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Cone C-class Function with Common Fixed Point Theorems for Cone *b*-metric Space

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Abstract. In this paper, we discuss the common fixed point theorem obtain sufficient conditions for the existence of common fixed points of a pair of mapping satisfying generalized contraction involving rational expressions in cone b metric spaces via cone C class functions

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1. Introduction and mathematical preliminaries

In 1089, Bakhtin [2] introduced b-metric space as a generalization of metric space. He proved the contraction principle in b-metric spaces that generalized the famous contraction principle in metric space. In 2007, Huang and Zhang [4] are introduce the concept of cone metric space and also they discussed some properties of the convergence of sequences and proved the fixed point theorems of a contraction mapping for cone metric spaces; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \le k < 1$, the inequality $d(Tx,Ty) \le kd(x,y), \forall x, y \in X$ has a unique fixed point. In 2011, Hussain and Saha [6] introduced the concept of cone b-metric space as a generalization of b-metric spaces and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space. Note on $\varphi-\psi$-contractive type mappings and related fixed point are proved by Ansari [11].

In this paper, we investigate the common fixed point theorem obtain sufficient conditions for the existence of common fixed points of a pair of mapping satisfying generalized contraction involving rational expressions in cone b metric spaces via cone C class functions.

Definition 1.1. *Let* E *be the real Banach space. A subset* P *of* E *is called a cone if and only if:*

- P is closed, non empty and $P \neq \emptyset$
- $ax + by \in P$ for all $x, y \in P$ and non negative real numbers a, b
- $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

Example 1.2. *[10]Let* $K > 1$. *be given. Consider the real vector space with*

$$E = \{ax + b : a, b \in R; x \in 1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.3. *Let* X *be a nonempty set. A mapping* $d : X \times X \to E$ *is said to be cone metric if and only if, for all* $x, y, z \in X$ *, the following conditions are satisfied:*

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) + d(y, z)$.

Then (X, d) is called a cone metric space (CMS).

Example 1.4. *Let* $E = R^2$

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha |x - y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.5. *[6] Let* X *be a nonempty set and* $s \geq 1$ *be a given real number. A mapping* $d : X \times X \to E$ *is said to be cone* b*-metric if and only if, for all* $x, y, z \in X$ *, the following conditions are satisfied:*

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then (X, d) is called a cone metric space (CbCMS).

Example 1.6. *[5] Let* $E = R^2$

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x-y|^p, \alpha |x-y|^p)$$

where $\alpha \geq 0$ and $p > 1$ are two real constants. Then (X, d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space. In fact, we only need to prove (iii) in Definition 1.5 as follows:

Let $x, y, z \in X$. Set $u = x - z, v = z - y,$ so $x - y = u + v$. From the inequality

$(a+b)^p \leq (2\max\{a,b\})^p \leq 2^p(a^p + b^p)$ forall $a, b \geq 0.$

we have

$$|x-y|^p = |u+v|^p \leq (|u| + |v|)^p \leq 2^p(|u|^p + |v|^p) = 2^p(|x-z|^p + |z-y|^p)$$

which implies that $d(x, y) \leq s[d(x, z) + d(z, y)]$ with $s = 2^p > 1$. But

$$|x-y|^p \leq |x-z|^p + |z-y|^p$$

is impossible for all $x > z > y$. Indeed, taking account of the inequality

$$(a+b)^p > a^p + b^p \text{ forall } a, b \geq 0,$$

we arrive at

$$|x-y|^p = |u+v|^p = (u+v)^p > u^p + v^p = |x-z|^p + |z-y|^p$$

for all x > z > y. Thus, (c_3) in Definition 1.3 is not satisfied, i.e., (X, d) is not a cone metric space.

Example 1.7. *[5] Let* $X = l^p$ *with* $0 < p < 1$, *where* $l^p = \{\{x_n\} \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$.

Let $d : X \times X \to R_+$ *define by* $d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$, *where*

$x = \{x_n\}, y = \{y_n\} \in l^p$. *Then* (X, d) *is a* b*-metric space [3]. Put*

$E = l^1, P = \{\{x_n\} \in E : x_n \geq 0, \text{forall } n \geq 1\}$. *Letting the mapping* $d^* : X \times X \to E$ *be defined by* $d^*(x, y) = \{\frac{d(x, y)}{2^n}\} n \geq 1$, *we conclude that* (X, d^*) *is a cone* b*-metric space with the coefficient* $s = 2^{\frac{1}{p}} > 1$, *but it is not a cone metric space.*

Definition 1.8. *[6] Let* (X, d) *be a cone b-metric space,* $x \in X$ *and* $\{x_n\}$ *be a sequence in* X. *Then*

1. $\{x_n\}$ converges to x whenever, for every $c \in E$ with $0 = c$, there is a natural number N such that $d(x_n, x) = c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x (n \to \infty)$.

2. $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $0 = c$, there is a natural number N such that $d(x_n, x_m) = c$ for all $n, m \geq N$

3. (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Definition 1.9. *A function $\psi : P \to P$ is called an altering distance function if the following properties are satisfied:*

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.10. *An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : P \to P$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.*

We denote this set with Φ_u

Definition 1.11. *[12] A mapping $F : P^2 \to P$ is called cone C-class function if it is continuous and satisfies following axioms:*

1. $F(s,t) \leq s$;
2. $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in P$.

We denote cone C-class functions as $\mathcal{C}$.

Example 1.12. *[12] The following functions $F : P^2 \to P$ are elements of $\mathcal{C}$, for all $s, t \in 0, \infty)$:*

1. $F(s,t) = s - t$,
2. $F(s,t) = ks$, where $0 < k < 1$,
3. $F(s,t) = s\beta(s)$, where $\beta : [0, \infty) \to [0,1)$,
4. $F(s,t) = \Psi(s)$, where $\Psi : P \to P$, $\Psi(0) = 0$, $\Psi(s) > 0$ for all $s \in P$ with $s \neq 0$ and $\Psi(S) \leq s$ for all $s \in P$.,
5. $F(s,t) = s - \varphi(s)$, where $\varphi : [0, \infty) \to 0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
6. $F(s,t) = s - h(s,t)$, where $h : [0, \infty) \times 0, \infty) \to 0, \infty)$ is a continuous function such that $h(s,t) = 0 \Leftrightarrow t = 0$ for all $t, s > 0$.
7. $F(s,t) = \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0, \infty) \to [0, \infty)$ is a upper semi continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

Lemma 1.13. *Let ψ and φ are altering distance and ultra altering distance functions respectively , $F \in \mathcal{C}$ and $\{s_n\}$ a decreasing sequence in P such that*

$$\psi(s_{n+1}) \leq F(\psi(s_n), \varphi(s_n))$$

for all $n \geq 1$. Then $\lim_{n \to \infty} s_n = 0$.

2. Main result

Theorem 2.1. *Let* (X,d) *be a complete cone b-metric space* $(CCbMS)$ *with the co-efficient* $s \geq 1$ *and* P *be a normal cone with normal constant* K *. Suppose that the mappings* $S,T: X \to X$ *satisfy:*

$$\begin{aligned}\psi(d(Sx,Ty)) \quad &\leq F(\psi(a_1 d(x,y) + a_2 \frac{[1+d(x,Sx)]d(y,Ty)}{1+d(x,y)} + a_3[d(x,Sx)+d(y,Ty)] \\ &+ a_4[d(x,Ty)+d(y,Sx)]), \varphi(a_1 d(x,y) + a_2 \frac{[1+d(x,Sx)]d(y,Ty)}{1+d(x,y)} \\ &+ a_3[d(x,Sx)+d(y,Ty)] + a_4[d(x,Ty)+d(y,Sx)]))\end{aligned} \tag{2.1}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are non negative reals with $sa_1 + a_2 + (s+1)a_3 + s(s+1) \quad a_4 < 1$. ψ and φ are altering distance and ultra altering distance functions respectively , $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s)$. Then S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point X and define

$x_{2k+1} = Sx_{2k}, x_{2k+2} = Tx_{2k+1}, k = 0,1,2,\cdots$

Then form 2.1 we have

$$\begin{aligned}&\psi((x_{2k+1}, x_{2k+2})) \\ &= \psi(d(Sx_{2k,} Tx_{2k+1})) \\ &\leq F(\psi(a_1 d(x_{2k}, x_{2k+1}) + a_2 \frac{[1+d(x_{2k}, Sx_{2k})]d(x_{2k}, Tx_{2k+1})}{1+d(x_{2k}, x_{2k+1})} \\ &+ a_3[d(x_{2k}, Sx_{2k}) + d(x_{2k+1,} Tx_{2k+1})] + a_4[d(x_{2k,} Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})]), \\ &\varphi(a_1 d(x_{2k}, x_{2k+1}) + a_2 \frac{[1+d(x_{2k}, Sx_{2k})]d(x_{2k}, Tx_{2k+1})}{1+d(x_{2k}, x_{2k+1})} \\ &+ a_3[d(x_{2k}, Sx_{2k}) + d(x_{2k+1,} Tx_{2k+1})] + a_4[d(x_{2k,} Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})])) \\ &\qquad = F(\psi(a_1 d(x_{2k}, x_{2k+1}) + a_2 \frac{[1+d(x_{2k}, x_{2k+1})]d(x_{2k}, x_{2k+2})}{1+d(x_{2k}, x_{2k+1})} \\ &\qquad + a_3[d(x_{2k}, x_{2k+1}) + d(x_{2k+1,} x_{2k+2})] + a_4[d(x_{2k,} x_{2k+2}) + d(x_{2k+1}, x_{2k+1})]), \\ &\qquad \varphi(a_1 d(x_{2k}, x_{2k+1}) + a_2 \frac{[1+d(x_{2k}, x_{2k+1})]d(x_{2k}, x_{2k+2})}{1+d(x_{2k}, x_{2k+1})} \\ &\qquad + a_3[d(x_{2k}, x_{2k+1}) + d(x_{2k+1,} x_{2k+2})] + a_4[d(x_{2k,} x_{2k+2}) + d(x_{2k+1}, x_{2k+1})]))\end{aligned}$$

$$= F(\psi((a_1+a_3)d(x_{2k},x_{2k+1})+(a_2+a_3)d(x_{2k+1},x_{2k+2})+a_4 d(x_{2k},x_{2k+2})),$$
$$\varphi((a_1+a_3)d(x_{2k},x_{2k+1})+(a_2+a_3)d(x_{2k+1},x_{2k+2})+a_4 d(x_{2k},x_{2k+2})))$$
$$\leq F(\psi((a_1+a_3)d(x_{2k},x_{2k+1})+(a_2+a_3)d(x_{2k+1},x_{2k+2})$$
$$+sa_4[d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2})]),\varphi((a_1+a_3)d(x_{2k},x_{2k+1})$$
$$+(a_2+a_3)d(x_{2k+1},x_{2k+2})+sa_4[d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2})]))$$
$$= F(\psi((a_1+a_3+sa_4)d(x_{2k},x_{2k+1})+(a_2+a_3+sa_4)d(x_{2k+1},x_{2k+2})),$$
$$\varphi((a_1+a_3+sa_4)d(x_{2k},x_{2k+1})+(a_2+a_3+sa_4)d(x_{2k+1},x_{2k+2})))$$

This implies that $d(x_{2k+1},x_{2k+2}) \leq \dfrac{a_1+a_3+sa_4}{1-a_1-a_3-sa_4} d(x_{2k},x_{2k+1})$. Similarly, we have

$$\psi(d(x_{2k+2},x_{2k+3}))$$
$$= \psi(d(Sx_{2k+1,}Tx_{2k+2}))$$
$$\leq F(\psi(a_1 d(x_{2k+1},x_{2k+2})+a_2 \frac{[1+d(x_{2k+1},Sx_{2k+1})]d(x_{2k+2},Tx_{2k+2})}{1+d(x_{2k+1},x_{2k+2})}$$
$$+a_3[d(x_{2k+1},Sx_{2k+1})+d(x_{2k+2,}Tx_{2k+2})]+a_4[d(x_{2k+1,}Tx_{2k+2})+d(x_{2k+2},Sx_{2k+1})]),$$
$$\varphi(a_1 d(x_{2k+1},x_{2k+2})+a_2 \frac{[1+d(x_{2k+1},Sx_{2k+1})]d(x_{2k+2},Tx_{2k+2})}{1+d(x_{2k+1},x_{2k+2})}$$
$$+a_3[d(x_{2k+1},Sx_{2k+1})+d(x_{2k+2,}Tx_{2k+2})]+a_4[d(x_{2k+1,}Tx_{2k+2})+d(x_{2k+2},Sx_{2k+1})]))$$
$$= F(\psi(a_1 d(x_{2k+1},x_{2k+2})+a_2 \frac{[1+d(x_{2k+1},x_{2k+2})]d(x_{2k+2},x_{2k+3})}{1+d(x_{2k+1},x_{2k+2})}$$
$$+a_3[d(x_{2k+1},x_{2k+2})+d(x_{2k+2,}x_{2k+3})]+a_4[d(x_{2k+1,}x_{2k+3})+d(x_{2k+2},x_{2k+2})]),$$
$$\varphi(a_1 d(x_{2k+1},x_{2k+2})+a_2 \frac{[1+d(x_{2k+1},x_{2k+2})]d(x_{2k+2},x_{2k+3})}{1+d(x_{2k+1},x_{2k+2})}$$
$$+a_3[d(x_{2k+1},x_{2k+2})+d(x_{2k+2,}x_{2k+3})]+a_4[d(x_{2k+1,}x_{2k+3})+d(x_{2k+2},x_{2k+2})]))$$
$$= F(\psi((a_1+a_3)d(x_{2k+1},x_{2k+2})+(a_2+a_3)d(x_{2k+2},x_{2k+3})+a_4 d(x_{2k+1},x_{2k+3})),$$
$$\varphi((a_1+a_3)d(x_{2k+1},x_{2k+2})+(a_2+a_3)d(x_{2k+2},x_{2k+3})+a_4 d(x_{2k+1},x_{2k+3})))$$
$$\leq F(\psi((a_1+a_3)d(x_{2k+1},x_{2k+2})+(a_2+a_3)d(x_{2k+2},x_{2k+3})$$
$$+sa_4[d(x_{2k+1},x_{2k+2})+d(x_{2k+2},x_{2k+3})]),$$
$$\varphi((a_1+a_3)d(x_{2k+1},x_{2k+2})+(a_2+a_3)d(x_{2k+2},x_{2k+3})$$
$$+sa_4[d(x_{2k+1},x_{2k+2})+d(x_{2k+2},x_{2k+3})]))$$
$$= F(\psi((a_1+a_3+sa_4)d(x_{2k+1},x_{2k+2})+(a_2+a_3+sa_4)d(x_{2k+2},x_{2k+3})),$$
$$\varphi((a_1+a_3+sa_4)d(x_{2k+1},x_{2k+2})+(a_2+a_3+sa_4)d(x_{2k+2},x_{2k+3}))).$$

This implies that

$$d(x_{2k+2},x_{2k+3}) \leq \frac{a_1+a_3+sa_4}{1-a_1-a_3-sa_4} d(x_{2k+1},x_{2k+2}).$$

Putting $\lambda = \dfrac{a_1 + a_3 + sa_4}{1 - a_1 - a_3 - sa_4}$

As $sa_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1,$ it is clear that $< 1/s,$ we have ,

$$d(x_{n+1}, x_{n+2}) \le \lambda(x_n, x_{n+1}) \le \cdots \le \lambda^{n+1} d(x_0, x_1).$$

Hence for any $m > n,$ we have

$$\begin{aligned}
d(x_n, x_m) &\le s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
&= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\
&\le sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
&= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\
&\le sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\
&\le s\lambda^n d(x_1, x_0) + s^2 \lambda^{n+1} d(x_1, x_0) + s^3 \lambda^{n+2} d(x_1, x_0) + \cdots + s^m \delta^{n+m-1} d(x_1, x_0) \\
&= s\lambda^n [1 + s\lambda + s^2\lambda^2 + s^3\lambda^3 + \cdots + (s\lambda)^{m-1}] d(x_1, x_0) \\
&\le [\frac{s\lambda^n}{1 - s\lambda}] d(x_1, x_0).
\end{aligned}$$

Since P is a normal cone with normal constant K, so we get

$$\| d(x_n, x_m) \| \le K \frac{s\lambda^n}{1 - s\lambda} \| d(x_1, x_0) \| .$$

This implies $\| d(x_n, x_m) \| \to 0$ as $n, m \to \infty$ since $0 < s\lambda < 1.$

Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone b-metric space, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Now, since

$$\begin{aligned}
&\psi(d(p, Tp)) \le \psi(s[d(p, x_{2n+1}) + d(x_{2n+1}, Tp)]) \\
&= \psi(sd(Sx_{2n}, Tp) + sd(p, x_{2n+1})) \\
&= \psi(sd(Sx_{2n}, Tp)) + \psi(sd(p, x_{2n+1})) \\
&\le F(\psi(sd(p, x_{2n+1}) + s[a_1 d(x_{2n}, p) + a_2 \frac{[1 + d(x_{2n}, Sx_{2n})] d(p, Tp)}{1 + d(x_{2n}, p)} \\
&+ a_3(d(x_{2n}, x_{2n+1}) + d(p, Tp)) + a_4(d(x_{2n}, Tp) + d(p, Sx_{2n}))]), \\
&(sd(p, x_{2n+1}) + s[a_1 d(x_{2n}, p) + a_2 \frac{[1 + d(x_{2n}, Sx_{2n})] d(p, Tp)}{1 + d(x_{2n}, p)} \\
&+ a_3(d(x_{2n}, x_{2n+1}) + d(p, Tp)) + a_4(d(x_{2n}, Tp) + d(p, Sx_{2n}))])) + \psi(sd(p, x_{2n+1})) \\
&= F(\psi(sd(p, x_{2n+1}) + s[a_1 d(x_{2n}, p) + a_2 \frac{[1 + d(x_{2n}, x_{2n+1})] d(p, Tp)}{1 + d(x_{2n}, p)} \\
&+ a_3 d(x_{2n}, x_{2n+1} + d(p, Tp)) + a_4(d(x_{2n}, Tp) + d(p, Sx_{2n}))]),
\end{aligned}$$

$$\varphi((sd(p,x_{2n+1})+s[a_1d(x_{2n},p)+a_2\frac{[1+d(x_{2n},x_{2n+1})]d(p,Tp)}{1+d(x_{2n},p)}$$
$$+a_3d(x_{2n},x_{2n+1}+d(p,Tp))+a_4(d(x_{2n},Tp)+d(p,Sx_{2n}))]])))+\psi(sd(p,x_{2n+1}))$$

$$\leq F(\psi(sd(p,x_{2n+1})+s[a_1d(x_{2n},p)+a_2\frac{[1+d(x_{2n},x_{2n+1})]d(p,Tp)}{1+d(x_{2n},p)}$$
$$+a_3d(x_{2n},x_{2n+1}+d(p,Tp))+a_4(d(x_{2n},Tp)+d(p,Sx_{2n}))]),$$
$$\varphi((sd(p,x_{2n+1})+s[a_1d(x_{2n},p)+a_2\frac{[1+d(x_{2n},x_{2n+1})]d(p,Tp)}{1+d(x_{2n},p)}$$
$$+a_3d(x_{2n},x_{2n+1}+d(p,Tp))+a_4(d(x_{2n},Tp)+d(p,Sx_{2n}))]])))$$

As $x_n \to p$ and $x_{n+1} \to p$ as $n \to \infty$, we get

$$(1-sa_2-sa_3-sa_4)\|d(p,Tp)\| \leq K[sa_1\|d(x_{2n},p)\|+s(a_4)\|d(p,x_{2n+1})\|] \to 0$$

as $n \to \infty$.

Hence $\|d(Tp,p)\| = 0$. Since $(1-sa_2-sa_3-sa_4) > 0$.

Thus we get $Tp = p$, that is, p is a fixed point of T.

Uniqueness:

Let q be another fixed point common to S and T, that is $Sq = Tq = q$ such that $p \neq q$.

Then from 2.1 we have

$$\psi(d(p,q))$$
$$= \psi(d(Sp,Tp))$$
$$\leq F(\psi(a_1d(p,Tp)+a_2\frac{[1+d(p,sp)]d(q,Tp)}{1+d(p,q)}+a_3[d(p,Sp)+d(q,Tp)]$$
$$+a_4(d(p,Tq)+d(q,Sp))),\varphi(a_1d(p,Tp)+a_2\frac{[1+d(p,sp)]d(q,Tp)}{1+d(p,q)}$$
$$+a_3[d(p,Sp)+d(q,Tp)]+a_4(d(p,Tq)+d(q,Sp))))$$
$$= F(\psi(a_1d(p,q)+a_2\frac{[1+d(p,p)]d(q,q)}{1+d(p,q)}+a_3[d(p,p)+d(q,q)]$$
$$+a_4(d(p,q)+d(q,p))),$$
$$\varphi(a_1d(p,q)+a_2\frac{[1+d(p,p)]d(q,q)}{1+d(p,q)}+a_3[d(p,p)+d(q,q)]$$
$$+a_4(d(p,q)+d(q,p))))$$
$$= F(\psi((a_1+2a_4)d(p,q)),\varphi((a_1+2a_4)d(p,q)))$$
$$\leq F(\psi((sa_1+a_2+(s+1)a_3+s(s+1)a_4)d(p,q)),$$
$$\varphi((sa_1+a_2+(s+1)a_3+s(s+1)a_4)d(p,q)))$$
$$< \psi(d(p,q))$$

which is a contradiction. Hence $\|d(p,q)\| = 0$ and so $p = q.$ Thus p is a unique common fixed point of S and $T.$

This completes the proof. Putting $S = T,$ we have the following results.

Corollary 2.2. *Let (X,d) be a complete cone b-metric space $(CCbMS)$ with the co-efficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $S,T : X \to X$ satisfy:*

$$\begin{aligned}\psi(d(Tx,Ty)) \leq\ & F(\psi(a_1 d(x,y) + a_2 \frac{[1+d(x,Tx)d(y,Ty)]}{1+d(x,y)} + a_3[d(x,Tx)+d(y,Ty)] \\ & + a_4[d(x,Ty)+d(y,Tx)]), \varphi(a_1 d(x,y) + a_2 \frac{[1+d(x,Tx)d(y,Ty)]}{1+d(x,y)} \\ & + a_3[d(x,Tx)+d(y,Ty)] + a_4[d(x,Ty)+d(y,Tx)]))\end{aligned} \tag{2.2}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are non negative reals with $sa_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1$. ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s).$ Then T has a unique common fixed point in $X.$

Proof: The proof of corollary (2.2) immediately follows from Theorem (2.1) by taking $S = T$. This completes the proof.

Corollary 2.3. *Let (X,d) be a complete cone b-metric space $(CCbMS)$ with the co-efficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $S,T : X \to X$ satisfy:*

$$\begin{aligned}& \psi(d(T^n x, T^n y)) \\ & \leq F(\psi(a_1 d(x,y) + a_2 \frac{[1+d(x,T^n x)d(y,T^n y)]}{1+d(x,y)} + a_3[d(x,T^n x)+d(y,T^n y)] \\ & + a_4[d(x,T^n y)+d(y,T^n x)]), \varphi(a_1 d(x,y) + a_2 \frac{[1+d(x,T^n x)d(y,T^n y)]}{1+d(x,y)} \\ & + a_3[d(x,T^n x)+d(y,T^n y)] + a_4[d(x,T^n y)+d(y,T^n x)]))\end{aligned} \tag{2.3}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are non negative reals with $sa_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1$. ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s).$ Then T has a unique common fixed point in $X.$

Proof: By Corollary (2.2) there exists $u \in X$ such that $T^n u = u.$ Then

$$
\begin{aligned}
\psi(d(Tu,u)) \quad &= \psi(d(TT^n u, T^n u)) \\
&= \psi(d(T^n Tu, T^n u)) \\
&\leq F(\psi(a_1 d(Tu,u) + a_2 \frac{[1+d(Tu,T^n Tu)]d(u,T^n u)}{1+d(Tu,u)} \\
&+ a_3[d(Tu,T^n Tu) + d(u,T^n u)] + a_4[d(Tu,T^n u) + d(u,T^n Tu)]), \\
&\varphi(a_1 d(Tu,u) + a_2 \frac{[1+d(Tu,T^n Tu)]d(u,T^n u)}{1+d(Tu,u)} \\
&+ a_3[d(Tu,T^n Tu) + d(u,T^n u)] + a_4[d(Tu,T^n u) + d(u,T^n Tu)])) \\
&\leq F(\psi(a_1 d(Tu,u) + a_2 \frac{[1+d(Tu,TT^n u)]d(u,T^n u)}{1+d(Tu,u)} \\
&+ a_3[d(Tu,TT^n u) + d(u,T^n u)] + a_4[d(Tu,T^n u) + d(u,TT^n u)]), \\
&\varphi(a_1 d(Tu,u) + a_2 \frac{[1+d(Tu,TT^n u)]d(u,T^n u)}{1+d(Tu,u)} \\
&+ a_3[d(Tu,TT^n u) + d(u,T^n u)] + a_4[d(Tu,T^n u) + d(u,TT^n u)])) \\
&= F(\psi(a_1 d(Tu,u) + a_2 \frac{[1+d(Tu,Tu)]d(u,u)}{1+d(Tu,u)} \\
&+ a_3[d(Tu,Tu) + d(u,u)] + a_4[d(Tu,u) + d(u,Tu)]), \\
&\varphi(a_1 d(Tu,u) + a_2 \frac{[1+d(Tu,Tu)]d(u,u)}{1+d(Tu,u)} \\
&+ a_3[d(Tu,Tu) + d(u,u)] + a_4[d(Tu,u) + d(u,Tu)])) \\
&= \psi((a_1 + 2a_4)d(Tu,u))
\end{aligned}
$$

and so $d(Tu,u) = 0$. Thus, $Tu = u$. This show that T has a unique fixed point X.

This completes the proof.

Putting $a_1 = k, a_2 = a_3 = a_4 = 0$ in Corollary (2.2) then we have the following result.

Corollary 2.4. *Let (X,d) be a complete cone b-metric space $(CCbMS)$ with the co-efficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $S,T : X \to X$ satisfy:*

$$\psi(d(Tx,Ty)) \leq F(\psi(kd(x,y)), \varphi(kd(x,y)))$$

For all $x, y \in X$, where $k \in (0,1)$ is a constant with $sk < 1$. ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s)$. Then T has a unique common fixed point in X.

Note 2.5. *Corollary 2.4 extends well known Banach contraction principle from complete metric space to that setting of complete cone b-metric space via cone $\mathcal{C}$-class function consider in this paper.*

Putting $a_3 = k, a_1 = a_2 = a_4 = 0$ in corollary (2.2), then we have the following result.

Corollary 2.6. *Let* (X, d) *be a complete cone b-metric space* $(CCbMS)$ *with the co-efficient* $s \geq 1$ *and* P *be a normal cone with normal constant* K. *Suppose that the mappings* $S, T : X \to X$ *satisfy:*

$$\psi(d(Tx, Ty)) \leq F(\psi(k[d(x, Tx) + d(y, Ty)]), \varphi(k[d(x, Tx) + d(y, Ty)]))$$

For all $x, y \in X$, where $k \in (0, \frac{1}{s+1})$ is a constant with $k(s+1) < 1$. ψ and φ are altering distance and ultra altering distance functions respectively , $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s)$. Then T has a unique common fixed point in X.

Note 2.7. *Corollary 2.6 extends well known Kannan contraction principle from complete metric space to that setting of complete cone* b*-metric space via cone* $\mathcal{C}$*-class function consider in this paper.*

Putting $a_4 = k, a_1 = a_2 = a_3 = 0$ in corollary (2.2), then we have the following result.

Corollary 2.8. *Let* (X, d) *be a complete cone b-metric space* $(CCbMS)$ *with the co-efficient* $s \geq 1$ *and* P *be a normal cone with normal constant* K. *Suppose that the mappings* $S, T : X \to X$ *satisfy:*

$$\psi(d(Tx, Ty)) \leq F(\psi(k[d(x, Ty) + d(y, Tx)]), \varphi(k[d(x, Ty) + d(y, Tx)]))$$

For all $x, y \in X$, where $k \in (0, \frac{1}{s+1})$ is a constant with $k(s+1) < 1$. ψ and φ are altering distance and ultra altering distance functions respectively , $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s)$. Then T has a unique common fixed point in X.

Note 2.9. *Corollary 2.8 extends well known Kannan contraction principle from complete metric space to that setting of complete cone* b*-metric space via cone* $\mathcal{C}$*-class function consider in this paper.*

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