Journal of Mathematics and Informatics
Vol. 5, 2016, 1-9
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 29 February 2016
www.researchmathsci.org

Journal of
Mathematics and Informatics

An Approximate Technique for Solving Second Order Strongly Nonlinear Differential Systems with High Order Nonlinearity in Presence of Small Damping

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Received 15 January 2016; accepted 3 February 2016

Abstract. An approximate analytical technique has been extended for solving second order strongly nonlinear differential systems with high (nine) order nonlinearity in presence of small damping based on the He's homotopy perturbation and the extended form of the Krylov-Bogoliubov-Mitropolskii (KBM) methods. In general, it is too much difficult to solve the strongly nonlinear differential systems with high order nonlinearity in presence of small damping. The advantage of the presented method is that, it is capable to tackle both strongly and weakly nonlinear differential systems with high order nonlinearity in presence of small damping but the classical perturbation methods are failed to handle in that situations. The method has been justified by an example.

Keywords: Homotopy perturbation and KBM methods, damped nonlinear differential equation with high order nonlinearity.

AMS Mathematics Subject Classification (2010): 34E05

1. Introduction

Most of the phenomena in the real world are essentially nonlinear and described by nonlinear differential systems. So, the study of nonlinear differential systems is very important in all areas of applied mathematics, physics, engineering, medical science, economics and other disciplines. In general, it is too much difficult to handle nonlinear problems and it is often very difficult to get an analytical solution than a numerical one. Common methods for constructing approximate analytical solutions to the nonlinear differential equations are the perturbation techniques. Some well known perturbation techniques are the Krylov-Bogoliubov-Mitropolskii (KBM)[1-3] method, the Lindstedt-Poincare (LP) method [4, 5], and the method of multiple time scales [4]. Almost all perturbation methods are based on an assumption that the small parameters must exist in the equations, which is too strict to find wide range of application of the classical perturbation techniques. It determines not only the accuracy of the perturbation approximations, but also the validity of the perturbation methods itself. However, in science and engineering, there exist many nonlinear differential systems which do not contain any small parameter, especially those appear in nature with strong and high order nonlinearity in presence of small damping. Therefore, many new techniques have been proposed to eliminate the "small parameter" assumption, such as the homotopy perturbation method (HPM) [6-8, 11, 18-21], variational iteration method [9-10], harmonic balance method [22], energy balance method [23], etc. In recent years, He [6] has developed some new approaches to Duffing equation with strongly and high order non-linearity without damping. In another paper, He [7] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. He [8] has presented a new interpretation of homotopy perturbation method. He [9-10] has presented variational iteration method for strongly nonlinear differential systems without damping. Belendez et al. [11] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator without damping. Ganji et al. [12] have presented approximate solutions to van der Pol damped nonlinear oscillators by means of He's energy balance method. Lim and Wu [13] have developed a new analytical approach to the Duffing- harmonic oscillator without damping. Alam et al. [14] have developed a general Struble's technique for solving an nth order weakly nonlinear differential system with small damping. Bojadziev [15] has presented an analytical method to damped nonlinear oscillations modeled by a 3-dimensional differential system. Arya and Bojadziev [16] have presented time depended oscillating systems with small damping, slowly varying parameters and delay. Sachs et al. [17] have presented a simple ODE models of tumor growth and anti-angiogenic or radiation treatment. Uddin et al. [18-20] have presented an approximate technique for solving strongly cubic and quadratic nonlinear differential systems with damping effects. Uddin et al. [21] have developed an approximate analytical technique for solving second order strongly nonlinear generalized Duffing equation with small damping. Ghadimi and Kaliji [22] have presented an application of the harmonic balance method on nonlinear equations. From our study, it has been seen that the most of the authors have studied nonlinear differential systems with cubic nonlinearity and without considering damping effects. But most of the physical and engineering problems occur in nature in the form of nonlinear differential systems with small damping. In this article, we are interested to present an approximate analytical technique for solving second order strongly nonlinear differential systems with high $(9th)$ order nonlinearity in presence of small damping based on the He's homotopy perturbation and the extended form of the KBM methods. The presented method transforms a difficult problem under simplification, into a simple problem which is easy to solve, especially with high order nonlinearity. The advantage of the presented method is that the first approximate solutions show a good agreement with the corresponding numerical solutions for both strongly and weakly nonlinear differential systems with high order nonlinearity.

2. The method.

Let us consider the nonlinear differential systems modeling with high order nonlinearity in presence of small damping in the following form:

$$\ddot{x} + 2k(\tau)\,\dot{x} + \upsilon^2 x = -\varepsilon_1\, f(x,\dot{x}), \qquad (1)$$

where the over dots denote differentiations with respect to time t, υ is a constant, ε_1 is a parameter which is not necessarily $(\varepsilon_1 = 1.0)$ small, $k \geq 0$, $2k$, is the linear damping coefficient, $\tau = \varepsilon t$, is the slowly varying time, ε is a small positive parameter and the coefficients in Eq.(1) are varying slowly in that their time derivatives are proportional to ε, $f(x, \dot{x})$ is a given high order nonlinear function which satisfies the following condition:

$$f(-x, -\dot{x}) = -f(x, \dot{x}). \tag{2}$$

We are going to use the following transformation to change the dependent variable

$$x = y(t) e^{-kt}. \tag{3}$$

Now differentiating Eq. (3) twice with respect to time t and substituting the values of $\ddot{x}$, $\dot{x}$ and x into Eq. (1) and then simplifying we obtain

$$\ddot{y} + (\upsilon^2 - k^2) y = -\varepsilon_1 e^{kt} f(y e^{-kt}, (\dot{y} - k y) e^{-kt}). \tag{4}$$

According to the homotopy perturbation method [6-8, 18-21] Eq. (4) can be re-written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon_1 e^{kt} f(y e^{-kt}, (\dot{y} - k y) e^{-kt}), \tag{5}$$

where

$$\omega^2 = \upsilon^2 - k^2 + \lambda. \tag{6}$$

Here ω is known as the angular frequency of the nonlinear differential systems and is a constant for undamped nonlinear oscillators. But for the damped nonlinear differential systems, ω is a time dependent function and it varies slowly with time t and λ is an unknown function which can be evaluated by eliminating the secular terms. To handle this situation, we are interested to use the extended form of the KBM [1, 2] method by Mitropolskii [3]. According to this method, the solution of Eq. (5) can be chosen in the following form:

$$y = a \cos \varphi, \tag{7}$$

where a and φ vary slowly with time t. In physical problems, a and φ are known as the amplitude and phase variables respectively and they keep an important role to the nonlinear physical systems. The following first order differential equations are satisfied by amplitude a and phase variable φ:

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a, \tau) + \varepsilon^2 A_2(a, \tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + \varepsilon B_1(a, \tau) + \varepsilon^2 B_2(a, \tau) + \cdots. \end{aligned} \tag{8}$$

Now differentiating Eq.(7) twice with respect to time t with the help of Eq. (8) and substituting the values of $\ddot{y}$, $\dot{y}$, y into Eq.(5) and then equating the coefficients of $\sin\varphi$ and $\cos\varphi$,we obtain

$$A_1 = -a\,\omega' / (2\,\omega),\ B_1 = 0, \tag{9}$$

where prime denotes differentiation with respect to τ. Now inserting Eq. (7) into Eq. (3) and Eq. (9) into Eq. (8), we obtain the following equations:

$$x = a e^{-kt} \cos \varphi, \tag{10}$$

$$\dot{a} = -\varepsilon\,\omega'\,a/2\,\omega,$$
$$\dot{\varphi} = \omega(\tau). \qquad (11)$$

First approximate solution of Eq. (1) is given by Eq. (10) with help of Eq. (11) by the presented method. Usually the integration of Eq. (11) is performed by the well-known techniques of calculus [4-5], but sometimes they are calculated by a numerical procedure [14-21]. Thus, the first approximate solution of Eq. (1) is completed.

3. Example

As an example of the above procedure, let us consider the strongly nonlinear differential systems with high ($9th$) order [6, 7] nonlinearity in presence of small damping in the following form:

$$\ddot{x} + 2k(\tau)\,\dot{x} + \upsilon^2\,x = -\varepsilon_1\,x^9, \qquad (12)$$

where $f(x,\dot{x}) = x^9$. Now using the transformation Eq. (3) into Eq. (12) and then simplifying them, we obtained

$$\ddot{y} + (\upsilon^2 - k^2)\,y = -\varepsilon_1\,y^9\,e^{-8\,k\,t}. \qquad (13)$$

According to the homotopy perturbation [6-8, 18-21] technique, Eq. (13) can be written as

$$\ddot{y} + \omega^2\,y = \lambda\,y - \varepsilon\,y^9\,e^{-8\,k\,t}, \qquad (14)$$

where ω is calculated from Eq. (6). According to the extended form of the KBM [1-3] method, the solution of Eq. (14) is obtained from Eq. (7).
From the trigonometric identity, we obtain

$$\cos^9\varphi = (\cos 9\varphi + 9\cos 7\varphi + 36\cos 5\varphi + 84\cos 3\varphi + 126\cos\varphi)/256. \qquad (15)$$

For avoiding the secular terms in particular solution of Eq. (14), we need to impose that the coefficient of the $\cos\varphi$ term is zero. Setting this term to zero, we obtain,

$$\lambda\,a - \frac{126\varepsilon_1\,a^9\,e^{-8kt}}{256} = 0, \qquad (16)$$

which leads to

$$\lambda = \frac{63\varepsilon_1\,a^8\,e^{-8kt}}{128}. \qquad (17)$$

Putting the value of λ from Eq. (17) into Eq. (6), we obtain the following frequency equation:

$$\omega^2 = \upsilon^2 - k^2 + \frac{63\varepsilon_1\,a^8\,e^{-8kt}}{128}. \qquad (18)$$

From Eq. (18), it is clear that the frequency of the damped nonlinear physical systems depends on both amplitude a and time t. When $t \to 0$ then Eq. (18) yields

$$\omega_0 = \omega(0) = \sqrt{\upsilon^2 - k^2 + \frac{63\varepsilon_1\,a_0^8}{128}}, \qquad (19)$$

where a_0 is known as the initial amplitude and ω_0 represents the initial frequency of the nonlinear physical systems.

Integrating the first equation of Eq. (11), we get

$$a = a_0 \sqrt{\frac{\omega_0}{\omega}}, \tag{20}$$

where a_0 is a constant of integration and known as the initial amplitude in literature.

Now putting Eq. (20) into Eq. (18), we obtain a six degree polynomial in ω in the following form:

$$\omega^6 + p\,\omega^2 + r = 0, \tag{21}$$

where

$$p = k^2 - \upsilon^2,\ r = -\frac{63\varepsilon\, a_0^8\, \omega_0^4\, e^{-8kt}}{128}. \tag{22}$$

Finaly, the first order analytical approximate solution of Eq. (12) is obtained as follows:

$$x = a\, e^{-kt} \cos\varphi \tag{23}$$

$$a = a_0 \sqrt{\frac{\omega_0}{\omega}} \qquad \dot{\varphi} = \omega(\tau)\,, \tag{24}$$

where ω_0 is obtained by Eq. (19) ; ω is calculated from Eq. (21) by using the well-known **Newton–Raphson** method and a and φ are given by Eq. (24).

4. Results and discussions

In this paper, we have extended He's homotopy perturbation method for solving second order typical [6, 7] strongly nonlinear problems with high order nonlinearity in presence of small damping. It is almost impossible to solve the strongly nonlinear physical problems, especially with high order nonlinearity in presence of damping by the classical perturbation methods [1-5, 14-17]. But the suggested method has been successfully applied to solve strongly nonlinear differential systems with high ($9th$) order nonlinearity in presence of small damping. The first order approximate solutions of Eq. (12) is computed with high order nonlinearity in presence of small damping by Eq. (23) and the corresponding numerical solutions are obtained by using fourth order **Runge-Kutta method.** The variational equations of the amplitude and phase variables appeared in a set of first order differential equations. The integration of these variational equations is performed by the well-known techniques of calculus [4, 5]. In the lack of analytical solutions, numerical procedure [10-21] is applied to solve them. The amplitude and phase variables change slowly with time t. The behavior of amplitude and phase variables characterizes the oscillating processes and amplitude tends to zero in presence of small damping as $t \to \infty$. Presented technique can take full advantage of the classical perturbation method. It is also noticed that the presented method is also capable to handle the typical second order weakly $(\varepsilon_1 = 0.1)$ nonlinear differential systems with high order nonlinearity in presence of small damping. Comparison is made between the solutions obtained by the presented technique and those obtained by the numerical procedure in

Figs. 1-2 for both strongly $(\varepsilon_1 = 1.0)$ and weakly $(\varepsilon_1 = 0.1)$ nonlinear differential systems with high order nonlinearity. In **Figs.1-2,** it is seen that the solutions obtained by the presented method show a good agreement with those solution obtained by the numerical procedure with several small damping effects.

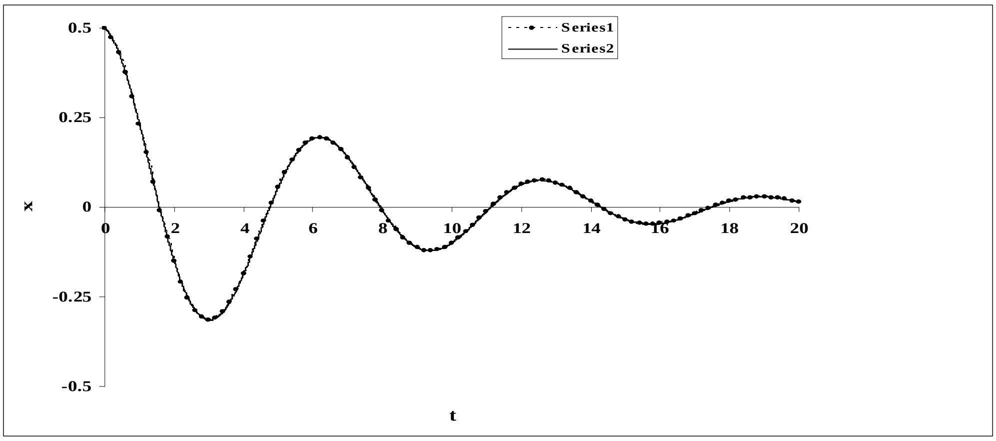

Figure 1(a): First approximate solution of Eq. (12) is denoted by dotted lines $(-\bullet-)$ by the presented analytical technique with the initial conditions $a_0 = 0.5, \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07853]$ with $\nu = 1.0,\ k = 0.15,\ \varepsilon_1 = 1.0,\ \varepsilon = 0.1$ and $f = x^9$ and the corresponding numerical solution is denoted by solid line $(-)$.

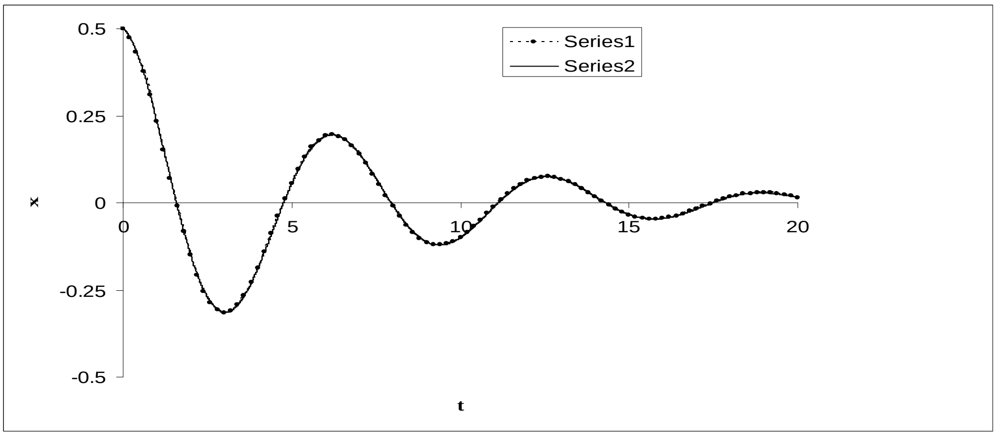

Figure 1(b): First approximate solution of Eq. (12) is denoted by dotted lines $(-\bullet-)$ by the presented analytical technique with the initial conditions $a_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07535]$ with $\nu = 1.0,\ k = 0.15,\ \varepsilon_1 = 0.1,\ \varepsilon = 0.1$ and $f = x^9$ and the corresponding numerical solution is denoted by solid line $(-)$.

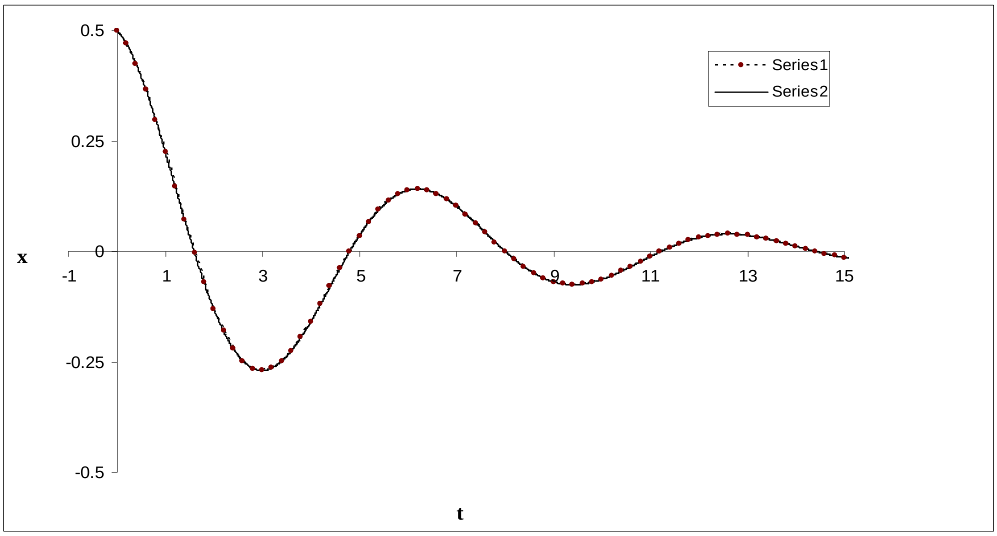

Figure 2(a): First approximate solution of Eq. (12) is denoted by dotted lines $(-\bullet-)$ by the presented analytical technique with the initial conditions $a_0 = 0.5, \varphi_0 = 0$ or $[x(0) = 0.5, \; \dot{x}(0) = -0.10360]$ with $\nu = 1.0, k = 0.2, \varepsilon_1 = 1.0, \varepsilon = 0.1$ and $f = x^9$ and the corresponding numerical solution is denoted by solid line $(-)$.

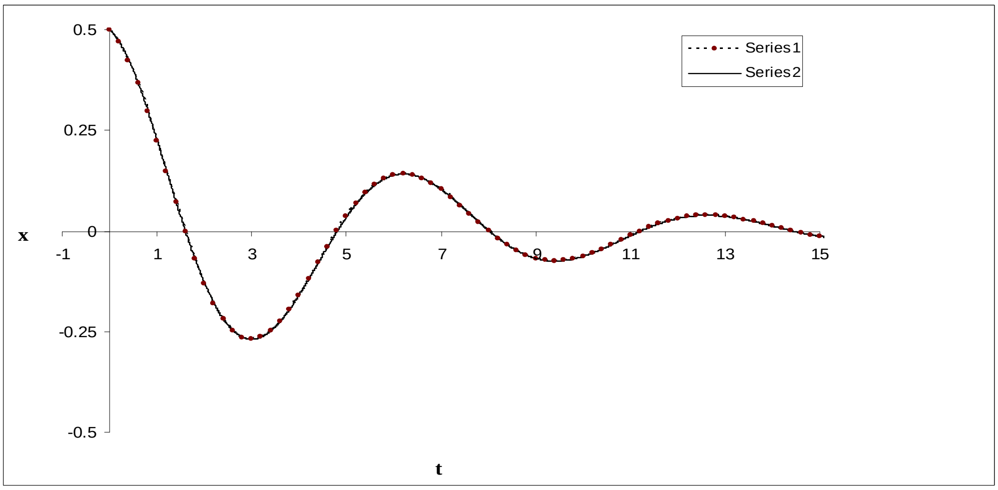

Figure 2(b): First approximate solution of Eq. (12) is denoted by dotted lines $(-\bullet-)$ by the presented analytical technique with the initial conditions $a_0 = 0.5, \varphi_0 = 0$ or $[x(0) = 0.5, \; \dot{x}(0) = -0.10036]$ with $\nu = 1.0, k = 0.2, \varepsilon_1 = 0.1, \varepsilon = 0.1$ and $f = x^9$ and the corresponding numerical solution is denoted by solid line $(-)$.

5. Conclusion

The presented method does not require a small parameter in the equation like the classical one. The method has been successfully implemented to illustrate the effectiveness and convenience of the suggested procedure and it is noticed that the first approximate solutions show a good agreement with those solutions obtained by the numerical procedure with high order nonlinearity in presence of small damping for both strongly $(\varepsilon_1 = 1.0)$ and weakly $(\varepsilon_1 = 0.1)$ nonlinearity.

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