Journal of Mathematics and Informatics
Vol. 6, 2016, 31-39
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 22 September 2016
www.researchmathsci.org

Journal of
Mathematics and
Informatics

On Best Approximation in $L^p(\mu, X)$ and $L^\phi(\mu, X), 1 \le p \le \infty$

A.A.Hakawati[1] **and S.A.Dwaik**[2]

Department of Mathematics, An-Najah National University
Nablus, Palestine
[1]E-mail: aahakawati@najah.edu; [2]E-mail: sawsana-dw@hotmail.com

Received 3 September 2016; accepted 13 September 2016

Abstract. In this paper, we prove that the property of being proximinally additive in Banach spaces is inherited by the space $L^\phi(\mu, G)$ in $L^\phi(\mu, X)$. Furthermore, and as an extension of our main result in [1], we prove that: With this property assumed, the subspace G is proximinal in the Banach space X if and only if, for $1 \le p \le \infty$, $L^p(\mu, G)$ is proximinal in $L^p(\mu, X)$ if and only if $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$ for every modulus function ϕ and any finite measure space (T, μ).

Keywords: Proximinally additive

AMS Mathematics Subject Classification (2010): 41A50

1. Introduction

For the subset G of the normed linear space (X, $\|.\|$). We define, for x $\in$ X, d (x , G) = inf $\{\|x - g\| : g \in G\}$. If G is a subspace of X, an element $g_\circ \in G$ is called a best approximant of x in G if $\|x - g_\circ\|$ = d(x,G). We shall denote the set of all best approximants of x in G as P(x,G). If for each x$\in$ X, the set $P(x, G) \neq \phi$ then G is said to be proximinal in X, and if P (x,G) is a singleton for each x $\in$X an G is called a Chebychev subspace.

An increasing function ϕ: [0 , ∞)→ [0 , ∞) is said to be a modulus function if it vanishes at zero, and is subadditive. This means that $\phi(x+y) \le \phi(x) + \phi(y)$ for all x and y in [0, ∞). Examples of modulus functions are: x^p, $0 < p \le 1$, and ln(1+x). Furthermore, if ϕ is a modulus function, then ϕ(x)= $\frac{\phi(x)}{1+\phi(x)}$ is again modulus.

It is also evident that the composition of two modulus functions is a modulus function, [3.p.159].

Let X be a real Banach space and let (T, μ) be a finite measure space. For a modulus function ϕ, we define the Orlicz space $L^\phi(\mu, X)$ as the set

$$\left\{ f : T \to X \ \ such\, that \int_T \phi(\|f(t)\|) d\mu(t) < \infty \right\}.$$

The function d : $L^{\phi}(\mu, X) \times L^{\phi}(\mu, X) \rightarrow [0,\infty)$ given by:

d(f,g) = $\int_T \phi(\|f(t) - g(t)\|) d\mu(t)$ turns $L^{\phi}(\mu, X)$ into a complete metric space [3].

For f $\in L^{\phi}(\mu, X)$, we write $\|f\|_{\phi} = \int_T \phi(\|f(t)\|)\, du(t)$. In what follows, when ϕ is mentioned, it is to be assumed a modulus function. We would also like to mention that in the literature, except for what we partly did in [1], we did not find conditions under which the proximinality of G in X is equivalent to the proximinality of $L^{\phi}(\mu, G)$ in $L^{\phi}(\mu, X)$ and to the proximinality of $L^{p}(\mu, G)$ in $L^{p}(\mu, X)$, $1 \le p \le \infty$. Here we show that the condition of proximinal additivity, again, gives the required equivalence. This, of course makes an extension to our restricted classification of the case *p=1* which we got in [1].

In the present time, researchers are working on the extensions of classical results in which they consider Haar subspaces for approximating sets, For reference One may consider [7]. Convenient tries can also be found in [5,8].

2. Proximinal additivity

Definition 2.1. A subspace G of a Banach space X is said to proximinally additive if G is closed and $z_1 + z_2 \in P(x_1 + x_2, G)$ whenever $z_1 \in P(x_1, G)$ and $z_2 \in P(x_2, G)$.

Example 2.2. Let $X = R^2$, and let G $= \{(x,0) : x \in R\}$. Then G is proximinally additive in X, with the Euclidean norm.

It turns out that proximinal additivity is transformed from G to the Orlicz space $L^{\phi}(\mu, G)$. Specifically, we have the following:

Theorem 2.3. Let X be a Banach space in which G is a proximinally additive subspace. Then $L^{\phi}(\mu, G)$ is proximinally additive in $L^{p}(\mu, G)$ $L^{\phi}(\mu, X)$.

Theorem 2.3. Let X be a Banach space in which G is a proximinally additive subspace. Then $L^{\phi}(\mu, G)$ is proximinally additive $L^{\phi}(\mu, X)$.

Poof: Let $g_1 \in P(f_1, L^{\phi}(\mu, G))$ and $g_2 \in P(f_2, L^{\phi}(\mu, G))$

By $[5, p.73]$, $g_1(t) \in P(f_1(t), G)\ a.e. t \in T$ (1)

Also, $g_2(t) \in P(f_2(t), G) a.e. t \in T$ (2)

Since G is proximinally additive, from (1) and (2) , we get that:

$$(g_1 + g_2)(t) \in P(f_1 + f_2)(t), G)\ a.e. t \in T.$$

Hence, $d((f_1 + f_2)(t), G) = \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|$

So $\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \le \|(f_1 + f_2)(t) - y\|$ a.e.t and for all $y \in G$.

In particular, one has :

$\|(f_1+f_2)(t)-(g_1+g_2)(t)\| \le (\|f_1+f_2)(t)-h(t)\|$ a.e.t and for all $h\in L^\phi(\mu,G)$

Since ϕ is strictly increasing,

$\phi(\|(f_1+f_2)(t)-(g_1+g_2)(t)\| \le \phi(\|(f_1+f_2)(t)-h(t)\|)$ *a.e.t and for all* $h\in L^\phi(\mu,G)$

Integrating the last inequality yields:

$\|(f_1+f_2)-(g_1+g_2)\|_\phi \le \|(f_1+f_2)-h\|_\phi$ for all $h\in L^\phi(\mu,G))$

Hence $d((f_1+f_2),L^\phi(\mu,G)) = \|(f_1+f_2)-(g_1+g_2)\|_\phi$

Therefore $g_1+g_2 \in P(f_1+f_2, L^\phi(\mu,G)$

Thus $L^\phi(\mu,G)$ is proximinally additive.

For the next result, we need the following theorem which was proved in [9]. Here, we give a simpler proof.

For this, we need to recall from [11,p.279] that:

a closed subspace G of a Banach space X is called an $L^p - summand, 1\le p<\infty$ if there is a bounded projection E : X $\rightarrow$ G which is onto and $\|x\|^p = \|(x)\|^P + \|x-E(x)\|^p$ for all $x\in X$. We present the following result.

Lemma 2.4. If G is an $L^p - summand$ of a Banach space X, then G is proximinal $(1\le p<\infty)$.

Proof: Let $x\in X$ .For each $g\in G$, we have :

$$\|x-g\|^p = \|E(x-g)\|^p + \|x-g-E(x-g)\|^p$$

$$= \|E(x)-E(g)\|^p + \|x-E(x)\|^p$$

$$\ge \|x-E(x)\|^p$$

Theorem 2.5. Let G be any closed subspace of a Hilbert space (x,<.>), then G is Chebyshev.

Proof: By [10,p.96] , X = $G\oplus G^\perp$ where $G^\perp = \{z\in X : z\perp G\}$.

Hence, for every $x\in X$, there is a unique representation x = g+z where $g\in G$ and $z\in G^\perp$.

Now, we define the projection E on X as E (x) =g.

Clearly E is onto and bounded.

Also, if x = g +z and $z\perp g$ then

$\|z+g\|^2 = \|z\|^2 + \|g\|^2$, *so*

Thus G is an L^2-summand of X .

Now, for $x\in X$, suppose g_1 , and g_2 are best approximates of x in G .

By the parallelogram law (applied to $\frac{1}{2}(x-g_1)$ and $\frac{1}{2}(x-g_2)$.

One gets:

$$\left\|\frac{1}{2}(x-g_1)+\frac{1}{2}(x-g_2)\right\|^2+\left\|\frac{-1}{2}(g_1-g_2)\right\|^2=2\left\|\frac{1}{2}(x-g_1)\right\|^2+2\left\|\frac{1}{2}(x-g_2)\right\|^2,$$

which gives that:

$$\left\|x-\frac{1}{2}(g_1+g_2)\right\|^2<\frac{1}{2}\|x-g_1\|^2+\frac{1}{2}\|x-g_2\|^2=[d(x,G)]^2, thus \left\|x-\frac{1}{2}(g_1+g_2)\right\|<d(x,G)$$

This contradicts the definition of d (x,G) , unless $g_1 = g_2$
Hence G is Chebyshev.

Corollary 2.6. Any closed subspace of $\Re^n$, or of C^n is Chebyshev.

3. Main results

Theorem 3.1. Let G be a closed subspace of a Hilbert space X. Then $L^\phi(\mu,G)$ is proximinal in $L^\phi(\mu,X)$.

Proof : By[[1], proposition 2.8], G is proximinally additive.
By theorem (2.5) G is Chebyshev, and in particular it is proximinal in X. By [[1], theorem (3.7)] $L^\phi(\mu,G)$ is proximinal in $L^\phi(\mu,X)$.

The following theorem is essential for our next main result. To prove we need the following lemma.

Lemma 3.2. Let G be a subspace of a normed space X.
For $x \in X$:

i) $if\ z\in P(x,G), then\ \alpha z\in P(\alpha x,G)$ for all scalars α
ii) $if\ z\in P(x,G), then\ z+g\in p(x+g,G)\ for\ all\ g\in G$

Proof: For (i); if $z\in P(x,G), and\ \alpha\neq 0$ is a scalar, then

$$\|\alpha x-g\|=|\alpha|\left\|x-\frac{1}{\alpha}g\right\|\geq|\alpha|\|x-g\|=\|\alpha x-\alpha z\|$$

so $\alpha z\in P(\alpha x,G)$

For (ii): If $g_\circ\in G$, we have

$$\|x+g-g_\circ\|\geq\|x-g\|=\|x+g-(z+g)\|$$

So $z+g\in p(x+g,G)$.

Theorem 3.3. Suppose G is a semi-Chebyshev hyperplane in a Banach space X which passes through the origin. Then G is proximinally additive.

Proof: Case (1): G is proximinal.

Let $f\in X^*$ be so that G = $\{x\in X : f(x)=0\}$

Fix $z\in x\setminus G,\ so\ f(z)\neq 0$.

Put $y_\circ=x-(f(x)/f(z))z$ where $x\in X$.

So, f ($y_\circ$) = 0, whence $y_\circ\in G$.

Consequently, $X = G \oplus W$ where $W = \{w = \alpha z : \alpha \text{ is a scalar}\}$ (1)
Now, let $z_1 \in P(x_1, G), z_2 \in P(x_2, G)$. It will be shown that $z_1 + z_2 \in P(x_1 + x_2, G)$. By (*) , every $x_1, x_2 \in X$ can be written uniquely as: $x_1 = g_1 + \alpha_1 z_1$ and $x_2 = g_2 + \alpha_2 z_2$
where g$_1$, g$_2$ $\in$ G and α_1, a_2 scalars (2)
Now, assume that $g_\circ \in P(x_1 + x_2, G)$. Then by (2) $g_\circ \in P((g_1 + g_2) + (\alpha_1 + \alpha_2)z, G)$.
By lemma (3.2) $g_\circ = ((g_1 + g_2) + (\alpha_1 + \alpha_2)$ where $w \in P(z, G)$.
So g$_\circ$ = $g_1 + \alpha_1 w + g_2 + \alpha_2 w$ which,
Again lemma (3.2) implies that :
$g_1 + \alpha_1 w \in P(g_1 + \alpha_1 z, G) = P(x_1, G)$ and $g_2 + \alpha_2 w \in P(g_2 + \alpha_2 z, G) = P(x_2, G)$.
Hence, $g_1 + \alpha_1 w = z_1$, and $g_2 + \alpha_2 w = z_2$, so g$_\circ$=z$_1$+z$_2$.
Therefore, $z_1 + z_2 \in P(x_1 + x_2, G)$.
Case (2): G is not proximinal.

By [6, p.93], $P(x, G) = \phi \ \forall x \in X \setminus G$.
Thus G is vacuously proximinally additive.

Now, we will introduce, with proofs, a sequence of propositions which will lead to our proposed extension result.

Theorem 3.4. Let G be a Chebyshev hyperplane in a Banach space X which passes through the origin then $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$.
Proof: By theorem (3.3), G is proximinally additive.
By [(3.7) of [1]], $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$.

Lemma 3.5. Let G be a closed subspace of a Banach space X. If $L^\infty(\mu, G)$ is proximinal in $L^\infty(\mu, X)$, then G is proximinal in X. ((T, μ) is a finite measure space).
Proof: Consider, for $x \in X$ the constant function f(t)= x , defined on T. Then $f \in L^\infty(\mu, X)$. Hence , by assumption , there is $g \in L^\infty(\mu, G)$ such that:
$\|f - g\|_\infty = d(f, L^\infty(\mu, G)$. By [10,p.36] , One has: $\|f - g\|_\infty = \sup_T d(f(t), G)$. Thus,
$\|f - g\|_\infty = \sup_T d(x, G) = d(x, G) = \sup_T \{\|x - g\|(t)\}$.

Therefore, $\|x - g(t)\| \le d(x, G)$ *for all* $t \in T$, and hence G is proximinal in X.

Theorem 3.6. Let G be a closed subspace which is proximinally additive in a Banach space X. Then $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$ if and only if $L^\infty(\mu, G)$ is proximinal in $L^\infty(\mu, X)$.

Proof: If $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$ then by [2,p.528], $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$.

Conversely, if $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$ then by lemma (3.5), G is proximinal in X. By theorem (3.7 of [1]), $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$.

Now, this is our extension theorem:

Theorem 3.7. Let G be a subspace which is proximinally additive in X, then the followings are equivalent for any finite measure space (T,μ):

(i) G is proximinal in X.

(ii) $L^\phi(\mu,G)$ is proximinal in $L^\phi(\mu,X)$.

(iii) $L^p(\mu,G)$ is proximinal in $L^p(\mu,X)$ for all $1\le p\le\infty$.

Proof: We only need to recall from [4, p.297] the fact that:

For $1<p<\infty$, $L^p(\mu,G)$ is proximinal in $L^p(\mu,X)$ if and only if $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$.

We close this paper by an example, which shows that being proximinally additive in X , the subspace G need not be proximinal.

Example 3.8. Let X = $c_\circ$, the space of null sequences, equipped with the sup. norm. Let

$$G=\left\{x\in c_\circ : \sum_{n=1}^{\infty}2^{-n}x_n=0\right\}$$

Clearly G is the hyperplane generated by $f(x)=\sum_{n=1}^{\infty}2^{-n}x_n$ and $\|f\|=1,[10,p.32]$, so in particular G is closed. Let x = $e^{(1)}$=(1,0,0,….),so $x\in c_\circ$, and d(x,G) = $\frac{1}{2}$ by [6,p.24].

Now, if there is $g\in G$ such that $\|x-g\|=\frac{1}{2}$ then:

$$|1-g_1|\le\frac{1}{2}\, and\, |g_n|\le\frac{1}{2}\; for\;\; all\, n\ge 2$$

Since $\sum_{n=1}^{\infty}2^{-n}g_n=0$, we get that :

$$\frac{1}{4}\le\frac{1}{2}|g_1|=\left|\sum_{n\ge2}2^{-n}g_n\right|\le\sum_{n\ge2}(2^{-n}|g_n|)\le\frac{1}{2}\sum_{n=2}^{\infty}2^{-n}=\frac{1}{4},$$

and this happens only if $|g_n|=\frac{1}{2}$ for all *n.*

But this contradicts the assumption that $g \in c_\circ$. Thus G could not have been proximinal in X.

4. Sets approximated by zero

In what follows, if G is a subspace of a normad space X, then we define the set $P_G^{-1}(0) = \{x \in X : 0 \in p(x,G)\}$. This set is referred to as the set approximated by zero.

Theorem 4.1. Let X be a Branch space, and G be a closed subspace of X. If G is proximinally additive, then, up to sets of measure zero ;

$$P^{-1}_{L^\phi(\mu, G)}(0) = L^\phi(\mu, P_G^{-1}(0)).$$

Proof : Let f ε $L^\phi(\mu, P_G^{-1}(0))$. This means f(t) ε $P_G^{-1}(0)$ and that $\|f\|_\phi < \infty$.

Now

f(t) ε $P_G^{-1}(0)$; so, 0 ε P(f(t),G); hence $d(f(t), G) = \|f(t)\|$.

i.e. $\|f(t)\| \leq \|f(t) - g\| \ \ \forall \ g \ \varepsilon \ G$.

In particular,

$\|f(t)\| \leq \|f(t) - h(t)\| \ \ \forall \ h \ \varepsilon \ L^\phi(\mu, G)$

Since ϕ is strictly increasing, then

$\phi(\|f(t)\|) \leq \phi(\|f(t) - h(t)\|) \forall \ h \ \varepsilon \ L^\phi(\mu, G)$

Integrating both sides we get

$$\|f\|_\phi \leq \|f - h\|_\phi \forall \ h \ \varepsilon \ L^\phi(\mu, G)$$

Hence $d(f, L^\phi(\mu, G)) = \|f\|_\phi$

Therefore, $0 \ \varepsilon P(f, L^\phi(\mu, G)) \Rightarrow f \ \varepsilon \ P^{-1}_{L^\phi(\mu, G)}(0)$

Thus

$$L^\phi(\mu, P_G^{-1}(0)) \subset P^{-1}_{L^\phi(\mu, G)}(0).$$

Conversely , let f $\in P^{-1}_{L^\phi(\mu,G)}(0)$.

So, the zero function $0 \in P(f, L^\phi(\mu, g))$.

Hence, by [5.p.73] $0 \in P(f(t), G) a.e. t \in T$.

So, $f(t) \in p_G^{-1}(0) \ \ a.e. t \in T$

Now, define g(t) = $\begin{cases} f(t) & if \ f(t) \in p_G^{-1}(0) \\ f(t) - g_t & otherwise \end{cases}$

where g_t is the unique best approximant of f(t) in G.

By [7, Lemma 3.1 and Theorem 3.2], and since
$(\forall t \in T) f(t) = g_t + (f(t) - g_t)$, we conclude that $g(t) \in P_G^{-1}(0) \forall t \in T$

Finally, by the very definition of g ; f = g a.e.t $\in$ T and $\|g\|_\phi < \infty$, *so* $g \in L^\phi(\mu, P_G^{-1}(0))$.

4. A note on optimization theory

Optimization is a mathematical technique that concerns the finding of maxima or minima of functions within some feasible region. A diversity of optimization techniques fight for the best solution. Particle Swarm Optimization (PSO) is a comparatively new, current, and dominant method of advanced optimization technique that has been empirically shown to perform well on many of these optimization problems. It is lucidly and widely used to find the global optimum solution in a complex search space. This, in a sense, is another face of best approximation theory, each in its field of application. The difference is in the fact that, optimal solutions occur as values of functions while proximinal maps have the basic problem of non-being linear. This in part shortens the scope of invoking such maps in the theory of best approximation. For further development, we would like to refer the reader to [13,14,15].

REFERENCES

1. A.A.Hakawati and S.A.Dwaik, On best approximation in $L^1(\mu, X)$ and $L^\phi(\mu, X)$, *Annals of Pure and Applied Mathematics,* 12(1) (2016) 1-8.
2. W.Deeb and R.Khalil, Best approximation in *L*(*X,Y*), *Math . Proc. Cambridge Philos. Soc.*, 104 (1988) 527-531.
3. W.Deeb, Multipliers and isometries of orlicz spaces, In "Proceedings of the conference on Mathematical Analysis and its Applications, (Kuwait, 1985), Volume 3 of KFAS proc. ser. Pages 159-165. Perganon Oxford 1988.
4. R.Khalil and W.Deeb, Best approximation in $L_p(\mu,X)$-II, *J. Approx. Theory,* 86 (3) (1989) 296-299.
5. W.Deeb and R.Khalil, Best approximation in $L_p(\mu,X)$, $0 < p < 1$, *J. Approx. Theory,* 58 (1989) 68-77.
6. I.Singer, *Best Approximation In Normed Linear Spaces By Elements of linear Subspaces,* Springer-Verlay, New York.
7. W.Garidi and G.Dongyue, Some problems on best approximation in orlicz spaces, *J. Applied Mathematics,* 3 (2012) 322-324.
   dx.doi.org/10.4236/am.2012.34048.
8. H.Al-Minawi and S.Ayesh, Best approximation in orlicz spaces, *Internat. J Math. and Math. Sci.*, 14(2) (1991) 245-252.
9. E.Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons, New York, 1978.
10. A.H.Siddiqi, *Functional Analysis with Applications*, McGraw-Hill, New York, 1986.
11. D.K.Biswas and S.C.Panja, Advanced optimization technique, *Annals of Pure and Applied Mathematics*, 5(1) (2013) 82-89.
12. K.Pramila and G.Uthra, Optimal solution of an intuitionistic fuzzy transportation problem, *Annals of Pure and Applied Mathematics,* 8(2) (2014) 67-73.

13. C.Taylor and P.Hood, A numerical solution of the Navier-Stokes equations using finite element technique, *Computer and Fluids*, 1(1) (1973) 73-89.
14. P.Dechaumphai, *Finite Element Method in Engineering*, Second edition, Chulalong - korn University Press, Bangkok, 1999.
15. M.Pal, *Numerical Analysis for Scientists and Engineers: Theory and C Programs*, Alpha Science International , Oxford, U K, 2007.