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Fixed Point Theorems for Various Types of Compatible Mappings of Integral Type in Modular Metric Spaces

***Aklesh Pariya*[1], *Prerna Pathak*[2], *V.H. Badshah*[3] and *Nirmala Gupta*[4]**

[1]Department of Mathematics, Medi-caps University, Indore (M.P.), India
Email: akleshpariya3@yahoo.co.in

[2,3]School of Studies in Mathematics Vikram University, Ujjain (M.P.), India
Email: prernapathak20@yahoo.com

[4]Department of Mathematics, Govt. Girls Degree College, Ujjain (M.P.), India
Email:gupta.nirmala70@gmail.com
[2]Corresponding author. Email: prernapathak20@yahoo.com

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Abstract. In this paper, we study and prove the existence of fixed point theorems for fourcompatible, weakly compatible and occasionally weakly compatible mapping of integral type in modular metric spaces and generalized the result of Azadifar et al. [2] Rahimpoor et al. [16] and Rashwan and Hammad [17].

Keywords: Modular, modular metric space, occasionally weakly compatible, fixed point, contraction, integral type inequality

AMS Mathematics Subject Classification (2010): 47H09, 47H10, 46A80

1. Introduction

The study of fixed and common fixed points of mappings satisfying a certain metrical contractive conditions attracted many researchers. Let (X,d) be a metric space. A mapping $T: X \to X$is a contraction if $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \epsilon X$, where $0 \leq k < 1$. The Banach's contraction mapping principle appeared in explicit form in Banach's thesis in 1922 [3]. Since its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. Banach contraction principle has been extended in many different directions; see [6-14]. The notion of modular spaces, as a generalization of metric spaces, was introduced by Nakano [14] and was intensively developed by Koshi and Shimogaki [8] Yamamuro [20] and others. The main idea behind this new concept is the physical interpretation of the modular. Informally speaking whereas a metric on a set represent finite nonnegative distances between two points of the set, a modular on a set attributes a non negative (possibly, infinite valued) 'field of (generalized) velocities': to each 'time' $\lambda > 0$ (the absolute value of), an average velocity $\omega_\lambda(x, y)$ is associated in such way that in order to cover the 'distance' between points $x, y \epsilon X$, it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. A lot of mathematicians are interested fixed points of modular spaces.

Further the most complete development of these theories are due to Luxemburg [9], Musielk and Orlicz [10], Mazur [13], Turpin [19] and there collaborators.
In 2008, Chistyakov [4] introduced the notion of modular metric spaces generated by F-modular and developed the theory of this space. In 2010 Chistyakov [5] defined the notion of modular on an arbitrary set and develop the theory of metric spaces generated by modular such that called the modular metric spaces. Chistyakov [4, 6] introduced and studied the concept ofmodular metric spaces and proved fixed point theorems for contractive map in Modular spaces. It is related to contracting rather “generalized average velocities” than metric distances, and the successive approximations of fixed points converge to the fixed points in a weaker sense as compared to metric convergence. Recently, Mongkolkeha et al. [11,12] has introduced some notions and established some fixed point results in modular metric spaces. Rahimpoor et al. [16] established some fixed point results in modular metric spaces for weakly compatible mappings, Azadifar et al. [2,11,12] proved some fixed point results in modular metric spaces for compatible mappings of integral type in Modular Spaces. Razani and Moradi [18] proved common fixed point results of integral type in modular spaces. Rashwan and Hammad [17], proved common fixed point results for weak contraction of integral type in Modular Spaces.

2. Experimental details, methods, materials, basic definitions and preliminaries

We will start with a brief recollection of basic concepts and facts in modular spaces and modular metric spaces (see [4, 5, 6]).

Definition 2.1. Let X be a vector space over R (or C). A functional $\rho : X \to [0, \infty]$ is called a modular if for arbitrary x and y, elements of X satisfying the following three conditions:
(A.1) $\rho(x) = 0$ $if and\ only\ if\ x = 0$.
(A.2) $\rho(\alpha x) = \rho(x) for\ all\ scalar\ \alpha\ with\ |\ \alpha\ |= 1;$
(A.3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y), whenever\ \alpha, \beta \geq 0, \alpha + \beta = 1.$
If we replace (A.3) by
(A.4) $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y), for\ \alpha, \beta \geq 0, \alpha^s + \beta^s = 1\ with\ an\ s\epsilon(0,1],$
then the modular ρ is called s-convex modular, and if s = 1, ρ is called a convex modular.
If ρ is modular in X, then the set defined by

$$X_\rho = \{x\epsilon X :\ \rho(\lambda x) \to 0\ as\ \lambda \to 0^+\} \quad (2.1)$$

is called a modular space. $X_\rho$is a vector subspace of X it can be equipped with an F - norm defined by setting $\|x\|_\rho$= inf $\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq \lambda\}$, $x\epsilon X_\rho$. (2.2)
In addition, if ρ is convex, then the modular space X_ρ coincides with

$$X_\rho^* = \{x\epsilon X :\ \exists\lambda = \lambda(x) > 0\ such\ that\ \rho(\lambda x) < \infty\} \quad (2.3)$$

and the functional $\|x\|_\rho^*$= inf $\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}$ is an ordinary norm on X_ρ^* which is equivalence to $\|x\|_\rho$(see [13]).

Let X be a non empty set, $\lambda \in (0, \infty)$ and due to the disparity of the arguments, function $\omega : (0, \infty)$ x X x X$\to [0, \infty]$ will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y\ \epsilon X$.

Definition 2.2. Let X be a non empty set. A function $\omega : (0, \infty)$ x X x X$\to [0, \infty]$ is said to be a metric modular on X if it satisfies the following three axioms:

(i) given $x, y \epsilon X, \omega_\lambda(x,y) = 0\ for\ all\ \lambda > 0\ if\ and\ only\ if\ x = y$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)\ for\ all\ \lambda > 0$ and $x, y \epsilon X$;

(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y) for\ all\ \lambda, \mu > 0$ and $x, y, z \epsilon X$.

If instead of (i), we have only the condition

(i') $\omega_\lambda(x,x) = 0\ for\ all\ \lambda > 0$ and $x \epsilon X$, then ω is said to be a (metric) pseudo modular on X and if ω satisfies (i') and

(i_s) given x, yϵX, if there exists a number $\lambda > 0$, possibly depending on x and y, such that $\omega_\lambda(x,y) = 0$, then x = y, with this condition ω is called a strict modular on X.

A modular (pseudo modular, strict modular) w on X is said to be convex if, instead of (iii) we replace the following condition :

(iv) for all $\lambda > 0$, μ > 0 and x, y, z ∈X it satisfies the inequality

$$\omega_{\lambda+\mu}(x,y) = \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y) \text{ for all } \lambda, \mu > 0 \text{ and } x, y, z \in X.$$

Clearly, if ω is a strict modular, then ω is a modular, which in turn implies ω is a pseudo modular on X, and similar implications hold for convex ω.

The essential property of a (pseudo) modularω on a set X is a following given $x, y \epsilon X$, the function $0 < \lambda \to \omega_\lambda(x,y) \epsilon [0,\infty]$ is non increasing on $(0,\infty)$. In fact, if $0 < \mu < \lambda$, then (iii), (i') and (ii) imply

$$\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y) \quad (2.4)$$

It follows that at each point $\lambda > 0$ the right limit $\omega_{\lambda+0}(x,y) \coloneqq \lim_{\varepsilon\to+0}\omega_{\lambda+\varepsilon}(x,y)$ and the left limit $\omega_{\lambda-0}(x,y) \coloneqq \lim_{\varepsilon\to+0}\omega_{\lambda-\varepsilon}(x,y)$ exist in [0,∞] and the following two inequalities hold:

$$\omega_{\lambda+0}(x,y) \le \omega_\lambda(x,y) \le \omega_{\lambda-0}(x,y) \quad (2.5)$$

From [2.4, 2.5], we know that, if $x_0 \in X$, the set

$$X_\omega = \{x \in X : \lim_{\lambda\to\infty}\omega_\lambda(x,x_0) = 0\}$$

is a metric space, called a modular space, whose metric is given by

$d_\omega^0(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le \lambda\} for\ all\ x, y \in X_\omega$.

Moreover, if ω is convex, the modular set X_ω is equal to

$X_\omega^* = \{x \in X : \exists\lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}$

And metrizable by $d_\omega^*(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\} for\ all\ x, y \in X_\omega^*$.

We know that if X is a real linear space, $\rho : X \to [0,\infty]$ and

$$\omega_\lambda(x,y) = \rho\left(\frac{x-y}{\lambda}\right) for\ all\ \lambda > 0\ and\ x, y \in X, \quad (2.6)$$

Then ρ is modular (convex modular) on X in the sense of (A.1) - (A.4) if and only if ω is metric modular (convex metric modular, respectively) on X. On the other hand, ifω satisfy the following two conditions:

(i) $\omega_\lambda(\mu x, 0) = \omega_{\lambda/\mu}(x,0)\ for\ all\ \lambda, \mu > 0\ and\ x \in X$,

(ii) $\omega_\lambda(x+z, y+z) = \omega_\lambda(x,y)\ for\ all\ \lambda > 0\ and\ x, y, z \in X$, if we set $\rho(x) = \omega_1(x,0)$ with (2.6) holds, where $x \in X$, then

(a) $X_\rho = X_\omega$ is a linear subspace of X and the functional $\|x\|_\rho = d_\omega^0(x,0), x \epsilon X_\rho$, is an F-norm on X_ρ;

(b) If ω is convex, $X_\rho^* \equiv X_\omega^*(0) = X_\rho$ is a linear subspace of X and the functional $\|x\|_\rho = d_\omega^*(x,0), x \epsilon X_\rho^*$, is an norm on X_ρ^*.

Similar assertions hold if replace the word modular by pseudo modular. If ω is metric modular in X, we called the set X_ω is modular metric space.

By the idea of property in metric spaces and modular spaces, we defined the following:

Definition 2.3. [14] Let X_ω be a modular metric space.

(1) The sequence $(x_n)_{n\epsilon N}$ in X_ω is said to be convergent to $x \epsilon X_\omega$ if

$$\omega_\lambda(x_n, x) \to 0 \; as \; n \to \infty \; for \; all \; \lambda > 0.$$

(2) The sequence $(x_n)_{n\epsilon N}$ in X_ω is said to be Cauchy if

$$\omega_\lambda(x_m, x_n) \to 0 \; as m, n \to \infty \; for \; all \; \lambda > 0.$$

(3) A subset C of X_ω is said to be closed if the limit of the convergent sequence of C always belong to C.

(4) A subset C of X_ω is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit in C.

(5) A subset C of X_ω is said to be bounded if for all $\lambda > o \delta_\omega(C) = \sup\{\omega_\lambda(x, y); x, y \epsilon C\} < \infty$.

We recall the following definitions in metric spaces.

Definition 2.4. Let X be a set, *f*, *g* self maps of X. A point x in X is called a coincidence point of f and g iff *fx= gx*. We shall call *w = fx= gx,* a point of coincidence of *f* and *g*.

Definition 2.5. Two maps S and T are said to be weakly compatible if they commute at coincidence points.

Al-Thagafi and Shahzad [1] gave a proper generalization of nontrivial weakly compatible maps which have a coincidence point.

Definition 2.6. [1] Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

We shall also need the following lemma from Jungck and Rhoades [7].

Lemma 2.1. Let X be a set, f, g owc self maps of X. If f and g have a unique point of coincidence, w : =fx= gx, then w is a unique common fixed point of f and g.

Thus we define the above definitions in modular metric spaces as follows.

Definition 2.7. Let X_ω be a modular metric space. Let f, g self maps of X_ω. A point x in X_ω is called a coincidence point of f and g iff fx= gx. We shall call w = fx= gx a point of coincidence of f and g.

Definition 2.8. Let X_ω be a modular metric space. Two maps fand g of $X_\omega$are said to be weakly compatible if they commute at coincidence points.

Definition 2.9. Let X_ω be a modular metric space. Two self maps f and g of X_ω are occasionally weakly compatible (owc) iff there is a point x in X_ω which is a coincidence point of f and g at which f and g commute.

Definition 2.10. [2] Let X_ω be a modular metric space induced by metric modular ω. Two self mapping f, g of X_ω are ω-compatible if $\omega_\lambda(fgx_n, gfx_n) \to 0$, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X_ω such that $gx_n \to z$ and $Tx_n \to z$ for some point $z \epsilon X_\omega$ and for $\lambda > 0$.

Lemma 2.2. Let X_ω be a modular metric space and f, g owc self maps of X_ω. If f and g have a unique point of coincidence, w :=fx= gx, then w is a unique common fixed point of f and g.

3. Observations, results and discussion

Definition 3.1. [15] A function Ψ:$R^+ \to R^+$ is called a comparison function if it satisfies the following conditions:

(i) Ψ is monotone increasing, $\Psi(t) < t$ for some $t > 0$,

(ii) $\Psi(0) = 0$,

(iii) $\lim n\to\infty\ \Psi^n(t) = 0,\ \forall t \geq 0$.

Theorem 3.1. Let X_ω be a complete modular metric space and I, *J*, *S*, $T{:}X_\omega \to X_\omega$ be self mapping of a complete modular metric space X_ω into itself satisfying the conditions

(3.1.1) $S(X_\omega) \subset J(X_\omega)$ and $T(X_\omega) \subset I(X_\omega)$

(3.1.2) for all $x, y \epsilon X_\omega$ then there exist a non decreasing right continuous function $\psi \in \Psi$

$\psi : R^+ \to R^+$, $\psi(0) = 0$ and $\psi^n(t) < t$ for every $t > 0$ such that

$$\int_0^{\omega_\lambda(Sx,Ty)} \varphi(t)dt \leq \psi \int_0^{M(x,y)} \varphi(t)dt$$

where $M(x,y) = \max\{\omega_\lambda(Ix, Jy), \omega_\lambda(Sx, Ix), \omega_\lambda(Ty, Jy), \frac{1}{2}[\omega_\lambda(Sx, Jy) + \omega_{2\lambda}(Ty, Ix)]$

where $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non negative and for

all $\varepsilon > 0$; $\int_0^\varepsilon \varphi(t)dt > 0$ and $\lambda > 0$. If the pair (S, I) is compatible and (T, J) is weakly compatible on X, one of S or I is continuous then I, *J*, *S*, *T* have a unique common fixed point in X_ω.

Proof. Let x_0 be an arbitrary point in X_ω. Since $S(X_\omega) \subseteq J(X_\omega)$, choose a point x_1 in X_ω such that $Sx_0 = Jx_1$. Also since $T(X_\omega) \subseteq I(X_\omega)$, let x_2 be a point in X_ω such that $Tx_1 = Ix_2$. Using this argument repeatedly, we construct a sequence $\{y_n\}$ in X_ω such that $Sx_n = Jx_{n+1} = y_n$ and $Tx_{n+1} = Ix_{n+2} = y_{n+1}$ for all $n \geq 0$.

Now we take $x = x_n$, $y = x_{n+1}$ in (3.1.2), we get

$$\int_0^{\omega_\lambda(Sx,Ty)} \varphi(t)dt \leq \psi \int_0^{\max\{\omega_\lambda(Ix,Jy),\omega_\lambda(Sx,Ix),\omega_\lambda(Ty,Jy),\frac{1}{2}[\omega_\lambda(Sx,Jy)+\omega_{2\lambda}(Ty,Ix)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(Sx_n,Tx_{n+1})} \varphi(t)dt$$

$$\leq \psi \int_0^{max\{\omega_\lambda(Ix_n,Jx_{n+1}),\omega_\lambda(Sx_n,Ix_n),\omega_\lambda(Tx_{n+1},Jx_{n+1}),\frac{1}{2}[\omega_\lambda(Sx_n,Jx_{n+1})+\omega_{2\lambda}(Tx_{n+1},Ix_n)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(y_n,y_{n+1})} \varphi(t)dt$$

$$\leq \psi \int_0^{\max\{\omega_\lambda(y_{n-1},y_n),\omega_\lambda(y_n,y_{n-1}),\omega_\lambda(y_{n+1},y_n),\frac{1}{2}[\omega_\lambda(y_n,y_n)+\omega_{2\lambda}(y_{n+1},y_{n-1})]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(y_n,y_{n+1})} \varphi(t)dt$$

$$\leq \psi \int_0^{\max\{\omega_\lambda(y_{n-1},y_n),\omega_\lambda(y_n,y_{n-1}),\omega_\lambda(y_{n+1},y_n),\frac{1}{2}[\omega_\lambda(y_{n+1},y_n)+\omega_\lambda(y_n,y_{n-1})]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(y_n,y_{n+1})} \varphi(t)dt \quad \leq \psi \int_0^{\omega_\lambda(y_n,y_{n-1})} \varphi(t)dt \quad < \int_0^{\omega_\lambda(y_n,y_{n-1})} \varphi(t)dt$$

So we obtain by continuing in a similar manner

$$\int_0^{\omega_\lambda(y_n,y_{n+1})} \varphi(t)dt \quad \leq \psi^2 \int_0^{\omega_\lambda(y_{n-1},y_{n-2})} \varphi(t)dt \quad \ldots \leq \psi^n \int_0^{\omega_\lambda(y_0,y_1)} \varphi(t)dt$$

Thus taking limit n→ ∞ , and using the definition of $\psi \in \Psi$,we get

$\psi^n \int_0^{\omega_\lambda(y_0,y_1)} \varphi(t)dt \to 0$. This shows that $\{y_n\}$ is a Cauchy sequence inX_ω.

By the completeness of X_ω, there exist some t in X_ω such that sequence $\{y_n\}$ and its sub sequence$\{y_{n+1}\}$ and $\{y_{n+2}\}$ also converges to t in X_ω .

Now assuming the continuity of I,

We get, $I^2x_n \to It,\ \ ISx_n \to It.$

Also in view of compatibility of $(I,S),\ SIx_n \to It$.

Now using (3.1.2), we have consider

$$\int_0^{\omega_\lambda(Sx_n,Tx_{n+1})} \varphi(t)dt$$

$$\leq \psi \int_0^{max\{\omega_\lambda(I^2x_n,Jx_{n+1}),\omega_\lambda(SIx_n,I^2x_n),\omega_\lambda(Tx_{n+1},Jx_{n+1}),\frac{1}{2}[\omega_\lambda(SIx_n,Jx_{n+1})+\omega_{2\lambda}(Jx_{n+1},I^2x_n)]\}} \varphi(t)dt$$

Letting $n \to \infty$, we get

$$\int_0^{\omega_\lambda(It,t)} \varphi(t)dt \leq \psi \int_0^{max\{\omega_\lambda(It,t),\omega_\lambda(It,It),\omega_\lambda(t,t),\frac{1}{2}[\omega_\lambda(It,t)+\omega_{2\lambda}(t,It)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(It,t)} \varphi(t)dt \leq \psi \int_0^{max\{\omega_\lambda(It,t),0,0,\frac{1}{2}[\omega_\lambda(It,t)+\omega_{2\lambda}(t,It)]\}} \varphi(t)dt \int_0^{\omega_\lambda(It,t)} \varphi(t)dt \leq$$
$$\psi \int_0^{\omega_\lambda(It,t)} \varphi(t)dt < \int_0^{\omega_\lambda(It,t)} \varphi(t)dt,$$

So that $It = t$.

Again consider,

$$\int_0^{\omega_\lambda(St,Tx_{n+1})} \varphi(t)dt$$

$$\leq \psi \int_0^{max\{\omega_\lambda(It,Jx_{n+1}),\omega_\lambda(St,It),\omega_\lambda(Tx_{n+1},Jx_{n+1}),\frac{1}{2}[\omega_\lambda(St,Jx_{n+1})+\omega_{2\lambda}(Jx_{n+1},It)]\}} \varphi(t)dt$$

letting $n \to \infty$, we get

$$\int_0^{\omega_\lambda(St,t)} \varphi(t)dt \leq \psi \int_0^{max\{\omega_\lambda(t,t),\omega_\lambda(St,t),\omega_\lambda(t,t),\frac{1}{2}[\omega_\lambda(St,t)+\omega_{2\lambda}(t,t)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(St,t)} \varphi(t)dt \le \psi \int_0^{max\{0,\omega_\lambda(St,t),0,\frac{1}{2}[\omega_\lambda(St,t)+0]\}} \varphi(t)dt$$

$\int_0^{\omega_\lambda(St,t)} \varphi(t)dt \le \psi \int_0^{\omega_\lambda(St,t)} \varphi(t)dt < \int_0^{\omega_\lambda(St,t)} \varphi(t)dt$. Hence $St = t$.
Now $S(X_\omega) \subset J(X_\omega)$ and so their exist another point u in X_ω:$t = St = Ju$.
Now we will show that Tu= t. For this consider

$$\int_0^{\omega_\lambda(t,Tu)} \varphi(t)dt = \int_0^{\omega_\lambda(St,Ju)} \varphi(t)dt$$

$$\le \psi \int_0^{max\{\omega_\lambda(It,Ju),\omega_\lambda(St,It),\omega_\lambda(Tu,Ju),\frac{1}{2}[\omega_\lambda(St,Ju)+\omega_{2\lambda}(Ju,It)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(t,Tu)} \varphi(t)dt \le \psi \int_0^{max\{\omega_\lambda(t,t),\omega_\lambda(t,t),\omega_\lambda(Tu,t),\frac{1}{2}[\omega_\lambda(t,t)+\omega_{2\lambda}(t,t)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(t,Tu)} \varphi(t)dt \le \psi \int_0^{max\{0,0,\omega_\lambda(Tu,t),\frac{1}{2}[0+0]\}} \varphi(t)dt$$

$\int_0^{\omega_\lambda(t,Tu)} \varphi(t)dt \le \psi \int_0^{\omega_\lambda(t,Tu)} \varphi(t)dt < \int_0^{\omega_\lambda(t,Tu)} \varphi(t)dt$. So that $Tu = t$.
Since (T, J) are weakly compatible on X_ω and $Tu = Ju = t$ so that $TJu = JTu$

$$Tt = TJu = JTu = Jt$$

$$\int_0^{\omega_\lambda(t,Jt)} \varphi(t)dt \le \int_0^{\omega_\lambda(St,Tt)} \varphi(t)dt$$

$$\le \psi \int_0^{max\{\omega_\lambda(It,Jt),\omega_\lambda(St,Tt),\omega_\lambda(Tt,Jt),\frac{1}{2}[\omega_\lambda(St,Jt)+\omega_{2\lambda}(Tt,It)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(t,Jt)} \varphi(t)dt \le \psi \int_0^{max\{\omega_\lambda(t,t),\omega_\lambda(t,t),\omega_\lambda(t,Jt),\frac{1}{2}[\omega_\lambda(t,Jt)+\omega_{2\lambda}(t,t)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(t,Jt)} \varphi(t)dt \le \psi \int_0^{max\{0,0,\omega_\lambda(t,Jt),\frac{1}{2}[\omega_\lambda(t,Jt)+0]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(t,Jt)} \varphi(t)dt \le \psi \int_0^{\omega_\lambda(t,Jt)} \varphi(t)dt < \int_0^{\omega_\lambda(t,Jt)} \varphi(t)dt$$

So that $t = Jt$. Thus Jt = St =It= Tt = t, so that t is a common fixed point of I,J,S and T.
Uniqueness. To prove uniqueness, let z≠t be another common fixed point of I, J, S and T. Then by (3.1.2),

$$\int_0^{\omega_\lambda(Sz,Tt)} \varphi(t)dt \le \psi \int_0^{max\{\omega_\lambda(Iz,Jt),\omega_\lambda(Sz,Tz),\omega_\lambda(Tt,Jt),\frac{1}{2}[\omega_\lambda(Sz,Jt)+\omega_{2\lambda}(Tt,Iz)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(z,t)} \varphi(t)dt \le \psi \int_0^{max\{\omega_\lambda(z,t),\omega_\lambda(z,z),\omega_\lambda(t,t),\frac{1}{2}[\omega_\lambda(z,t)+\omega_{2\lambda}(t,z)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(z,t)} \varphi(t)dt \le \psi \int_0^{max\{\omega_\lambda(z,t),0,0,\frac{1}{2}[\omega_\lambda(z,t)+\omega_{2\lambda}(t,z)]\}} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(z,t)} \varphi(t)dt \leq \psi \int_0^{\omega_\lambda(z,t)} \varphi(t)dt < \int_0^{\omega_\lambda(z,t)} \varphi(t)dt$$

This is a contradiction. Hence t is a unique common fixed point of I, J,S and T.

Theorem.3.2. Let X_ω be a modular metric space and I, J, S, T :$X_\omega \to X_\omega$ be self mappings such that $S(X_\omega) \subseteq J(X_\omega)$, and $T(X_\omega) \subseteq I(X_\omega)$ and one of $I(X_\omega)$ or $J(X_\omega)$ be a ω-complete subspace of X_ω. Suppose there exists number $a, b, c, d \epsilon [0,1)$ with at least one of $a, b, c, d > 0$ such that the following assertion for all $x, y \epsilon X_\omega$ and $\lambda>0$ hold:

(3.2.1) $(a + b + c + 2d) < 1$ for all $0 \leq a, b, c, d < 1$

(3.2.2) $\int_0^{\omega_\lambda(Sx,Ty)} \varphi(t)dt \leq a \int_0^{\omega_\lambda(Ix,Jy)} \varphi(t)dt + b \int_0^{\omega_\lambda(Sx,Ix)} \varphi(t)dt + c \int_0^{\omega_\lambda(Ty,Jy)} \varphi(t)dt + d \int_0^{[\omega_\lambda(Sx,Jy)+\omega_{2\lambda}(Ty,Ix)]} \varphi(t)dt$

(3.1.3) $\omega_\lambda(Sx, Ty) < \infty$

Then S, T, I and J have a coincidence point. If the pairs (S, I) and (T, J) are occasionally weakly compatible then S, T, I and J have a common fixed point in X_ω.

Proof: Since the pair (S, I) and (T, J) are occasionally weakly compatible then there exist $u, v \epsilon X_\omega$

Such that $Su = Iu$ and $Jv = Tv$

Now we can assert that $Su = Tv$, if not then by (3.2.2)

$$\begin{aligned}\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt &\leq a \int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt + b \int_0^{\omega_\lambda(Su,Iu)} \varphi(t)dt \\ &+ c \int_0^{\omega_\lambda(Tv,Jv)} \varphi(t)dt \\ &+ d \left[\int_0^{\omega_\lambda(Su,Jv)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(Tv,Iu)} \varphi(t)dt\right]\end{aligned}$$

$$\begin{aligned}\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt &\leq a \int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt + b \int_0^{\omega_\lambda(Iu,Iu)} \varphi(t)dt \\ &+ c \int_0^{\omega_\lambda(Jv,Jv)} \varphi(t)dt \\ &+ d \left[\int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(Jv,Iu)} \varphi(t)dt\right]\end{aligned}$$

$$\begin{aligned}\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt &\leq a \int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt \\ &+ d \left[\int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(Jv,Iu)} \varphi(t)dt\right]\end{aligned}$$

By definition of metric modular and the inequality (2.4), we get

$$\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt \leq (a+d)\int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt + d\int_0^{\omega_{2\lambda}(Jv,Iu)} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt$$
$$\leq (a+d)\int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt$$
$$+ d\left[\int_0^{\omega_\lambda(Iu,Iu)} \varphi(t)dt + \int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt\right]$$

$$\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt \leq (a+d)\int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt + d\int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt$$

$$\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt \leq (a+2d)\int_0^{\omega_\lambda(Iu,Jv)} \varphi(t)dt$$

Or $(1-a-2d)\int_0^{\omega_\lambda(Su,Tv)} \varphi(t)dt < 0$, which is a contradiction.

Hence $Su = Tv$ and thus $Su = Iu = Tv = Jv$(3.2.1.1)

Moreover, if there is another fixed point of coincidence z such that $Sz = Iz$, and using condition (3.2.2)

$$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt$$
$$\leq a\int_0^{\omega_\lambda(Iz,Jv)} \varphi(t)dt + b\int_0^{\omega_\lambda(Sz,Iz)} \varphi(t)dt + c\int_0^{\omega_\lambda(Tv,Jv)} \varphi(t)dt$$
$$+ d\left[\int_0^{\omega_\lambda(Sz,Jv)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(Tv,Iz)} \varphi(t)dt\right]$$

$$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt$$
$$\leq a\int_0^{\omega_\lambda(Sz,Jv)} \varphi(t)dt + b\int_0^{\omega_\lambda(Sz,Sz)} \varphi(t)dt + c\int_0^{\omega_\lambda(Tv,Tv)} \varphi(t)dt$$
$$+ d\left[\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(Sz,Tv)} \varphi(t)dt\right]$$

$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt \leq (a+2d)\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt$, which is contradiction.

Hence, we get $Sz = Iz = Tv = Jv$ (3.2.1.2)

Thus from equation (3.2.1.1) and (3.2.1.2) it follows that $Sz = Su$.This implies$z = u$.

Hence $z = Su = Iu$ for some $z\epsilon X_\omega$ is the coincidence point of S and I.

Then by Lemma 2.2, z is a unique common fixed point of S and I.

Hence, $Sz = Iz = z$

Similarly, there is a another common fixed point ϵX_ω : $v = Tv = Jv$

Suppose $v \neq z$, then by (3.2.2) we have

$$\int_0^{\omega_\lambda(Sz,Tv)} \varphi(t)dt \leq a\int_0^{\omega_\lambda(Iz\,,Jv)} \varphi(t)dt + b\int_0^{\omega_\lambda(Sz,Iz)} \varphi(t)dt + c\int_0^{\omega_\lambda(Tv,Jv)} \varphi(t)dt + d[\int_0^{\omega_\lambda(Sz,Jv)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(Tv,Iz)} \varphi(t)dt]$$

$\int_0^{\omega_\lambda(z,v)} \varphi(t)dt \leq a\int_0^{\omega_\lambda(z\,,v)} \varphi(t)dt + d[\int_0^{\omega_\lambda(z,v)} \varphi(t)dt + \int_0^{\omega_{2\lambda}(v,z)} \varphi(t)dt]$,

we get $\int_0^{\omega_\lambda(z,v)} \varphi(t)dt \leq (a+2d)\int_0^{\omega_\lambda(z\,,v)} \varphi(t)dt$, which is contradiction. Hence $z = v$.

Hence, z is a unique common fixed point of S, T, I and J.

Theorem 3.3. Let X_ω be a modular metric space and S, T:$X_\omega \to X_\omega$ be self mappings such that $T(X_\omega) \subseteq S(X_\omega)$ and one of $S(X_\omega)$ or $T(X_\omega)$ be a ω-complete subspace of X_ω.

Suppose there exists number $a, b, c\,, d\epsilon[0,1)$ with at least one of $a, b, c\,, d> 0$ such that the following assertion for all$x, y\epsilon X_\omega$ and λ>0 hold:

(3.3.1)$(a + b + c + 2d) < 1$ for all $0 \leq a, b, c, d < 1$

(3.3.2)$\int_0^{\omega_\lambda(Tx,Ty)} \varphi(t)dt \leq$
$a\int_0^{\omega_\lambda(Sx\,,Sy)} \varphi(t)dt + b\int_0^{\omega_\lambda(Sx,Tx)} \varphi(t)dt + c\int_0^{\omega_\lambda(Sy,Ty)} \varphi(t)dt +$
$d\int_0^{[\omega_\lambda(Sx,Ty)+\omega_{2\lambda}(Sy,Tx)]} \varphi(t)dt$

(3.3.3) $\omega_\lambda(Sx, Ty) < \infty$

Then S,T have a coincidence point. Moreover if the pairs (S,T) is occasionally weakly compatible then S, T have a common fixed point in X_ω.

Proof: If we put I=J = Ix_ω where Ix_ω is an identity mapping on X_ω, the result follows from theorem 3.2.

Remark 3.1. The theorem 3.2, theorem 3.3 remains true if we put $\varphi(t) = 1$, we get the following corollaries:

Corollary 3.3.1. Let X_ω be a modular metric space and I, J, S, T :$X_\omega \to X_\omega$ be self mappings such that $S(X_\omega) \subseteq J(X_\omega)$, and $T(X_\omega) \subseteq I(X_\omega)$ and one of $I(X_\omega)$ or $J(X_\omega)$ be a ω-complete subspace of X_ω. Suppose there exist number $a, b, c\,, d\epsilon[0,1)$ with at least one of $a, b, c\,, d> 0$ such that the following assertion for all $x, y\epsilon X_\omega$ and λ>0 hold:

(3.3.1.1)$(a + b + c + 2d) < 1$ for all $0 \leq a, b, c, d < 1$

(3.3.1.2)$\omega_\lambda(Sx, Ty) \leq a\omega_\lambda(Ix\,, Jy) + b\omega_\lambda(Sx, Ix) + c\omega_\lambda(Ty, Jy) + d[\omega_\lambda(Sx, Jy) +$
$\omega_{2\lambda}(Ty, Ix)]$

(3.3.1.3) $\omega_\lambda(Sx, Ty) < \infty$

Then S, T, I and J have a coincidence point. If the pairs (S, I) and (T, J) are occasionally weakly compatible then S, T, I and J have a common fixed point in X_ω.

Corollary 3.3.2. Let $X_\omega$be a modular metric space and S, T:$X_\omega \to X_\omega$ be self mapping such that $T(X_\omega) \subseteq S(X_\omega)$ and S(X_ω) be a ω-complete subspace of X_ω. Suppose there exist number $a, b, c, d\epsilon[0,1)$ with at least one of a, b, c, d > 0 such that the following assertion for all $x, y\epsilon X_\omega$ and λ>0 hold

(3.3.2.1)$(a + b + c + 2d) < 1$ for all $0 \leq a, b, c, d < 1$

(3.3.2.2) $\omega_\lambda(Tx, Ty) \leq a\omega_\lambda(Sx, Sy) + b\omega_\lambda(Sx, Tx) + c\omega_\lambda(Sy, Ty) + d[\omega_\lambda(Sx, Ty) + \omega_{2\lambda}(Sy, Tx)]$

(3.3.2.3) $\omega_\lambda(Sx, Ty) < \infty$

Then S and T have a coincidence point. Moreover if the pair (S, T) is occasionally weakly compatible then S and T have a unique common fixed point in X_ω.

Remark 3.2. The theorem 3.2 remains true if the inequality (3.2.1), (3.2.2) are replaced by the following inequality-

(i) $\int_0^{\omega_\lambda(Sx,Ty)} \varphi(t)dt \leq a \int_0^{\omega_\lambda(Ix,Jy)} \varphi(t)dt + b \int_0^{\omega_\lambda(Sx,Ix)} \varphi(t)dt + c \int_0^{\omega_\lambda(Ty,Jy)} \varphi(t)dt + d \int_0^{\omega_\lambda(Sx,Jy)} \varphi(t)dt + e \int_0^{\omega_{2\lambda}(Ty,Ix)} \varphi(t)dt$

with$(a + b + c + d + e) < 1$ for all $0 \leq a, b, c, d, e < 1$

(ii)$\int_0^{\omega_\lambda(Sx,Ty)} \varphi(t)dt \leq a \int_0^{\omega_\lambda(Ix,Jy)} \varphi(t)dt + b \int_0^{\omega_\lambda(Sx,Ix)+\omega_\lambda(Ty,Jy)} \varphi(t)dt + c \int_0^{\omega_\lambda(Sx,Jy)+\omega_{2\lambda}(Ty,Ix)} \varphi(t)dt$ with $(a + 2b + 2c) < 1$ for all $0 \leq a, b, c < 1$

(iii) If we put b = c = d = e = 0, in inequality (3.2.1), (3.2.2) we have

$\int_0^{\omega_\lambda(Sx,Ty)} \varphi(t)dt \leq a \int_0^{\omega_\lambda(Ix,Jy)} \varphi(t)dt$, with for all $0 \leq a < 1$

(iv) If we put a=d=0, we get in inequality (3.2.1), (3.2.2)

$$\int_0^{\omega_\lambda(Sx,Ty)} \varphi(t)dt \leq b \int_0^{\omega_\lambda(Sx,Ix)} \varphi(t)dt + c \int_0^{\omega_\lambda(Ty,Jy)} \varphi(t)dt$$

with $(b + c) < 1$ for all $0 \leq b, c < 1$.

(v) If we put I=J == Ix_ω in (3.2.2), where Ix_ω is an identity mapping on X_ω,

$\int_0^{\omega_\lambda(Tx,Ty)} \varphi(t)dt \leq a \int_0^{\omega_\lambda(Sx,Sy)} \varphi(t)dt + b \int_0^{\omega_\lambda(Sx,Tx)} \varphi(t)dt + c \int_0^{\omega_\lambda(Sy,Ty)} \varphi(t)dt + d \int_0^{[\omega_\lambda(Sx,Ty)+\omega_{2\lambda}(Sy,Tx)]} \varphi(t)dt$ with $(a + b + c + 2d) < 1$ for all $0 \leq a, b, c, d < 1$.

Hence in a similar manner, if we put the values of a, b, c, d are zero respectively, we get different integral type inequality for two mappings. If we put for $\varphi(t) = 1$, we get the different type inequality for four and two mappings.

4. Conclusion

Some common fixed point theorems satisfying integral type contractive condition for compatible, weakly compatible and occasionally weakly compatible mappings in modular metric space are proved. Our main result in theorem 3.1, is a generalization of the results of Azadifar et al. [2], for a pair of compatible and weakly compatible mappings, theorem 3.3, is a generalization of the results of Rahimpoor et al. [16], for a pair of occasionally weakly compatible mappings of integral type, corollary 3.3.2 is a similar result due to Rahimpoor et al. [16], with slightly different contractive condition as mentioned in remark 3.2 condition (ii). Our results and corollaries are the real extension and generalizations of the corresponding results of Chistyakov [6], Mongkolkeha [11,12], and recent results in modular metric spaces.

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