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Coloring of Hypergraphs

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Abstract. The concept of coloring and related problems are the main problems in combinatorics. In the literature of combinatorics hypergraph coloring is a well studied problem. There are many applications of coloring in telecommunication, computer science and engineering. In this chapter, we present some examples to illustrate various types of coloring. Here we also discuss that hypergraphs generalize standard graphs by defining edges between multiple vertices instead of only two vertices. Hence some properties of hypergraph must be a generalization of graph properties.

Keywords: k-coloring, chromatic number, line graph,

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

A hypergraph is a generalization of graph where edges can connect more than two vertices and are called hyperedges. Hypergraphs arise naturally in important practical problem including circuit layout, numerical linear algebra etc., just as graphs naturally represent many kind of information in mathematical and computer science problems.

For all terminology and notation in hypergraph theory, not specifically defined here, we refer the reader to Berge [2].

A hypergraph H is a pair $H = (X, E)$ where X is a set of vertices and E a family of subsets of X, called hyperedges. A hypergraph whose each hyperedge contains only two vertices is a graph. H is linear if $|e_i - e_j| \leq 1$, for all distinct e_i, $e_j \in E$ and H is simple, if all edges are distinct. The degree of a vertex $\in X(H)$, denoted by $d_H(v)$, is the number of edges containing v in H. We denote $\Delta(H) = max_{v\in X} d_H(v)$ and $\delta(H) = min_{v\in X}\, d_H(v)$.

A hypergraph H is r-uniform when every edge of H contains exactly r vertices. A 2-uniform hypergraph is a graph. Two hypergraphs $H_1$and H_2 are isomorphic, if there exists a bijection $\varphi: X(H_1) \to X(H_2)$ such that

$$\{x_{1,}x_2, \dots\dots, x_p\} \in E(H_1) \Leftrightarrow \{\varphi(x_1), \varphi(x_2), \dots., \varphi(x_p)\} \epsilon E(H_2).$$

In this case, we write $H_1 \cong H_2$.

A k-edge coloring of H is an assignment of k-colors to the edges of H so that distinct intersecting edges receive different color. The chromatic index $\chi'(H)$ is the least number k of colors required for a k-edge coloring of H. A vertex coloring of H is an assignment of k-colors to the vertices H such that no edge is monochromatic. The chromatic number $\chi(H)$ is the least number k of colors required for a k-vertex coloring. Similarly, a strong k-vertex coloring of H is an assignment of k-colors the vertices of H in such a way that n edge contains two vertices of the same color. The strong chromatic number $\chi_s(H)$ is the least number k of colors required for a strong k-vertex coloring.

For all terminology and notation in hypergraph theory and graph theory, not specifically defined here, we refer the reader to Berge [2,3] and Harary [5]. Also, for hypergraph coloring we refer to Berge [4].

2. Graphs and hypergraphs

Given any hypergraph $H = (V, E)$, we can define various graphs derived from the hypergraph H. They are described below:

Line graph or representative graph of a hypergraph

Let $H = (V, E)$ be a hypergraph without repeated edge such that $E \neq \emptyset$. The line graph or representative graph of H is the graph $L(H) = (V', E')$ such that:

(i) $V' = E$

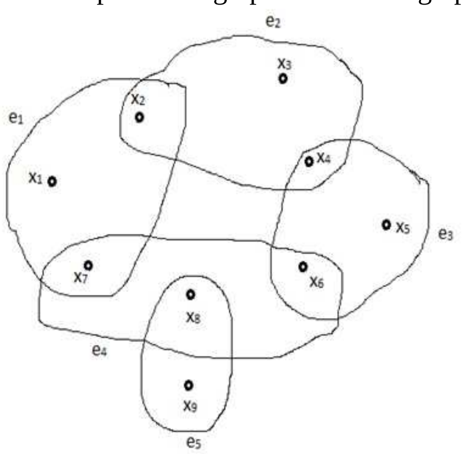
(ii) $\{i, j\} \in E'\ if\ and\ only\ if\ e_i \cap e_j \neq \emptyset, (i \neq j)$

Some properties of hypergraphs can be seen on the line-graph, for instance it is easy to show that:

I) The hypergraph H is connected if and only if L(H).

II) Any non trivial graph is the line graph of some linear hypergraph.

Here we consider a hypergraph and find out the line graph of the hypergraph. Also show that the particular graph is a cordial graph.



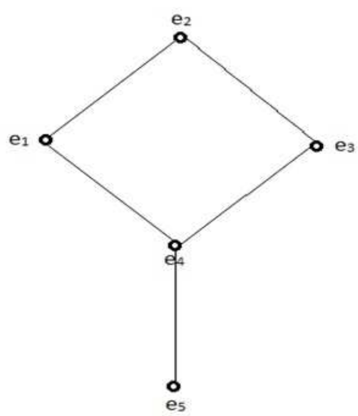


Figure 1(a): **Figure 1(b):**

Figure 1:

Figure 1(a) and (b) show a hypergraph $H = (V, E)$, where $V = \{x_1, x_2, \ldots\ldots, x_9\}$, $E = \{e_1, e_2, \ldots\ldots, e_5\}$ and its representative.

2-section of a hypergraph

Let $H = (V, E)$ be a hypergraph, the 2-section of H is the graph, denoted by $[H]_2$ whose vertices are the vertices of H and where two distinct vertices form an edge if and only if they are in the same hyperedge of H.

Illustration:

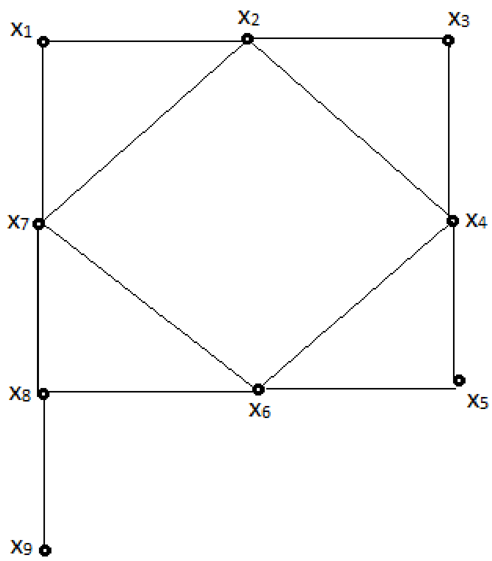


Figure 2:

Figure shows the 2-section of the hypergraph H described in Fig. 1(a).

Incidence Graph of a hypergraph

The incidence graph of a hypergraph $H = (V, E)$ is a bipartite graph $IG(H)$ with a vertex set $S = V \cup E$, and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in e$.

Illustration

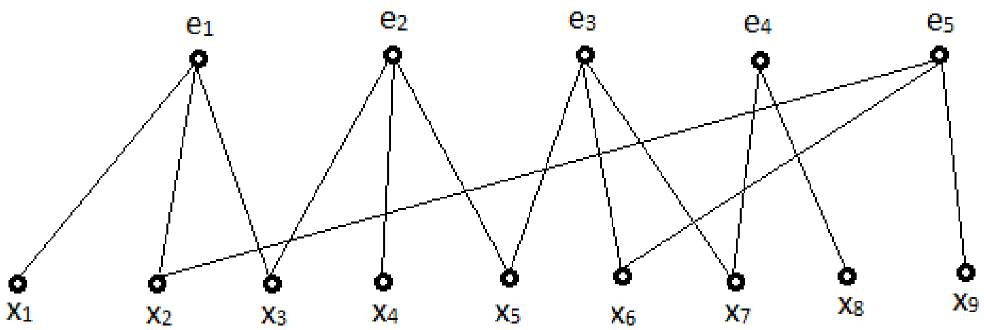


Figure 3:

Figure shows the incidence graph associated with the hypergraph H described in Fig1(a). We have seen several methods to associate a graph to a hypergraph, the converse can be done also. Suppose $G = (V, E)$ be a graph, we can associate a hypergraph H_G called neighbourhood to this graph such that

$$H_G = (V, (e_x = \{x\} \cup G(x): x \in V))$$

Illustration

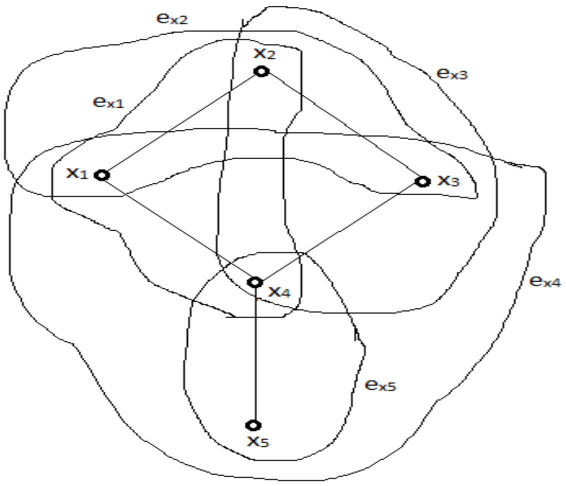


Figure 4:

The two known extensions of cycloatic number [1] of a graph to hypergraphs are cyclicity and cyclomatic number of a hypergraph. Many crucial properties of the two notions are the same, but they do not coincide for many hypergraphs and they behave differently in some aspects. A comparative study of the two notions has been done by France Dacar [6].

We can recall the known results from [1,7].

3. Hypergraph coloring

Let $H = (V, E)$ be a hypergraph and $k \geq 2$ be an integer. A k-coloring of the vertices of H is an assignment of colors t the vertices such that:

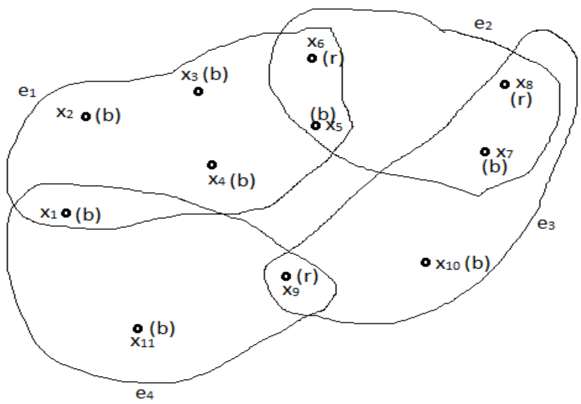


Figure 5:

(i) A vertex has just one color
(ii) We use k-colors to the vertices
(iii) No hyperedge with a cardinality more than 1 is monochromatic.

From this definition it is easyto see that any coloring induces a partition of the vertices in k-classes $(C_1, C_2, \dots\dots, C_k)$.

Figure 5 show a colored hypergraph H where (r) is red and (b) is blue.
There are following types of hypergraph coloring:

(I) Strong coloring
Let $H = (V, E)$ be a hypergraph, a strong k-coloring is a partition $(C_1, C_2, \dots\dots, C_k)$ of V such that the same color does not appear twice in the same hyperedge. In other words: $|e \cap C_i| \leq 1$ for any hyperedge and any element of the partition.

Illustration

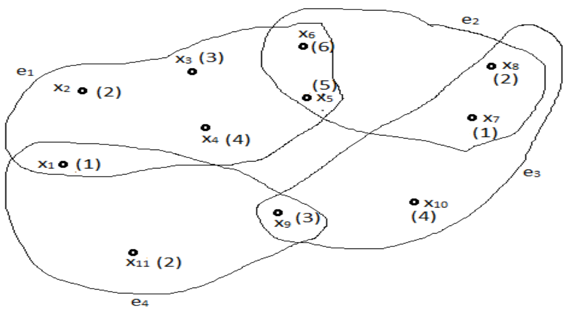


Figure 6:

(II) Equitable coloring
Let $H = (V, E)$ be a hypergraph, an equitable k-coloring is a k-partition $(C_1, C_2, \dots\dots, C_k)$ of V such that, in every hyperedge e, all the colors $\{1,2, \dots\dots, k\}$ appear the same number of times, to within one, if k does not divide $|e_i|$.

It is easy to see that a strong k-coloring is an equitable k-coloring.

Illustration

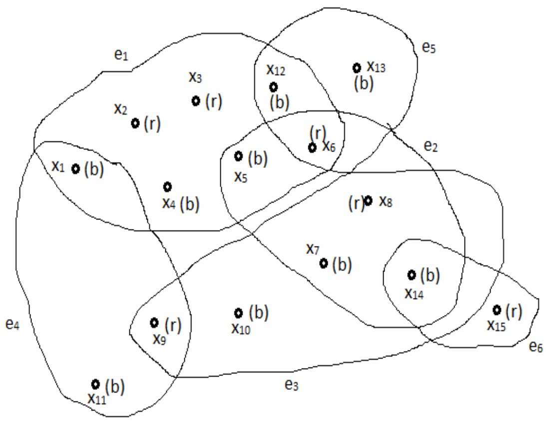


Figure 7:

(III) Good coloring

Let $H = (V, E)$ be a hypergraph, a good k-coloring is a partition $(C_1, C_2, \dots\dots, C_k)$ of V such that every hyperedge e contains the largest possible number of different colors, i.e. for every e$\in E$, the number of colors in e is min $\{|e|, k\}$.

Illustration

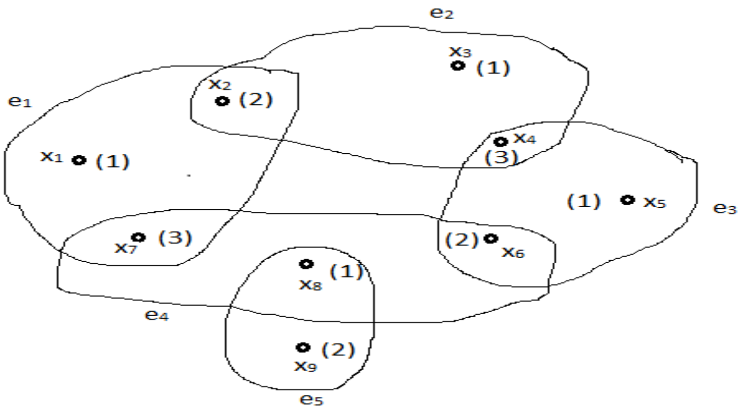


Figure 8:

(IV) Uniform coloring

Let $H = (V, E)$ be a hypergraph with $|V| = n$. A uniform k-coloring is a k-partition $(C_1, C_2, \dots\dots, C_k)$ of V such that the number of vertices the same color is always same, to within one, if k does not divide n.

On the basis of coloring one special type of hypergraph has been defined named Interval hypergraph.

Illustration

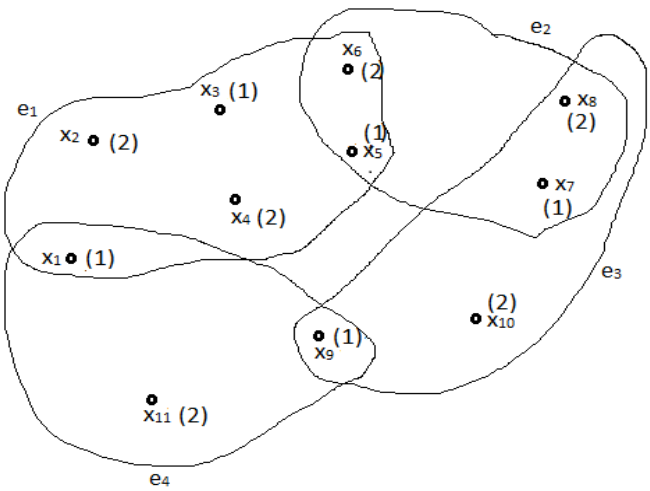


Figure 9:

Interval hypergraph

Let V be a non-empty set equipped with a total ordering $\leq$ on V, that is , for any two distinct elements $x, y \in V$, either $x \leq y$ or $y \leq x$. The couple $(V, \leq)$ is called a totally or linearly ordered set. Given any two distinct elements $x, y \in V$, we define the closed interval $[x, y]$ or $I(x, y)$ to the set $\{z \in V: x \leq z \leq y\}$.

A hypergraph $H = (V, E)$ with finite vertex set V and hyperedge family E is said to be ordered if there is a total order on V such that, for every hyperedge $e \in E$, there exist two distinct vertices $x, y \in V$ such that $e = I(x, y)$.

A hypergraph (without loop) is an interval hypergraph if its vertices can be labeled by $1,2, \dots \dots, n$ so that each hyperedge is made of vertices with consecutive label numbers, i.e. a hypergraph $H = (V, E)$ is an interval hypergraph if its vertices can be totally ordered so that every hyperedge $e \in E$ induced an interval in this ordering.

Illustration

Example of Interval Hypergraph: e_1={x_1≔1,x_2≔2,x_3≔3} , $e_2 = \{x_1 \coloneqq 1, x_2 \coloneqq 2\}$, $e_3 = \{x_2 \coloneqq 2, x_3 \coloneqq 3, x_4 \coloneqq 4,\}$, , $e_4 = \{x_4 \coloneqq 4, x_5 \coloneqq 5\}$.

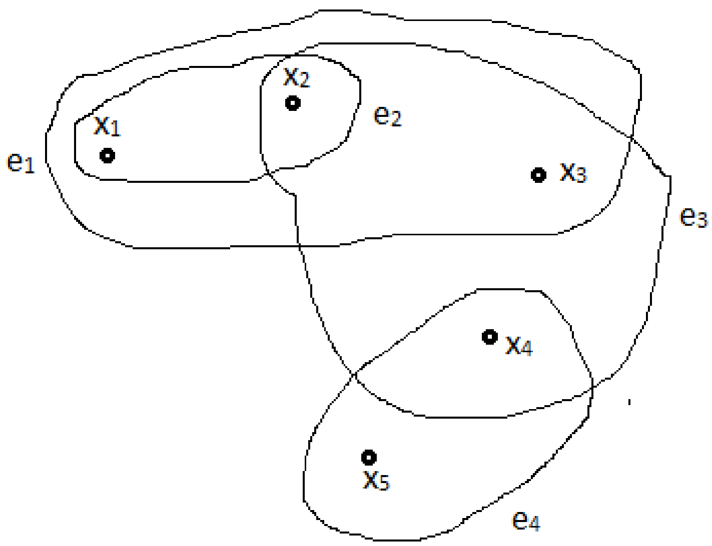


Figure 10:

Hyperedge coloring

Let $H = (V, E)$ be a hypergraph, a hyperedge k-coloring of H is a coloring of the hyperedges such that:

(i) A hyperedge has just one color

(ii) We use k-colors to color the hyperedges

(iii) Two distinct intersecting hyperedges receive two different colors.

4. Conclusion

Hypergraphs are the most general structures in discrete mathematics and like in most fruitful mathematical theories, the theory of hypergraphs has many applications. The concept of hypergraph provides an elegant framework for obtaining several general results on set systems. Hypergraph colorings model many practical problems in many different sciences.

Suppose a network for mobile phones. We can model this network by a hypergraph. If we model a frequency by a color, a good coloring gives us the minimal number of frequencies, k, we need so that communications do not interfere.

We find this theory in psychology, genetics and also in various human activities. Hypergraphs have shown their power as a tool to understand problems in a wide variety of scientific field. Moreover it well known that hypergraph theory is a very useful tool to resolve optimization problems such as scheduling problems, location problems and so on.

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