounded sublattice of L if and only if B is a bounded sublattice of L.
Proof. Straightforward, because $\chi_B = B_0^1$.

Definition 3.4. For any fuzzy subset μ of L and α ϵ [0, 1], the subset {x ϵ L: μ(x) ≥ α} of L, denoted by U(μ;α), is called a level subset of μ.

A fuzzy subset is a fuzzy bounded sublattice if and only if all its nonempty level subsets are bounded sublattices. But there is a fuzzy dot bounded sublattice with a nonempty level subset which is not a bounded sublattice. In fact, consider the fuzzy dot bounded sublattice ξ of the example 3.1; U(ξ;0,4) = {0,a,b,1} is not a bounded sublattice of L, because a,b ϵ U(ξ;0,4) and a∨b = c ∉ U(ξ;0,4).

Theorem 3.3. Let μ be a fuzzy subset of L. If μ is a fuzzy dot bounded sublattice of L, then U(μ;1) is either empty or a bounded sublattice of L.
Proof. Assume that μ is a fuzzy dot bounded sublattice of L and U(μ;1) is a nonempty subset of L. If x and y belong to U(μ;1), then
min{μ(x∧y),μ(x∨y)} ≥ min{μ(x∧y),μ(x∨y),μ(0),μ(1)} ≥ μ(x)·μ(y) = 1·1 = 1; thus, min{μ(x∧y),μ(x∨y)} = 1; i.e., μ(x∧y) = 1 and μ(x∨y) = 1; i.e., x∧y ϵ U(μ;1) and x∨y ϵ U(μ;1). Since there is a ϵ U(μ;1), we have $\mu(a) = 1$; thus, μ(0) = μ(1) = 1 by the corollary 3.1; i.e., 0 ϵ U(μ;1) and 1 ϵ U(μ;1).
Hence, U(μ;1) is a bounded sublattice of L.

Definition 3.5. Let $f: L \to P$ be a function from a set L to a set P and ν be a fuzzy subset of P. The preimage under f of ν, denoted by $f^{-1}[\nu]$, is the fuzzy subset of L defined by: $f^{-1}[\nu]$(x) = ν(f(x)) for all x ϵ L.

Theorem 3.4. Let $f: L \to P$ be a homomorphism of bounded lattices and ν be a fuzzy subset of P. If ν is a fuzzy dot bounded sublattice of P, then f^{-1}[ν] is a fuzzy dot bounded sublattice of L.
Proof. Assume that ν is a fuzzy dot bounded sublattice of P. For any x,y ϵ L**,**

$$\min\{f^{-1}[\nu](x\wedge y), f^{-1}[\nu](x\vee y)\, f^{-1}[\nu](0), f^{-1}[\nu](1)\} =$$
$$= \min\{\nu(f(x\wedge y)),\nu(f(x\vee y)),\nu(f(0)),\nu(f(1))\}$$
$$= \min\{\nu(f(x)\wedge f(y)),\nu(f(x)\vee f(y)),\nu(0),\nu(1)\}$$
$$\geq \min\{\nu(f(x))\cdot\nu(f(y)),\nu(f(x))\cdot\nu(f(y)),\nu(f(x))\cdot\nu(f(y)),\nu(f(x))\cdot\nu(f(y))\}$$
$$= \nu(f(x))\cdot\nu(f(y))$$
$$= f^{-1}[\nu](x)\cdot f^{-1}[\nu](y).$$

Hence, f^{-1}[ν] is a fuzzy dot bounded sublattice of L.

Definition 3.6. Let $f: L \to P$ be a function from a set L to a set P and μ be a fuzzy subset of L. The image under f of μ, denoted by $f[\mu]$, is the fuzzy subset of P defined by: $f[\mu](y) = \begin{cases} sup_{a\epsilon f^{-1}(y)}\mu(a) & if\ f^{-1}(y) \neq \emptyset, \\ 0 & otherwise. \end{cases}$ for all $y \epsilon P$.

Theorem 3.5. Let $f: L \to P$ be an onto homomorphism of bounded lattices and μ be a fuzzy subset of L. If μ is a fuzzy dot bounded sublattice of L, then $f[\mu]$ is a fuzzy dot bounded sublattice of P.

Proof. Assume that μ is a fuzzy dot bounded sublattice of L.
Let x,y ϵ P, A = f^{-1}(x) and B = f^{-1}(y).
Let + ϵ {∧,∨} and C = f^{-1}(x+y). Consider the set

$$A+B = \{s \epsilon L: s=a+b \text{ for some } a \epsilon A \text{ and } b \epsilon B\}.$$

If s ϵ A+B, then s = a+b for some a ϵ A and b ϵ B; so that,

$$f(s) = f(a+b) = f(a)+f(b) = x+y;$$

that is, s ϵ f^{-1}(x+y) = C. Hence, A+B ⊆C. It follows that

$$\begin{aligned} f[\mu](x+y) &= \sup_{s\epsilon C}\mu(s) \\ &\geq \sup_{s\epsilon A+B}\mu(s) \\ &\geq \sup_{a\epsilon A,b\epsilon B}\mu(a+b) \\ &\geq \sup_{a\epsilon A,b\epsilon B}\mu(a)\cdot\mu(b) \\ &= (\sup_{a\epsilon A}\mu(a))\cdot(\sup_{b\epsilon B}\mu(b)) \\ &= f[\mu](x)\cdot f[\mu](y). \end{aligned}$$

For any e ϵ {0,1}, we have

$$\begin{aligned} f[\mu](e) &\geq \mu(e), \text{ because } e \epsilon f^{-1}(e) \\ &\geq \sup_{a\epsilon A,b\epsilon B}\mu(a)\cdot\mu(b) \\ &= (\sup_{a\epsilon A}\mu(a))\cdot(\sup_{b\epsilon B}\mu(b)) \\ &= f[\mu](x)\cdot f[\mu](y). \end{aligned}$$

Hence, min{f[μ](x∧y), f[μ](x∨y), f[μ](0), f[μ](1)} ≥ f[μ](x)·f[μ](y) for all x,y ϵ P; i.e., f[μ] is a fuzzy dot bounded sublattice of P.

Definition 3.7. Let σ be a fuzzy subset of L. The strong σ-relation on L is the fuzzy subset μ_σ of LxL defined by: $\mu_\sigma(x,y) = \sigma(x)\cdot\sigma(y)$ for all x,y ϵ L.

Theorem 3.6. If σ is a fuzzy dot bounded sublattice of L, then μ_σ is a fuzzy dot bounded sublattice of LxL.

Proof. Assume that σ is a fuzzy dot bounded sublattice of L.
For any + ϵ {∧, ∨} and x_1,x_2,y_1,y_2 ϵ L, we have

$$\begin{aligned} \mu_\sigma((x_1,y_1)+(x_2,y_2)) &= \mu_\sigma(x_1+x_2,y_1+y_2) \\ &= \sigma(x_1+x_2)\cdot\sigma(y_1+y_2) \\ &\geq (\sigma(x_1)\cdot\sigma(x_2))\cdot(\sigma(y_1)\cdot\sigma(y_2)) \\ &= (\sigma(x_1)\cdot\sigma(y_1))\cdot(\sigma(x_2)\cdot\sigma(y_2)) \\ &= \mu_\sigma(x_1,y_1)\cdot\mu_\sigma(x_2,y_2). \end{aligned}$$

For any e ∈ {0,1} and x_1,x_2,y_1,y_2 ϵ L, we have

$$\mu_\sigma(e,e) = \sigma(e)\cdot\sigma(e) \geq (\sigma(x_1)\cdot\sigma(y_1))\cdot(\sigma(x_2)\cdot\sigma(y_2)) = \mu_\sigma(x_1,y_1)\cdot \mu_\sigma(x_2,y_2).$$

Hence,
min{ $\mu_\sigma((x_1,y_1)\wedge(x_2,y_2)),\mu_\sigma((x_1,y_1)\vee(x_2,y_2)),\mu_\sigma(0,0),\mu_\sigma(1,1)\} \geq \mu_\sigma(x_1,y_1)\cdot\mu_\sigma(x_2,y_2)$ for all x_1,x_2,y_1,y_2 ϵ L; i.e., μ_σ is a fuzzy dot bounded sublattice of LxL.

Definition 3.8. Let σ be a fuzzy subset of L. A fuzzy relation μ on L is called a σ-product relation on L if μ(x,y) ≥ σ(x)·σ(y) for all x,y ϵL.

Definition 3.9. Let σ be a fuzzy subset of L. A fuzzy relation μ on L is called a left fuzzy relation on σ if $\mu(x,y) = \sigma(x)$ for all $x,y \in L$.
Similarly, we can define a right fuzzy relation on σ. Note that a left (resp. right) fuzzy relation on σ is a fuzzy σ-product relation on L.

Theorem 3.7. Let μ be a left fuzzy relation on a fuzzy subset σ of L. If μ is a fuzzy dot bounded sublattice of LxL, then σ is a fuzzy dot bounded sublattice of L.
Proof. Assume that μ is a fuzzy dot bounded sublattice of LxL.
For any $x_1,x_2,y_1,y_2 \in L$, we have

$$\begin{aligned}
&\min\{\sigma(x_1\wedge x_2),\sigma(x_1\vee x_2),\sigma(0),\sigma(1)\} = \\
&\quad = \min\{\mu(x_1\wedge x_2, y_1\wedge y_2), \mu(x_1\vee x_2 ,y_1\vee y_2), \mu(0,0),\mu(1,1)\} \\
&\quad = \min\{ \mu((x_1,y_1)\wedge(x_2,y_2)), \mu((x_1,y_1)\vee(x_2,y_2),\mu(0,0),\mu(1,1)\} \\
&\quad \geq \min\{ \mu(x_1,y_1)\cdot \mu(x_2,y_2), \mu(x_1,y_1)\cdot \mu(x_2,y_2), \mu(x_1,y_1)\cdot \mu(x_2,y_2), \mu(x_1,y_1)\cdot \mu(x_2,y_2)\} \\
&\quad = \mu(x_1,y_1)\cdot \mu(x_2,y_2) \\
&\quad = \sigma(x_1)\cdot\sigma(x_2).
\end{aligned}$$

Hence, $\min\{\sigma(x_1\wedge x_2),\sigma(x_1\vee x_2),\sigma(0),\sigma(1)\} \geq \sigma(x_1)\cdot\sigma(x_2)$ for all $x_1,x_2 \in L$; i.e., σ is a fuzzy dot bounded sublattice of L.

Theorem 3.8. Let μ be a fuzzy relation on L satisfying the inequality $\mu(x,y) \leq \min\{\mu(x,0),\mu(x,1),\mu(0,1),\mu(1,0)\}$ for all $x,y \in L$. Given $z \in L$, let σ_z be a fuzzy subset of L defined by $\sigma_z(x) = \mu(x,z)$ for all $x \in L$. If μ is a fuzzy dot bounded sublattice of LxL, then σ_z is a fuzzy dot bounded sublattice of L.
Proof. Assume that μ is a fuzzy dot bounded sublattice of LxL .
For any $x,y \in L$, we have

$$\begin{aligned}
\sigma_z(x\wedge y) &= \mu(x\wedge y,z) = \mu(x\wedge y,z\wedge 1) = \mu((x,z)\wedge(y,1)) \\
&\geq \mu(x,z)\cdot\mu(y,1) \geq \mu(x,z)\cdot\mu(y,z) = \sigma_z(x)\cdot \sigma_z(y) ,
\end{aligned}$$

$$\begin{aligned}
\sigma_z(x\vee y) &= \mu(x\vee y,z) = \mu(x\vee y,z\vee 0) = \mu((x,z)\vee(y,0)) \\
&\geq \mu(x,z)\cdot\mu(y,0) \geq \mu(x,z)\cdot\mu(y,z) = \sigma_z(x)\cdot \sigma_z(y) ,
\end{aligned}$$

$$\begin{aligned}
\sigma_z(0) &= \mu(0,z) = \mu(x\wedge 0,z\wedge 1) = \mu((x,z)\wedge(0,1)) \\
&\geq \mu(x,z)\cdot\mu(0,1) \geq \mu(x,z)\cdot\mu(y,z) = \sigma_z(x)\cdot \sigma_z(y) ,
\end{aligned}$$

$$\begin{aligned}
\sigma_z(1) &= \mu(1,z) = \mu(x\vee 1,z\vee 0) = \mu((x,z)\vee(1,0)) \\
&\geq \mu(x,z)\cdot\mu(1,0) \geq \mu(x,z)\cdot\mu(y,z) = \sigma_z(x)\cdot\sigma_z(y) .
\end{aligned}$$

Hence, $\min\{\sigma_z(x\wedge y),\sigma_z(x\vee y),\sigma_z(0),\sigma_z(1)\} \geq \sigma_z(x)\cdot\sigma_z(y)$ for all $x,y \in L$; i.e., σ_z is a fuzzy dot bounded sublattice of L.

Theorem 3.9. Let μ be a fuzzy subset of LxL and let σ_μ be a fuzzy subset of L defined by $\sigma_\mu(x)=\inf_{y\in L}\mu(x,y)\cdot\mu(y,x)$ for all $y \in L$. If μ is a fuzzy dot bounded sublattice of LxL satisfying the equality $\min\{\mu(x,0),\mu(0,x),\mu(x,1),\mu(1,x)\}=1$ for all $x \in L$, then σ_μ is a fuzzy dot bounded sublattice of L.
Proof. For any $x,y,z \in L$, we have

$$\begin{aligned}
\mu(x\wedge y, z)&=\mu(x\wedge y,z\wedge 1)=\mu((x,z)\wedge(y,1)) \\
&\geq\mu(x,z)\cdot\mu(y,1)= \mu(x,z)\cdot 1= \mu(x,z)
\end{aligned}$$

and

$$\begin{aligned}
\mu(z,x\wedge y) &= \mu(z\wedge 1,x\wedge y)=\mu((z,x)\wedge(1,y)) \\
&\geq \mu(z,x)\cdot\mu(1,y)= \mu(z,x)\cdot 1= \mu(z,x);
\end{aligned}$$

it follows that

$$\begin{aligned}
\mu(x\wedge y, z)\cdot\mu(z,x\wedge y) &\geq \mu(x,z)\cdot\mu(z,x) \\
&\geq (\mu(x,z)\cdot\mu(z,x))\cdot(\mu(y,z)\cdot\mu(z,y)) .
\end{aligned}$$

So that, for any $x,y \in L$, $\sigma_\mu(x\wedge y) = \inf_{z\in L}\mu(x\wedge y,z)\cdot\mu(z,x\wedge y)$
$\geq \inf_{z\in L}(\mu(x,z)\cdot\mu(z,x))\cdot(\mu(y,z)\cdot\mu(z,y))$
$\geq \inf_{z\in L}(\inf_{z\in L}\mu(x,z)\cdot\mu(z,x))\cdot(\inf_{z\in L}\mu(y,z)\cdot\mu(z,y))$
$= (\inf_{z\in L}\mu(x,z)\cdot\mu(z,x))\cdot(\inf_{z\in L}\mu(y,z)\cdot\mu(z,y))$
$= \sigma_\mu(x)\cdot\sigma_\mu(y)$.

For any $x,y,z \in L$, we have

$$\mu(x\vee y,z) = \mu(x\vee y,z\vee 0) = \mu((x,z)\vee(y,0)) \geq \mu(x,z)\cdot\mu(y,0) = \mu(x,z)\cdot 1= \mu(x,z)$$

and

$$\mu(z,x\vee y)= \mu(z\vee 0,x\vee y) = \mu((z,x)\vee(0,y)) \geq \mu(z,x)\cdot\mu(0,y) = \mu(z,x)\cdot 1= \mu(z,x);$$

it follows that $\mu(x\vee y, z)\cdot\mu(z,x\vee y) \geq \mu(x,z)\cdot\mu(z,x)$
$\geq (\mu(x,z)\cdot\mu(z,x))\cdot(\mu(y,z)\cdot\mu(z,y))$.

So that, for any $x,y \in L$, $\sigma_\mu(x\vee y) = \inf_{z\in L}\mu(x\vee y,z)\cdot\mu(z,x\vee y)$
$\geq \inf_{z\in L}(\mu(x,z)\cdot\mu(z,x))\cdot(\mu(y,z)\cdot\mu(z,y))$
$\geq (\inf_{z\in L}\mu(x,z)\cdot\mu(z,x))\cdot(\inf_{z\in L}\mu(y,z)\cdot\mu(z,y))$
$= \sigma_\mu(x)\cdot\sigma_\mu(y)$.

For any $x,y \in L$, $\sigma_\mu(0) = \inf_{z\in L}\mu(0,z)\cdot\mu(z,0) = \inf_{z\in L}1\cdot 1=1 \geq \sigma_\mu(x)\cdot\sigma_\mu(y)$

and

$\sigma_\mu(1) = \inf_{z\in L}\mu(1,z)\cdot\mu(z,1) = \inf_{z\in L}1\cdot 1 = 1 \geq \sigma_\mu(x)\cdot\sigma_\mu(y)$.

Hence, $\min\{\sigma_\mu(x\wedge y),\sigma_\mu(x\vee y),\sigma_\mu(0),\sigma_\mu(1)\} \geq \sigma_\mu(x)\cdot\sigma_\mu(y)$ for all $x,y \in L$; i.e., σ_μ is a fuzzy dot bounded sublattice of L.

Definition 3.10. (see, [4]) A fuzzy map f from a set L to a set P is an ordinary map f from L to the set F(P) of all fuzzy subsets of P satisfying the following conditions:

i. for any $x \in L$, there exists $y_x \in P$ such that $(f(x))(y_x) = 1$,
ii. for any $x \in L$, $f(x)(y_1) = f(x)(y_2)$ implies $y_1 = y_2$.

One observes that a fuzzy map f from L to P gives rise to a unique ordinary map μ_f from LxP to [0, 1] given by $\mu_f(x,y) = f(x)(y)$ for all $x \in L$ and $y \in P$. One also notes that a fuzzy map from L to P gives a unique f_1 from L to P defined as $f_1(x) = y_x$.

Now we can generalize the notion of homomorphism of bounded lattices to the notion of fuzzy homomorphism of bounded lattices.

Definition 3.11. A fuzzy map f from a bounded lattice L to a bounded lattice P is called a fuzzy homomorphism if the following conditions are satisfied:

i. $\mu_f(a\wedge b,y) = \sup_{y=u\wedge v}(\mu_f(a,u)\cdot\mu_f(b,v))$ for all $a,b \in L$ and $y \in P$,
ii. $\mu_f(a\vee b,y) = \sup_{y=u\vee v}(\mu_f(a,u)\cdot\mu_f(b,v))$ for all $a,b \in L$ and $y \in P$,
iii. $\mu_f(0,0) = \mu_f(1,1) = 1$.

One note that if f is an ordinary map, then the above definition reduces to an ordinary homomorphism. One also observes that if a fuzzy map is a fuzzy homomorphism, then the ordinary map f_1 is an ordinary homomorphism.

Theorem 3.10. Let f be a fuzzy homomorphism from a bounded lattice L to a bounded lattice P. Then for any $a,b \in L$ and $x,y \in P$, we have

$$\min\{\mu_f(a\wedge b,x\wedge y),\mu_f(a\vee b,x\vee y),\mu_f(0,0), \mu_f(1,1)\} \geq \mu_f(a,x)\cdot\mu_f(b,y).$$

Proof. For any $a,b \in L$ and $x,y \in P$, we have

$\min\{\mu_f(a\wedge b,x\wedge y),\mu_f(a\vee b,x\vee y),\mu_f(0,0),\mu_f(1,1)\}=$

$= \min\{\mu_f(a\wedge b,x\wedge y),\mu_f(a\vee b,x\vee y),1,1\}$
$= \min\{\mu_f(a\wedge b,x\wedge y),\mu_f(a\vee b,x\vee y)\}$
$= \min\{\sup_{x\wedge y=u\wedge v}(\mu_f(a,u)\cdot\mu_f(b,v)),\sup_{x\vee y=u\vee v}(\mu_f(a,u)\cdot\mu_f(b,v))\}$
$\geq \min\{\mu_f(a,x)\cdot\mu_f(b,y),\mu_f(a,x)\cdot\mu_f(b,y)\}$, because $x\wedge y=x\wedge y$ and $x\vee y=x\vee y$
$= \mu_f(a,x)\cdot\mu_f(b,y)$;

thus, $\min\{\mu_f(a\wedge b,x\wedge y),\mu_f(a\vee b,x\vee y),\mu_f(0,0),\mu_f(1,1)\} \geq \mu_f(a,x)\cdot\mu_f(b,y)$.

Definition 3.12. Let μ and ν be two fuzzy subsets of L. The fuzzy subsets $\mu{\curlywedge}\nu$ and $\mu{\curlyvee}\nu$ of L are respectively defined by:

$(\mu{\curlywedge}\nu)(x) = \sup_{x=a\wedge b}\mu(a)\cdot\nu(b)$ and $(\mu{\curlyvee}\nu)(x) = \sup_{x=a\vee b}\mu(a)\cdot\nu(b)$ for all $x \in L$.

Remark 3.2. The binary operations $\curlywedge$ and $\curlyvee$ preserve the fuzzy set order.

Proposition 3.2. The binary operations $\curlywedge$ and $\curlyvee$ are associative.

Proof. Let μ, ν and δ be three fuzzy subsets of L. For any $z \in L$, we have

$((\mu{\curlywedge}\nu){\curlywedge}\delta)(z) = \sup_{z=a\wedge b}(\mu{\curlywedge}\nu)(a)\cdot\delta(b) = \sup_{z=a\wedge b}(\sup_{a=p\wedge q}\mu(p)\wedge\nu(q))\cdot\delta(b)$
$= \sup_{z=a\wedge b} \sup_{a=p\wedge q}(\mu(p)\cdot\nu(q))\cdot\delta(b) = \sup_{z=(p\wedge q)\wedge b}(\mu(p)\cdot\nu(q))\cdot\delta(b)$
$= \sup_{z=p\wedge(q\wedge b)}\mu(p)\cdot(\nu(q)\cdot\delta(b)) = \sup_{z=p\wedge a}\mu(p)\cdot(\sup_{a=q\wedge b} \nu(q)\cdot\delta(b))$
$= \sup_{z=p\wedge a} \mu(p)\cdot(\nu{\curlywedge}\delta)(a) = (\mu{\curlywedge}(\nu{\curlywedge}\delta))(z)$.

Thus, $(\mu{\curlywedge}\nu){\curlywedge}\delta = \mu{\curlywedge}(\nu{\curlywedge}\delta)$.
Hence, $\curlywedge$ is associative.
Similarly we may show that $\curlyvee$ is associative.

Proposition 3.3. The binary operations $\curlywedge$ and $\curlyvee$ are commutative.

Proof. Let μ and ν be two fuzzy subsets of L. For any $z \in L$, we have
$(\mu{\curlywedge}\nu)(z) = \sup_{z=a\wedge b}\mu(a)\cdot\nu(b) = \sup_{z=b\wedge a}\nu(b)\cdot\mu(a) = (\nu{\curlywedge}\mu)(z)$. Thus, $\mu{\curlywedge}\nu = \nu{\curlywedge}\mu$.
Hence, $\curlywedge$ is commutative.
Similarly we may show that $\curlyvee$ is commutative.

Definition 3.13. Let μ and ν be two fuzzy subsets of L. The fuzzy subsets $\mu^0\nu$ and $\mu^1\nu$ of L are respectively defined by:

$$(\mu^0\nu)(x) = \begin{cases}\sup_{a,b\in L}\mu(a)\cdot\nu(b) \text{ if } x = 0,\\ 0 \text{ if } x \neq 0.\end{cases}$$

and

$$(\mu^1\nu)(x) = \begin{cases}\sup_{a,b\in L}\mu(a)\cdot\nu(b) \text{ if } x = 1,\\ 0 \text{ if } x \neq 1.\end{cases}.$$

Proposition 3.4. Let μ, ν and δ be three fuzzy subsets of L. Then

$$(\mu^0\nu)^0\delta = \mu^0(\nu^0\delta) \text{ and } (\mu^1\nu)^1\delta = \mu^1(\nu^1\delta).$$

Proof. For any $x \neq 0$, we have $((\mu^0\nu)^0\delta)(x) = 0 = (\mu^0(\nu^0\delta))(x)$; more, we have

$((\mu^0\nu)^0\delta)(0)= \sup_{a,b\in L}(\mu^0\nu) (a)\cdot\delta(b) = \sup_{b\in L}(\mu^0\nu) (0)\cdot\delta(b)$
$= \sup_{b\in L}(\sup_{c,d\in L}\mu(c)\cdot\nu(d)) \cdot\delta(b)$
$= \sup_{b\in L}\sup_{c,d\in L}(\mu(c)\cdot\nu(d)) \cdot\delta(b)$
$= \sup_{b,c,d\in L}(\mu(c)\cdot\nu(d)) \cdot\delta(b)$
$= \sup_{c\in L}\sup_{b,d\in L} \mu(c)\cdot(\nu(d) \cdot\delta(b))$
$= \sup_{c\in L}\mu(c)\cdot\sup_{b,d\in L}\nu(d) \cdot\delta(b)$
$= \sup_{c\in L}\mu(c)\cdot(\nu^0\delta)(0)$
$= \sup_{c,a\in L}\mu(c)\cdot(\nu^0\delta)(a)$
$= (\mu^0(\nu^0\delta))(0)$.

Hence, $((\mu^0\nu)^0\delta)(x) = (\mu^0(\nu^0\delta))(x)$ for all $x \in L$; i.e., $(\mu^0\nu)^0\delta = \mu^0(\nu^0\delta)$.

Similarly we may show that $(\mu^1\nu)^1\delta = \mu^1(\nu^1\delta)$.

Proposition 3.5. Let μ and ν be two fuzzy subsets of L. Then $\mu^0\nu = \nu^0\mu$ and $\mu^1\nu = \nu^1\mu$.

Proof. $(\mu^0\nu)(0) = \sup_{a,b\in L}\mu(a)\cdot\nu(b) = \sup_{b,a\in L}\nu(b)\cdot\mu(a) = (\nu^0\mu)(0)$ and $(\mu^0\nu)(x) = 0 = (\nu^0\mu)(x)$ for all $x \neq 0$. Hence, $(\mu^0\nu)(x) = (\nu^0\mu)(x)$ for all $x \in L$; i.e., $\mu^0\nu = \nu^0\mu$.

Similarly we may show that $\mu^1\nu = \nu^1\mu$.

Theorem 3.11. Let μ be a fuzzy subset of L. Then the following are equivalent.

a. μ is a fuzzy dot bounded sublattice of L.
b. $\mu^\wedge\mu \leq \mu$, $\mu^\vee\mu \leq \mu$, $\mu^0\mu \leq \mu$ and $\mu^1\mu \leq \mu$.

Proof. (a→b) Assume that μ is a fuzzy dot bounded sublattice of L.

For any $x \in L$, we have $\mu(a)\cdot\mu(b) \leq \mu(x)$ for all $a,b \in L$ such that $x = a\wedge b$; i.e., $\sup_{x=a\wedge b}\mu(a)\cdot\mu(b) \leq \mu(x)$; i.e., $(\mu^\wedge\mu)(x) \leq \mu(x)$. Thus, $\mu^\wedge\mu \leq \mu$.

Similarly we may show $\mu^\vee\mu \leq \mu$.

$\mu(a)\cdot\mu(b) \leq \mu(0)$ for all $a,b \in L$; i.e., $\sup_{a,b\in L}\mu(a)\cdot\mu(b) \leq \mu(0)$; i.e., $(\mu^0\mu)(0) \leq \mu(0)$.

More, $(\mu^0\mu)(x) = 0 \leq \mu(x)$ for all $x \neq 0$. Thus, $(\mu^0\mu)(x) \leq \mu(x)$ for all $x \in L$; i.e., $\mu^0\mu \leq \mu$.

Similarly we may show $\mu^1\mu \leq \mu$.

Hence, $\mu^\wedge\mu \leq \mu$, $\mu^\vee\mu \leq \mu$, $\mu^0\mu \leq \mu$ and $\mu^1\mu \leq \mu$.

(b→a) Assume that $\mu^\wedge\mu \leq \mu$, $\mu^\vee\mu \leq \mu$, $\mu^0\mu \leq \mu$ and $\mu^1\mu \leq \mu$.

For any $x,y \in L$, $(\mu^\wedge\mu)(x\wedge y) \leq \mu(x\wedge y)$, $(\mu^\vee\mu)(x\vee y) \leq \mu(x\vee y)$, $(\mu^0\mu)(0) \leq \mu(0)$ and $(\mu^1\mu)(1) \leq \mu(1)$; i.e., $\sup_{x\wedge y=a\wedge b}\mu(a)\cdot\mu(b) \leq \mu(x\wedge y)$, $\sup_{x\vee y=a\vee b}\mu(a)\cdot\mu(b) \leq \mu(x\vee y)$, $\sup_{a,b\in L}\mu(a)\cdot\mu(b) \leq \mu(0)$ and $\sup_{a,b\in}\mu(a)\cdot\mu(b) \leq \mu(1)$; thus, $\mu(x)\cdot\mu(y) \leq \mu(x\wedge y)$, $\mu(x)\cdot\mu(y) \leq \mu(x\vee y)$, $\mu(x)\cdot\mu(y) \leq \mu(0)$ and $\mu(x)\cdot\mu(y) \leq \mu(1)$; i.e., $\min\{\mu(x\wedge y),\mu(x\vee y),\mu(0),\mu(1)\} \geq \mu(x)\cdot\mu(y)$.

Hence, μ is a fuzzy dot bounded sublattice of L.

Definition 3.14. Let $x \in L$ and $\alpha \in [0, 1]$. The fuzzy subset x_α of L, defined by $x_\alpha(y) = \begin{cases}\alpha \text{ if } y = x,\\ 0 \text{ if } y \neq x.\end{cases}$, is called a fuzzy point of L .

Definition 3.15. A fuzzy point x_α of L is said to be contained in a fuzzy subset μ of L, denoted by $x_\alpha \in \mu$, if $\alpha \leq \mu(x)$.

Proposition 3.6. Let x_p and y_q be two fuzzy points of L. Then

a. $x_p{}^\wedge y_q = (x\wedge y)_{p\cdot q}$ and $x_p{}^\vee y_q = (x\vee y)_{p\cdot q}$
b. $x_p{}^0 y_q = 0_{p\cdot q}$ and $x_p{}^1 y_q = 1_{p\cdot q}$.

Proof. a. For any $z \neq x\wedge y$, we have $(x_p{}^\wedge y_q)(z) = \sup_{z=a\wedge b}x_p(a)\cdot y_q(b) = \sup_{z=a\wedge b}0\cdot 0 = \sup_{z=a\wedge b}0 = 0$; more, $(x_p{}^\wedge y_q)(x\wedge y) = \sup_{x\wedge y=a\wedge b}x_p(a)\cdot y_q(b) = x_p(x)\cdot y_q(y) = p\cdot q$. Thus, $x_p{}^\wedge y_q = (x\wedge y)_{p\cdot q}$.

Similarly we may show $x_p{}^\vee y_q = (x\vee y)_{p\cdot q}$.

b. $(x_p{}^0 y_q)(0) = \sup_{a,b\in L} x_p(a)\cdot y_q(b) = x_p(x)\cdot y_q(y) = p\cdot q$; more, $(x_p{}^0 y_q)(z) = 0$ for all $z \neq 0$. Thus, $x_p{}^0 y_q = 0_{p\cdot q}$.

Similarly we may show $x_p{}^1 y_q = 1_{p\cdot q}$.

Corollary 3.4. Let x_p, y_q and z_r be three fuzzy points of L. Then

a. $x_p{}^\wedge x_q = x_p{}^\vee x_q = x_{p\cdot q}$ and $x_p{}^\wedge x_p = x_p{}^\vee x_p = x_{p^2}$
b. $(x_p{}^\wedge y_q)^\wedge z_r = x_p{}^\wedge(y_q{}^\wedge z_r) = (x\wedge y\wedge z)_{p\cdot q\cdot r}$ and $(x_p{}^\vee y_q)^\vee z_r = x_p{}^\vee(y_q{}^\vee z_r) = (x\vee y\vee z)_{p\cdot q\cdot r}$
c. $(x_p{}^\wedge x_q)^\vee x_p = (x_p{}^\vee x_q)^\wedge x_p = x_{q.p^2}$.

Proof. a. $x_p \wedge x_q = (x\wedge x)_{p.q} = x_{p\cdot q} = (x\vee x)_{p.q} = x_p \vee x_q$ and $x_p\wedge x_p = x_{p\cdot p} = x_{p^2} = x_{p\cdot p} = x_p \vee x_p$.
b. $(x_p\wedge y_q)\wedge z_r = (x\wedge y)_{p.q}\wedge z_r = ((x\wedge y)\wedge z)_{(p.q).r} = (x\wedge(y\wedge z))_{p.(q.r)} = x_p\wedge(y\wedge z)_{q\cdot r} = x_p\wedge(y_q\wedge z_r)$.
Similarly we may show $(x_p\vee y_q)\vee z_r = x_p\vee(y_q\vee z_r) = (x\vee y\vee z)_{p\cdot q\cdot r}$.
c. $(x_p\wedge x_q)\vee x_p = x_{p.q}\vee x_p = x_{(p\cdot q)\cdot p} = x_{q.p^2} = x_{p\cdot(p\cdot q)} = x_p\vee x_{p\cdot q} = x_p\vee(x_p\wedge x_q)$.

Proposition 3.7. Let μ and ν be two fuzzy subsets of L. Then

a. $\mu\wedge\nu = \sup\{x_p\wedge y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$ and $\mu\vee\nu = \sup\{x_p\vee y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$
b. $\mu{}^0\nu = \sup\{x_p{}^0y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$ and $\mu{}^1\nu = \sup\{x_p{}^1y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$.

Proof. a. Let $z \epsilon$ L.

$(\mu\wedge\nu)(z) = \sup_{z=a\wedge b}\mu(a)\cdot\nu(b) = \sup_{z=a\wedge b}a_{\mu(a)}(a)\cdot b_{\nu(b)}(b)$
$\leq \sup\{\sup_{z=a\wedge b}x_p(a)\cdot y_q(b): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$, because $a_{\mu(a)}\epsilon\mu$ and $b_{\nu(b)}\epsilon\nu$;
$= \sup\{(x_p\wedge y_q)(z): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$
$= (\sup\{x_p\wedge y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\})(z)$.

For any $x_p \epsilon \mu$ and $y_q \epsilon \nu$, for any $z=a\wedge b$, we have $x_p(a) \leq \mu(a)$ and $y_q(b) \leq \nu(b)$;
thus, $\sup_{z=a\wedge b}x_p(a)\cdot y_q(b) \leq \sup_{z=a\wedge b}\mu(a)\cdot\nu(b)$ for all $x_p \epsilon \mu$ and $y_q \epsilon \nu$;
i.e., $\sup_{z=a\wedge b}x_p(a)\cdot y_q(b) \leq (\mu\wedge\nu)(z)$ for all $x_p \epsilon \mu$ and $y_q \epsilon \nu$;
i.e., $\sup\{\sup_{z=a\wedge b}x_p(a)\cdot y_q(b): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\} \leq (\mu\wedge\nu)(z)$;
i.e., $\sup\{(x_p\wedge y_q)(z): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\} \leq (\mu\wedge\nu)(z)$.

Hence, $(\mu\wedge\nu)(z) = (\sup\{x_p\wedge y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\})(z)$ for all $z \epsilon$ L; i.e.,
$\mu\wedge\nu = \sup\{x_p\wedge y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$.
Similarly we may show $\mu\vee\nu = \sup\{x_p\vee y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$.
b. For any $z \epsilon$ L such that $z \neq 0$, we have

$(\sup\{x_p{}^0y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\})(z) = \sup\{(x_p{}^0y_q)(z): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$
$= \sup\{0: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$
$= 0$
$= (\mu{}^0\nu)(z)$.

$(\mu{}^0\nu)(0) = \sup_{a,b\epsilon L}\mu(a)\cdot\nu(b) = \sup_{a,b\epsilon L} a_{\mu(a)}(a)\cdot b_{\nu(b)}(b)$
$\leq \sup\{\sup_{a,b\epsilon L}x_p(a)\cdot y_q(b): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$, because $a_{\mu(a)}\epsilon\mu$ and $b_{\nu(b)}\epsilon\nu$;
$= \sup\{(x_p{}^0y_q)(0): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$
$= (\sup\{x_p{}^0y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\})(0)$.

For any $x_p \epsilon \mu$ and $y_q \epsilon \nu$, for any $a,b \epsilon$ L, we have $x_p(a) \leq \mu(a)$ and $y_q(b) \leq \nu(b)$;
thus, $\sup_{a,b\epsilon L}x_p(a)\cdot y_q(b) \leq \sup_{a,b\epsilon L}\mu(a)\cdot\nu(b)$ for all $x_p \epsilon \mu$ and $y_q \epsilon \nu$;
i.e., $\sup_{a,b\epsilon L}x_p(a)\cdot y_q(b) \leq (\mu{}^0\nu)(z)$ for all $x_p \epsilon \mu$ and $y_q \epsilon \nu$;
i.e., $\sup\{\sup_{a,b\epsilon L}x_p(a)\cdot y_q(b): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\} \leq (\mu{}^0\nu)(z)$;
i.e., $\sup\{(x_p{}^0y_q)(z): x_p\epsilon\mu \text{ and } y_q\epsilon\nu\} \leq (\mu{}^0\nu)(z)$.

Hence, $(\mu{}^0\nu)(z) = (\sup\{x_p{}^0y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\})(z)$ for all $z \epsilon$ L; i.e.,
$\mu{}^0\nu = \sup\{x_p{}^0y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$.
Similarly we may show $\mu{}^1\nu = \sup\{x_p{}^1y_q: x_p\epsilon\mu \text{ and } y_q\epsilon\nu\}$.

Theorem 3.12. Let μ be a fuzzy subset of L. The following are equivalent.

a. μ is a fuzzy dot bounded sublattice of L.
b. $x_p\wedge y_q\epsilon\mu$, $x_p\vee y_q\epsilon\mu$, $x_p{}^0y_q\epsilon\mu$ and $x_p{}^1y_q\epsilon\mu$ for all $x_p\epsilon\mu$ and $y_q\epsilon\mu$.

Proof. (a.→b.) Assume that μ is a fuzzy dot bounded sublattice of L. For any $x_p\epsilon\mu$ and $y_q\epsilon\mu$, we have $\mu(x) \geq p$ and $\mu(y) \geq q$; thus, $\mu(x)\cdot\mu(x) \geq p\cdot q$; thus, $\min\{\mu(x\wedge y),\mu(x\vee y),\mu(0),\mu(1)\}\geq p\cdot q$; i.e., $\mu(x\wedge y) \geq p\cdot q$, $\mu(x\vee y) \geq p\cdot q$, $\mu(0) \geq p\cdot q$ and $\mu(1) \geq p\cdot q$; i.e., $(x\wedge y)_{p\cdot q}\epsilon\mu$, $(x\vee y)_{p\cdot q}\epsilon\mu$, $0_{p\cdot q}\epsilon\mu$ and $1_{p\cdot q}\epsilon\mu$; i.e., $x_p\wedge y_q\epsilon\mu$, $x_p\vee y_q\epsilon\mu$, $x_p{}^0 y_q\epsilon\mu$ and $x_p{}^1y_q\epsilon\mu$.
(b.→a.) Assume that $x_p\wedge y_q\epsilon\mu$, $x_p\vee y_q\epsilon\mu$, $x_p{}^0y_q\epsilon\mu$ and $x_p{}^1y_q\epsilon\mu$ for all $x_p, y_q\epsilon\mu$.
Let $x,y \epsilon$ L. Since $x_{\mu(x)},y_{\mu(y)}\epsilon\mu$, we have $x_{\mu(x)}\wedge y_{\mu(y)}\epsilon\mu$, $x_{\mu(x)}\vee y_{\mu(y)}\epsilon\mu$, $x_{\mu(x)}{}^0y_{\mu(y)}\epsilon\mu$ and $x_{\mu(x)}{}^1y_{\mu(y)}\epsilon\mu$; i.e., $(x\wedge y)_{\mu(x)\cdot\mu(y)}\epsilon\mu$, $(x\vee y)_{\mu(x)\cdot\mu(y)}\epsilon\mu$, $0_{\mu(x)\cdot\mu(y)}\epsilon\mu$ and$1_{\mu(x)\cdot\mu(y)}\epsilon\mu$; i.e.,

$\mu(x\wedge y) \geq \mu(x)\cdot\mu(y)$, $\mu(x\vee y) \geq \mu(x)\cdot\mu(y)$, $\mu(0) \geq \mu(x)\cdot\mu(y)$, $\mu(1) \geq \mu(x)\cdot\mu(y)$; i.e., $\min\{\mu(x\wedge y),\mu(x\vee y),\mu(0),\mu(1)\} \geq \mu(x)\cdot\mu(y)$.
Hence, μ is a fuzzy dot bounded sublattice of L.

Notation 3.4. For any $x,y \in L\backslash\{0,1\}$ such that $x \neq y$ and $\alpha,\beta \in [0, 1]$, $\widehat{x_\alpha, y_\beta}$ denotes the fuzzy subset of L defined by:

$$\widehat{x_\alpha, y_\beta}(t) = \begin{cases} \alpha^2 \ if\ t \in \{0,1\}, \\ \alpha \ if\ t = x, \\ \beta \ if\ t = y, \\ \alpha\cdot\beta \ \ if\ \ t \in \{x\wedge y, x\vee y\} \setminus \{0,x,y,1\}, \\ 0 \ otherwise. \end{cases}$$

for all $t \in L$.

Example 3.2. Let L be the bounded lattice of the example 3.1. The fuzzy subset $\widehat{a_{0,3}, b_{0,5}}$ of L is defined by:

$$\widehat{a_{0,3}, b_{0,5}}(t) = \begin{cases} 0{,}09 \text{ if } t \in \{0,1\}, \\ 0{,}3 \text{ if } t = a, \\ 0{,}5 \text{ if } t = b, \\ 0{,}15 \text{ if } t = c. \end{cases}.$$

For any any $x,y \in L\backslash\{0,1\}$ such that $x \neq y$ and $\alpha,\beta \in [0, 1]$, $\widehat{x_\alpha, y_\beta}$ is not necessary a fuzzy dot bounded sublattice of L. In fact, the fuzzy subset $\widehat{a_{0,3}, b_{0,5}}$ of the example 3.2 is not a fuzzy dot bounded sublattice, because

$$\widehat{a_{0,3}, b_{0,5}}(0)=0{,}09 \ngeq 0{,}25=\widehat{a_{0,3}, b_{0,5}}(b)\cdot\widehat{a_{0,3}, b_{0,5}}(b).$$

Theorem 3.13. Let $x,y \in L\backslash\{0, 1\}$ such that $x \neq y$ and $\alpha,\beta \in [0, 1]$ such that $\beta \leq \alpha$. Then $\widehat{x_\alpha, y_\beta}$ is the smallest fuzzy dot bounded sublattice of L containing both x_α and y_β; i.e., the fuzzy dot bounded sublattice of L generated by $\{x_\alpha,y_\beta\}$.

Proof. Since $x_\alpha(x) = \alpha = \widehat{x_\alpha, y_\beta}(x)$ and $x_\alpha(a) = 0 \leq \widehat{x_\alpha, y_\beta}(a)$ for all $a \in L\backslash\{x\}$, it follows that $x_\alpha(a) \leq \widehat{x_\alpha, y_\beta}(a)$ for all $a \in L$; i.e., $\widehat{x_\alpha, y_\beta}$ contains x_α. Similarly we may prove that $\widehat{x_\alpha, y_\beta}$ contains y_β. Therefore, $\widehat{x_\alpha, y_\beta}$ contains both x_α and y_β. Next we show that $\widehat{x_\alpha, y_\beta}$ is a fuzzy dot bounded sublattice of L.

For any $a,b \in L$ such that $a \notin \{0,x,y,x\wedge y,x\vee y,1\}$ or $b \notin \{0,x,y,x\wedge y,x\vee y,1\}$, we have

$\min\{\widehat{x_\alpha, y_\beta}(a\wedge b), \widehat{x_\alpha, y_\beta}(a\vee b), \widehat{x_\alpha, y_\beta}(0), \widehat{x_\alpha, y_\beta}(1)\} \geq 0 = \widehat{x_\alpha, y_\beta}(a)\cdot\widehat{x_\alpha, y_\beta}(b)$.

$\min\{\widehat{x_\alpha, y_\beta}(0\wedge 0), \widehat{x_\alpha, y_\beta}(0\vee 0), \widehat{x_\alpha, y_\beta}(0), \widehat{x_\alpha, y_\beta}(1)\} = \min\{\alpha^2,\alpha^2,\alpha^2,\alpha^2\}$
$= \alpha^2$
$\geq \alpha^2\cdot\alpha^2$
$= \widehat{x_\alpha, y_\beta}(0)\cdot\widehat{x_\alpha, y_\beta}(0)$.

$\min\{\widehat{x_\alpha, y_\beta}(0\wedge x), \widehat{x_\alpha, y_\beta}(0\vee x), \widehat{x_\alpha, y_\beta}(0), \widehat{x_\alpha, y_\beta}(1)\} = \min\{\alpha^2,\alpha,\alpha^2,\alpha^2\}$
$= \alpha^2$
$\geq \alpha^2\cdot\alpha$
$= \widehat{x_\alpha, y_\beta}(0)\cdot\widehat{x_\alpha, y_\beta}(x)$.

$\min\{\widehat{x_\alpha, y_\beta}(0\wedge y), \widehat{x_\alpha, y_\beta}(0\vee y), \widehat{x_\alpha, y_\beta}(0), \widehat{x_\alpha, y_\beta}(1)\} = \min\{\alpha^2,\beta,\alpha^2,\alpha^2\}$
$= \min\{\alpha^2,\beta\}$
$\geq \alpha^2\cdot\beta$
$= \widehat{x_\alpha, y_\beta}(0)\cdot\widehat{x_\alpha, y_\beta}(y)$.

$\min\{\widehat{x_\alpha, y_\beta}(0\wedge(x\wedge y)), \widehat{x_\alpha, y_\beta}(0\vee(x\wedge y)), \widehat{x_\alpha, y_\beta}(0), \widehat{x_\alpha, y_\beta}(1)\} =$
$= \min\{\alpha^2,\widehat{x_\alpha, y_\beta}(x\wedge y),\alpha^2,\alpha^2\}$
$= \min\{\alpha^2,\widehat{x_\alpha, y_\beta}(x\wedge y)\}$

$$\geq \alpha^2\cdot\widehat{x_\alpha, y_\beta}(x\wedge y)$$
$$= \widehat{x_\alpha, y_\beta}(0)\cdot\widehat{x_\alpha, y_\beta}(x\wedge y).$$

$$\min\{\widehat{x_\alpha, y_\beta}(0\wedge(x\vee y)),\ \widehat{x_\alpha, y_\beta}(0\vee(x\vee y)),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}=$$
$$= \min\{\alpha^2,\widehat{x_\alpha, y_\beta}(x\vee y),\alpha^2,\alpha^2\}$$
$$= \min\{\alpha^2,\widehat{x_\alpha, y_\beta}(x\vee y)\}$$
$$\geq \alpha^2\cdot\widehat{x_\alpha, y_\beta}(x\vee y)$$
$$= \widehat{x_\alpha, y_\beta}(0)\cdot\widehat{x_\alpha, y_\beta}(x\vee y).$$

$$\min\{\widehat{x_\alpha, y_\beta}(0\wedge 1),\ \widehat{x_\alpha, y_\beta}(0\vee 1),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\} = \min\{\alpha^2,\alpha^2,\alpha^2,\alpha^2\}$$
$$= \alpha^2$$
$$\geq \alpha^2\cdot\alpha^2$$
$$= \widehat{x_\alpha, y_\beta}(0)\cdot \widehat{x_\alpha, y_\beta}(1).$$

$$\min\{\widehat{x_\alpha, y_\beta}(x\wedge x),\ \widehat{x_\alpha, y_\beta}(x\vee x),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}= \min\{\alpha,\alpha,\alpha^2,\alpha^2\}$$
$$= \alpha^2$$
$$= \alpha\cdot\alpha$$
$$= \widehat{x_\alpha, y_\beta}(x)\cdot \widehat{x_\alpha, y_\beta}(x).$$

$$\min\{\widehat{x_\alpha, y_\beta}(x\wedge y),\ \widehat{x_\alpha, y_\beta}(x\vee y),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}\geq \min\{\alpha\cdot\beta,\alpha\cdot\beta,\alpha^2,\alpha^2\}$$
$$= \alpha\cdot\beta$$
$$= \widehat{x_\alpha, y_\beta}(x)\cdot \widehat{x_\alpha, y_\beta}(y).$$

$$\min\{\widehat{x_\alpha, y_\beta}(x\wedge(x\wedge y)),\widehat{x_\alpha, y_\beta}(x\vee(x\wedge y)),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}=$$
$$= \min\{\widehat{x_\alpha, y_\beta}(x\wedge y),\alpha,\alpha^2,\alpha^2\}$$
$$\geq \min\{\alpha\cdot\widehat{x_\alpha, y_\beta}(x\wedge y),\alpha,\alpha^2,\alpha^2\}$$
$$= \alpha\cdot\widehat{x_\alpha, y_\beta}(x\wedge y)$$
$$= \widehat{x_\alpha, y_\beta}(x)\cdot\widehat{x_\alpha, y_\beta}(x\wedge y).$$

$$\min\{\widehat{x_\alpha, y_\beta}(x\wedge(x\vee y)),\widehat{x_\alpha, y_\beta}(x\vee(x\vee y)),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}=$$
$$= \min\{\alpha,\widehat{x_\alpha, y_\beta}(x\vee y),\alpha^2,\alpha^2\}$$
$$\geq \min\{\alpha,\alpha\cdot\widehat{x_\alpha, y_\beta}(x\vee y),\alpha^2,\alpha^2\}$$
$$= \alpha\cdot\widehat{x_\alpha, y_\beta}(x\vee y)$$
$$= \widehat{x_\alpha, y_\beta}(x)\cdot\widehat{x_\alpha, y_\beta}(x\vee y).$$

$$\min\{\widehat{x_\alpha, y_\beta}(x\wedge 1),\ \widehat{x_\alpha, y_\beta}(x\vee 1),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}= \min\{\alpha,\alpha^2,\alpha^2,\alpha^2\}$$
$$= \alpha^2$$
$$\geq \alpha\cdot \alpha^2$$
$$= \widehat{x_\alpha, y_\beta}(x)\cdot \widehat{x_\alpha, y_\beta}(1).$$

$$\min\{\widehat{x_\alpha, y_\beta}(y\wedge y),\ \widehat{x_\alpha, y_\beta}(y\vee y),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}= \min\{\beta,\beta,\alpha^2,\alpha^2\}$$
$$\geq \beta^2$$
$$= \widehat{x_\alpha, y_\beta}(y)\cdot \widehat{x_\alpha, y_\beta}(y).$$

$$\min\{\widehat{x_\alpha, y_\beta}(y\wedge(x\wedge y)),\ \widehat{x_\alpha, y_\beta}(y\vee(x\wedge y)),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}=$$
$$= \min\{\widehat{x_\alpha, y_\beta}(x\wedge y),\beta,\alpha^2,\alpha^2\}$$
$$\geq \min\{\beta\cdot\widehat{x_\alpha, y_\beta}(x\wedge y),\beta,\alpha^2,\alpha^2\}$$
$$= \beta\cdot\widehat{x_\alpha, y_\beta}(x\wedge y)$$
$$= \widehat{x_\alpha, y_\beta}(y)\cdot \widehat{x_\alpha, y_\beta}(x\wedge y).$$

$$\min\{\widehat{x_\alpha, y_\beta}(y\wedge(x\vee y)),\ \widehat{x_\alpha, y_\beta}(y\vee(x\vee y)),\ \widehat{x_\alpha, y_\beta}(0),\ \widehat{x_\alpha, y_\beta}(1)\}=$$
$$= \min\{\beta,\ \widehat{x_\alpha, y_\beta}(x\vee y),\alpha^2,\alpha^2\}$$
$$\geq \min\{\beta,\beta\cdot\widehat{x_\alpha, y_\beta}(x\vee y),\alpha^2,\alpha^2\}$$
$$= \beta\cdot\widehat{x_\alpha, y_\beta}(x\vee y)$$
$$= \widehat{x_\alpha, y_\beta}(y)\cdot \widehat{x_\alpha, y_\beta}(x\vee y).$$

$\min\{\widehat{x_{\alpha},y_{\beta}}(y\wedge 1), \widehat{x_{\alpha},y_{\beta}}(y\vee 1), \widehat{x_{\alpha},y_{\beta}}(0), \widehat{x_{\alpha},y_{\beta}}(1)\} = \min\{\beta,\alpha^2,\alpha^2,\alpha^2\}$
$\geq \beta\cdot\alpha^2$
$= \widehat{x_{\alpha},y_{\beta}}(y)\cdot \widehat{x_{\alpha},y_{\beta}}(1).$

$\min\{\widehat{x_{\alpha},y_{\beta}}((x\wedge y)\wedge(x\wedge y)), \widehat{x_{\alpha},y_{\beta}}((x\wedge y)\vee(x\wedge y)), \widehat{x_{\alpha},y_{\beta}}(0), \widehat{x_{\alpha},y_{\beta}}(1)\} =$
$= \min\{\widehat{x_{\alpha},y_{\beta}}(x\wedge y), \widehat{x_{\alpha},y_{\beta}}(x\wedge y),\alpha^2,\alpha^2\}$
$\geq \min\{\widehat{x_{\alpha},y_{\beta}}(x\wedge y)\cdot \widehat{x_{\alpha},y_{\beta}}(x\wedge y), \widehat{x_{\alpha},y_{\beta}}(x\wedge y),\alpha^2,\alpha^2\}$
$= \widehat{x_{\alpha},y_{\beta}}(x\wedge y)\cdot \widehat{x_{\alpha},y_{\beta}}(x\wedge y).$

$\min\{\widehat{x_{\alpha},y_{\beta}}((x\wedge y)\wedge(x\vee y)), \widehat{x_{\alpha},y_{\beta}}((x\wedge y)\vee(x\vee y)), \widehat{x_{\alpha},y_{\beta}}(0), \widehat{x_{\alpha},y_{\beta}}(1)\}=$
$= \min\{\widehat{x_{\alpha},y_{\beta}}(x\wedge y), \widehat{x_{\alpha},y_{\beta}}(x\vee y),\alpha^2,\alpha^2\}$
$\geq \min\{\widehat{x_{\alpha},y_{\beta}}(x\wedge y)\cdot \widehat{x_{\alpha},y_{\beta}}(x\vee y), \widehat{x_{\alpha},y_{\beta}}(x\vee y),\alpha^2,\alpha^2\}$
$= \widehat{x_{\alpha},y_{\beta}}(x\wedge y)\cdot \widehat{x_{\alpha},y_{\beta}}(x\vee y).$

$\min\{\widehat{x_{\alpha},y_{\beta}}((x\wedge y)\wedge 1), \widehat{x_{\alpha},y_{\beta}}((x\wedge y)\vee 1), \widehat{x_{\alpha},y_{\beta}}(0), \widehat{x_{\alpha},y_{\beta}}(1)\}=$
$= \min\{\widehat{x_{\alpha},y_{\beta}}(x\wedge y),\alpha^2,\alpha^2,\alpha^2\}$
$\geq \widehat{x_{\alpha},y_{\beta}}(x\wedge y).\alpha^2$
$= \widehat{x_{\alpha},y_{\beta}}(x\wedge y)\cdot \widehat{x_{\alpha},y_{\beta}}(1).$

$\min\{\widehat{x_{\alpha},y_{\beta}}((x\vee y)\wedge(x\vee y)), \widehat{x_{\alpha},y_{\beta}}((x\vee y)\vee(x\vee y)), \widehat{x_{\alpha},y_{\beta}}(0), \widehat{x_{\alpha},y_{\beta}}(1)\}=$
$= \min\{\widehat{x_{\alpha},y_{\beta}}(x\vee y), \widehat{x_{\alpha},y_{\beta}}(x\vee y),\alpha^2,\alpha^2\}$
$\geq \min\{\widehat{x_{\alpha},y_{\beta}}(x\vee y)\cdot \widehat{x_{\alpha},y_{\beta}}(x\vee y), \widehat{x_{\alpha},y_{\beta}}(x\vee y),\alpha^2,\alpha^2\}$
$= \widehat{x_{\alpha},y_{\beta}}(x\vee y)\cdot \widehat{x_{\alpha},y_{\beta}}(x\vee y).$

$\min\{\widehat{x_{\alpha},y_{\beta}}((x\vee y)\wedge 1), \widehat{x_{\alpha},y_{\beta}}((x\vee y)\vee 1), \widehat{x_{\alpha},y_{\beta}}(0), \widehat{x_{\alpha},y_{\beta}}(1)\}=$
$= \min\{\widehat{x_{\alpha},y_{\beta}}(x\vee y),\alpha^2,\alpha^2,\alpha^2\}$
$\geq \widehat{x_{\alpha},y_{\beta}}(x\vee y).\alpha^2$
$= \widehat{x_{\alpha},y_{\beta}}(x\vee y)\cdot \widehat{x_{\alpha},y_{\beta}}(1).$

$\min\{\widehat{x_{\alpha},y_{\beta}}(1\wedge 1), \widehat{x_{\alpha},y_{\beta}}(1\vee 1), \widehat{x_{\alpha},y_{\beta}}(0), \widehat{x_{\alpha},y_{\beta}}(1)\} = \min\{\alpha^2,\alpha^2,\alpha^2,\alpha^2\}$
$= \alpha^2$
$\geq \alpha^2\cdot\alpha^2$
$= \widehat{x_{\alpha},y_{\beta}}(1)\cdot\widehat{x_{\alpha},y_{\beta}}(1).$

Therefore, $\widehat{x_{\alpha},y_{\beta}}$ is a fuzzy dot bounded sublattice of L. Finally we show that $\widehat{x_{\alpha},y_{\beta}}$ is the smallest fuzzy dot bounded sublattice of L containing both x_α and y_β.
So, let ν be a fuzzy dot bounded sublattice of L containing both x_α and y_β.
For any $a \in L\backslash\{0,x,y, x\wedge y,x\vee y,1\}$, we have $\nu(a) \geq 0 = \widehat{x_{\alpha},y_{\beta}}(a)$.
For any $e \in\{0,1\}$, we have $\nu(e) \geq \nu(x)^2 \geq (x_\alpha(x))^2 = \alpha^2 = \widehat{x_{\alpha},y_{\beta}}(e)$.
$\nu(x) \geq x_\alpha(x) = \alpha = \widehat{x_{\alpha},y_{\beta}}(x)$ and $\nu(y) \geq y_\beta(y) = \beta = \widehat{x_{\alpha},y_{\beta}}(y)$.
Let $+ \in \{\wedge,\vee\}$ and $x,y \in L$.
If $x + y \notin \{0, x, y, 1\}$, then $\nu(x+y) \geq \nu(x)\cdot\nu(y) \geq x_\alpha(x)\cdot y_\beta(y) = \alpha\cdot\beta = \widehat{x_{\alpha},y_{\beta}}(x+y)$.
If $x + y \in \{0,1\}$, then $\nu(x + y) \geq \nu(x)^2 \geq x_\alpha(x)^2 = \alpha^2 = \widehat{x_{\alpha},y_{\beta}}(x + y)$.
If $x + y = x$, then $\nu(x + y) = \nu(x) \geq x_\alpha(x) = \alpha = \widehat{x_{\alpha},y_{\beta}}(x) = \widehat{x_{\alpha},y_{\beta}}(x + y)$.
If $x + y = y$, then $\nu(x + y) = \nu(y) \geq y_\beta(y) = \beta = \widehat{x_{\alpha},y_{\beta}}(y) = \widehat{x_{\alpha},y_{\beta}}(x + y)$.
Therefore, $\nu(a) \geq \widehat{x_{\alpha},y_{\beta}}(a)$ for all $a \in L$; i.e., ν contains $\widehat{x_{\alpha},y_{\beta}}$.
Hence, $\widehat{x_{\alpha},y_{\beta}}$ is the smallest fuzzy dot bounded sublattice of L containing both x_α and y_β.

Remark 3.3. Let $x \in L\backslash\{0,1\}$ and $\alpha,\beta \in [0, 1]$ such that $\beta \leq \alpha$.

a) $\widehat{x_{\alpha},x_{\beta}} = \mathrm{Fsg}(x_\alpha)$.

b) $Fsg(x_\alpha)(t) = \begin{cases} \alpha^2 \; if \; t \in \{0,1\}, \\ \alpha \; if \; t = x, \\ 0 \; otherwise. \end{cases}$ for all $t \in L$.

c) $Fsg(0_\alpha)(t) = \begin{cases} \alpha^2 \; if \; t = 1, \\ \alpha \; if \; t = 0, \\ 0 \; otherwise. \end{cases}$ and $Fsg(1_\alpha)(t) = \begin{cases} \alpha^2 \; if \; t = 0, \\ \alpha \; if \; t = 1, \\ 0 \; otherwise. \end{cases}$ for all $t \in L$.

4. Conclusion

In the present paper, the notion of fuzzy dot bounded sublattices is introduced. Using t-norms T and s-norm S, these notions can further be generalized to T-fuzzy sets, S-fuzzy sets and intuitionistic (T,S)-fuzzy sets.

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