Journal of Mathematics and Informatics
Vol. 9, 2017, 1-7
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 6 October 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v9a1

Journal of
Mathematics and
Informatics

Tensor of the Type (0,4) in a Manifold Equipped with an almost Para Norden Contact Metric Manifold

***Shalini Singh*[1] and *S.D.Singh*[2]**

[1]Department of Applied Mathematics, JSS Academy of Technical Education
Noida-201301, India.
[2]Diesel Locomotive Works, Varanasi-221004, India.
E-mail: sdsingh@gmail.com
[1]Corresponding author. E-mail: shalinisingh1183@gmail.com

Received 9 September 2017; accepted 1 October 2017

Abstract. The present paper deals with various curvature tensors of the type (0,4) in an almost para norden contact metric manifold. In this paper, after defining almost para norden contact metric manifold I have defined $M_n{}^*$ and $M_n{}^{**}$ manifolds and the form of curvature tensor of the type (0,4) in this manifold. Several useful theorems on these manifolds have also been derived.

Keywords: C^∞-manifold, almost para norden contact metric manifold, curvature tensor.

AMS Mathematics Subject Classification (2010): 53A45

1. Introduction

Consider a differentiable manifold M_n of differentiability class C^∞. Let there exist in M_n a vector valued C^∞- linear function Φ, a C^∞- vector field η and a C^∞-one form ξ such that

(1.1) $\Phi^2(X) = -X + \xi(X)\eta$

(1.2) $\bar{\eta} = 0$

(1.3) $G(\bar{X},\bar{Y}) = -G(X,Y) + \xi(X)\xi(Y)$

Then the set (Φ,η,ξ,G) satisfying (1.1) to (1.3) is called an almost para norden contact metric structure and M_n equipped with an almost para norden contact metric structure will be called an almost para norden contact metric structure manifold.
It is easy to calculate in

(1.4) $\xi(\eta) = 1$

(1.5) $\xi(\bar{X}) = 0$

and

(1.6) $G(X,\eta) = \xi(X)$

Definition 1.1: A C^∞-manifold M_n, satisfying

(1.7) $$D_X \eta = \Phi(X) \underline{\underline{def}} \bar{X}$$

will be denoted by M_n^*

In M_n^*, we can easily show that

(1.8) $$(D_X \xi)(Y) = {}'\Phi(X,Y) = (D_Y \xi)(X)$$

where

(1.9) $${}'\Phi(X,Y) \underline{\underline{def}} G(\bar{X},Y) = G(X,\bar{Y})$$

Definition 1.2: A C^∞-manifold M_n, satisfying

(1.10) $${}'\Phi(Y,Z) = G(Y,Z) - \xi(Y)\xi(Z)$$

will be called M_n^{**} -manifold. Also in M_n^{**} we have

(1.11) $$\xi(K(X,Y,Z)) = -[G(Y,Z)\xi(X) - G(X,Z)\xi(Y)]$$

From (1.11) we have

(1.12) $$K(X,Y,Z) = -[G(Y,Z)X - G(X,Z)Y]$$

(1.13) $$K(Y,Z) = -(n-1)G(Y,Z)$$

(1.14) $$Ric(X,\eta) = -(n-1)\xi(X)$$

Also in M_n^{**}, the following results can be obtained

(1.15) $${}'K(X,Y,Z,U) = -[G(Y,Z)G(X,U) - G(X,Z)G(Y,U)]$$

(1.16) $${}'K(\eta,Y,Z,\eta) = [-G(Y,Z) + \xi(Y)\xi(Z)]$$

(1.17) $${}'K(X,Y,Z,\eta) = -[G(Y,Z)\xi(X) - G(X,Z)\xi(Y)]$$
$$= \xi(K(X,Y,Z))$$

(1.18) $${}'K(\eta,Y,Z,U) = -[G(Y,Z)\xi(U) - G(Y,U)\xi(Z)]$$

(1.19) $${}'K(X,\eta,Z,U) = -[G(X,U)\xi(Z) - G(X,Z)\xi(U)]$$

(1.20) $${}'K(X,U,Z,\eta) = -[G(Z,U)\xi(X) - G(X,Z)\xi(U)]$$

(1.21) $${}'K(X,Z,U,\eta) = -[G(Z,U)\xi(X) - G(X,U)\xi(Z)]$$

From (1.13), we have

(1.22) $$Ric(\bar{X},Y) + Ric(X,\bar{Y}) = 0$$

(1.23) $$Ric(\bar{X},\bar{Y}) = [Ric(X,Y) + (n-1)\xi(X)\xi(Y)]$$

We shall consider in M_n^{**} tensors W_1, W_2, W_3, W_4 of type $(0,4)$ defined by [1,2,3,4,5].

(1.24) $$W_1(X,Y,Z,U) \underline{\underline{def}} {}'K(X,Y,Z,U) + \frac{1}{(n-1)}[G(X,U)Ric(Y,Z) - G(Y,U)Ric(X,Z)]$$

(1.25) $$'W_2(X,Y,Z,U) \underline{\underline{def}}\, 'K(X,Y,Z,U) + \frac{1}{(n-1)}\left[G(X,Z)\,Ric(Y,U) - G(Y,Z)\,Ric(X,U)\right]$$

(1.26) $$'W_3(X,Y,Z,U) \underline{\underline{def}}\, 'K(X,Y,Z,U) + \frac{1}{(n-1)}\left[G(Y,Z)\,Ric(X,U) - G(Y,U)\,Ric(X,Z)\right]$$

(1.27) $$'W_4(X,Y,Z,U) \underline{\underline{def}}\, 'K(X,Y,Z,U) + \frac{1}{(n-1)}\left[G(X,Z)\,Ric(Y,U) - G(X,Y)\,Ric(Z,U)\right]$$

It is obvious that for an empty gravitational field characterized by $Ric(X,Y)=0$, the above four tensors are identical.

Since $'W_1$ and $'W_2$ are skew-symmetric in X and Y therefore, breaking $'W_1$ and $'W_2$ into symmetric and skew-symmetric parts with respect to Z and U, we get

(1.28) $$'\beta_1(X,Y,Z,U) \underline{\underline{def}}\, \frac{1}{2}\left[\, 'W_1(X,Y,Z,U) + 'W_1(X,Y,U,Z)\right]$$
$$= \frac{1}{2(n-1)}\left[G(X,U)\,Ric(Y,Z) - G(Y,U)\,Ric(X,Z)\right.$$
$$\left. + G(X,Z)\,Ric(Y,U) - G(Y,Z)\,Ric(X,U)\right]$$

(1.29) $$'\alpha_1(X,Y,Z,U) \underline{\underline{def}}\, \frac{1}{2}\left[\, 'W_1(X,Y,Z,U) - 'W_1(X,Y,U,Z)\right]$$
$$= 'K(X,Y,Z,U) + \frac{1}{2(n-1)}\left[G(X,U)\,Ric(Y,Z) - G(Y,U)\,Ric(X,Z)\right.$$
$$\left. - G(X,Z)\,Ric(Y,U) + G(Y,Z)\,Ric(X,U)\right]$$

(1.30) $$'\beta_2(X,Y,Z,U) \underline{\underline{def}}\, \frac{1}{2(n-1)}\left[G(X,Z)\,Ric(Y,U) - G(Y,Z)\,Ric(X,U)\right.$$
$$\left. + G(X,U)\,Ric(Y,Z) - G(Y,U)\,Ric(X,Z)\right]$$

(1.31) $$'\alpha_2(X,Y,Z,U) \underline{\underline{def}}\, 'K(X,Y,Z,U) + \frac{1}{2(n-1)}\left[G(X,Z)\,Ric(Y,U)\right.$$
$$\left. - G(Y,Z)\,Ric(X,U) - G(X,U)\,Ric(Y,Z) + G(Y,U)\,Ric(X,Z)\right]$$

Since $'W_3$ and $'W_4$ are skew-symmetric in Z and U, therefore, breaking $'W_3$ and $'W_4$ into symmetric and skew-symmetric parts with respect to X and Y, we get

(1.32) $$'\beta_3(X,Y,Z,U) \underline{\underline{def}}\, \frac{1}{2}\left[\, 'W_3(X,Y,Z,U) + 'W_3(Y,X,Z,U)\right]$$
$$= \frac{1}{2(n-1)}\left[G(Y,Z)\,Ric(X,U) - G(Y,U)\,Ric(X,Z)\right.$$
$$\left. + G(X,Z)\,Ric(Y,U) - G(X,U)\,Ric(Y,Z)\right]$$

(1.33) $$'\alpha_3(X,Y,Z,U) \underline{\underline{def}}\, \frac{1}{2}\left[\, 'W_3(X,Y,Z,U) - 'W_3(Y,X,Z,U)\right]$$
$$= 'K(X,Y,Z,U) + \frac{1}{2(n-1)}\left[G(Y,Z)\,Ric(X,U) - G(Y,U)\,Ric(X,Z)\right.$$

$$-G(X,Z)\,Ric(Y,U)+G(X,U)\,Ric(Y,Z)\Big]$$

(1.34) $$`\beta_4(X,Y,Z,U)\underline{\underline{def}}\frac{1}{2(n-1)}\Big[G(X,Z)\,Ric(Y,U)$$

$$-2G(X,Y)\,Ric(Z,U)+G(Y,Z)\,Ric(X,U)\Big]$$

(1.35) $$`\alpha_4(X,Y,Z,U)\underline{\underline{def}}\,`K(X,Y,Z,U)+\frac{1}{2(n-1)}\Big[G(X,Z)\,Ric(Y,U)-G(Y,Z)\,Ric(X,U)\Big]$$

2. W_1-Curvature tensor

Theorem 2.1. In M_n^{**}, we have

(2.1a) $$`W_1(X,Y,Z,\eta)=-2\big[G(Y,Z)\,\xi(X)-G(X,Z)\,\xi(Y)\big]$$

$$=`\alpha_1(X,Y,Z,\eta)$$

(2.1b) $$`W_1(\eta,Y,Z,U)=-2\big[G(Y,Z)\,\xi(U)-G(Y,U)\,\xi(Z)\big]$$

$$=`\alpha_1(\eta,Y,Z,U)$$

(2.1c) $$`W_1(\eta,Y,Z,\eta)=2\big[-G(Y,Z)+\xi(Y)\,\xi(Z)\big]$$

$$=`\alpha_1(\eta,Y,Z,\eta)$$

Proof: Replacing U by η in (1.24), we get

(2.2) $$`W_1(X,Y,Z,\eta)=`K(X,Y,Z,\eta)+\frac{1}{(n-1)}\Big[G(X,\eta)\,Ric(Y,Z)-G(Y,\eta)\,Ric(X,Z)\Big]$$

Using (1.6), (1.13) and (1.17) in the above equation, we get

(2.3) $$`W_1(X,Y,Z,\eta)=-2\big[G(Y,Z)\,\xi(X)-G(X,Z)\,\xi(Y)\big]$$

Replacing U by η in (1.29), we get

(2.4) $$`\alpha_1(X,Y,Z,\eta)=`K(X,Y,Z,\eta)+\frac{1}{2(n-1)}\Big[G(X,\eta)\,Ric(Y,Z)-G(Y,\eta)\,Ric(X,Z)$$

$$-G(X,Z)\,Ric(Y,\eta)+G(Y,Z)\,Ric(X,\eta)\Big]$$

Using (1.6), (1.13), (1.14) and (1.17) in the above equation, we get

(2.5) $$`\alpha_1(X,Y,Z,\eta)=-2\big[G(Y,Z)\,\xi(X)-G(X,Z)\,\xi(Y)\big]$$

From (2.3) and (2.4), we get (2.1a).

Replacing X by η in (1.24), we get

(2.6) $$`W_1(\eta,Y,Z,U)=`K(\eta,Y,Z,U)+\frac{1}{(n-1)}\Big[G(\eta,U)\,Ric(Y,Z)-G(Y,U)\,Ric(\eta,Z)\Big]$$

Using (1.6), (1.13), (1.14) and (1.18) in the above equation, we get

(2.7) $$`W_1(\eta,Y,Z,U)=-2\big[G(Y,Z)\,\xi(U)-G(Y,U)\,\xi(Z)\big]$$

Replacing X by η in (1.29), we get

(2.8) $$`\alpha_1(\eta,Y,Z,U)=`K(\eta,Y,Z,U)+\frac{1}{2(n-1)}\Big[G(\eta,U)\,Ric(Y,Z)-G(Y,U)\,Ric(\eta,Z)$$

$$-G(\eta, Z)\, Ric(Y,U) + G(Y,Z)\, Ric(\eta,U)\big]$$

Using (1.6), (1.13), (1.14) and (1.18) in the above equation, we get

(2.9) $\quad 'α_1(\eta,Y,Z,U) = -2\big[G(Y,Z)\,\xi(U) - G(Y,U)\,\xi(Z)\big]$

From (2.7) and (2.9), we get (2.1b).

Replacing X by η in (2.1a) and using (1.4) and (1.6), we get (2.1c).

Theorem 2.2. In M_n^{**}, we have

(2.10a) $\quad 'W_1(X,\eta,Z,U) = -2\big[G(X,U)\,\xi(Z) - G(X,Z)\,\xi(U)\big]$

(2.10b) $\quad \xi(Z)\, 'W_1(\eta,Y,U,\eta) - \xi(U)\, 'W_1(\eta,Y,Z,\eta) = 2\big[G(Y,Z)\,\xi(U) - G(Y,U)\,\xi(Z)\big]$

(2.10c) $\quad 'W_1(X,\eta,Z,U) + 'W_1(X,Z,U,\eta) + 'W_1(X,U,Z,\eta) = -4\big[G(Z,U)\xi(X) - G(X,Z)\xi(U)\big]$

Proof: Replacing Y by η in (1.24), we get

(2.11) $$'W_1(X,\eta,Z,U) = 'K(X,\eta,Z,U) + \frac{1}{(n-1)}\big[G(X,U)\, Ric(\eta,Z) - G(\eta,U)\, Ric(X,Z)\big]$$

Using (1.6), (1.13), (1.14) and (1.19) in the above equation, we get (2.10a).

Replacing Z by U in (2.1c) and multiplying by $\xi(Z)$, we get

(2.12) $\quad 'W_1(\eta,Y,U,\eta)\,\xi(Z) = 2\xi(Z)\big[-G(Y,U) + \xi(Y)\,\xi(U)\big]$

Multiplying (2.1c) by $\xi(U)$, we get

(2.13) $\quad 'W_1(\eta,Y,Z,\eta)\,\xi(U) = 2\xi(U)\big[-G(Y,Z) + \xi(Y)\,\xi(Z)\big]$

Subtracting (2.13) from (2.12), we get (2.10b).

Replacing Y by U and U by η in (1.24), we get

(2.14) $$'W_1(X,U,Z,\eta) = 'K(X,U,Z,\eta) + \frac{1}{(n-1)}\big[G(X,\eta)\, Ric(U,Z) - G(U,\eta)\, Ric(X,Z)\big]$$

Using (1.6), (1.13) and (1.20) in the above equation, we get

(2.15) $\quad 'W_1(X,U,Z,\eta) = -2\big[G(Z,U)\,\xi(X) - G(X,Z)\,\xi(U)\big]$

Replacing Y by Z, Z by U and U by η in (1.24), we get

(2.16) $$'W_1(X,Z,U,\eta) = 'K(X,Z,U,\eta) + \frac{1}{(n-1)}\big[G(X,\eta)\, Ric(Z,U) - G(Z,\eta)\, Ric(X,U)\big]$$

Using (1.6), (1.13) and (1.21) in the above equation, we get

(2.17) $\quad 'W_1(X,Z,U,\eta) = -2\big[G(Z,U)\,\xi(X) - G(X,U)\,\xi(Z)\big]$

Adding (2.10a), (2.15) and (2.17), we get (2.10c).

Corollary 2.1. In M_n^{**}, we have

(2.18a) $\quad 'W_1(\bar{X},\bar{Y},Z,\eta) - 'W_1(\eta,Y,\bar{Z},\bar{U}) = 0$

(2.18b) $\quad 'W_1(\eta,\bar{Y},Z,\eta) - 'W_1(\eta,Y,\bar{Z},\eta) = 0$

(2.18c) $\quad 'W_1(\eta,Y,\bar{Z},\eta)\,\xi(U) - 'W_1(\eta,Y,\bar{Z},U) = 0$

Proof: Barring X, Y in (2.1a) and Z, U in (2.1b) and using (1.5), we get (2.18a). Barring Y and Z in (2.1c) separately and adding them, we get (2.18b) due to (1.5) and (19). Barring Z in (2.1c) and multiplying by $\xi(U)$ and using (1.5), we get

(2.19) $$W_1(\eta, Y, \bar{Z}, \eta)\,\xi(U) = -2G(Y, \bar{Z})\,\xi(U)$$

Barring Z in (2.1b) and using (1.5), we get

(2.20) $$W_1(\eta, Y, \bar{Z}, U) = -\xi(U)\,G(Y, \bar{Z})$$

Subtracting (2.20) from (2.19), we get (2.18c).

Theorem 2.3. In M_n^{**}, we have

(2.21a) $$\grave{\beta}_1(\eta, Y, Z, \eta) = 0$$

(2.21b) $$\grave{\beta}_1(X, Y, Z, \eta) = 0$$

(2.21c) $$\grave{\beta}_1(\eta, Y, Z, U) = 0$$

Proof: Replacing X and U by η in (1.28), we get

(2.22) $$\grave{\beta}_1(\eta,Y,Z,\eta) = \frac{1}{2(n-1)}\Big[G(\eta,\eta)\,Ric(Y,Z) - G(Y,\eta)\,Ric(\eta,Z) + G(\eta,Z)\,Ric(Y,\eta) - G(Y,Z)\,Ric(\eta,\eta)\Big]$$

Using (1.6), (1.13) and (1.14) in the above equation, we get (2.21a).
Replacing U by η in (1.28) and using (1.6), (1.13) and (1.14), we get (2.21b).
Replacing X by η in (1.28) and using (1.6), (1.13) and (1.14), we get (2.21c).

Corollary 2.2. In M_n^{**}, we have

(2.23a) $$2\grave{\beta}_1(X, Y, Z, \eta) + W_1(X, Y, Z, \eta) = -2\big[G(Y,Z)\xi(X) - G(X,Z)\xi(Y)\big]$$

(2.23b) $$2\grave{\beta}_1(\bar{X}, Y, Z, \eta) + W_1(\bar{X}, Y, Z, \eta) = 2G(\bar{X}, Z)\,\xi(Y)$$

Proof: (2.1a) and (2.1b) yield (2.23a).
Barring X in (2.23a) and using (1.5), we get (2.23b).

Conclusion: In the present paper we have investigated the different properties on curvature tensor of the type (0,4) in the almost para norden contact metric manifold. The main results from the analysis of the paper are in M_n^{**}, the tensor W_1 and W_3 are identical so that all the results for W_1-curvature tensor will also hold for W_3–curvature tensor. Similar results can be obtained by considering W_2 and W_4.

REFERENCES

1. G.P.Pokharial and R.S.Mishra, Curvature tensor and their relativistic significance (I), *Yokohama, Math. J.*, 18 (1970) 105-108.
2. G.P.Pokharial and R.S.Mishra, Curvature tensor and their relativistic significance (II), *Yokohama, Math. J.*, 19 (1971) 97-103.

3. G.P.Pokharial and R.S.Mishra, Curvature tensor and their relativistic significance (III), *Yokohama, Math. J.,* 21 (1973) 115-119.
4. R.S.Mishra, Structure on a differentiable manifold and their applications, *Chandrama Prakashan*, Allahabad, India, (1984).
5. R.S.Mishra, A course in tensors with applications to Riemannian geometry, *Pothishala Private Limited, Allahabad*, 4th edition, (1995).
6. S.Singh, Semi-symmetric non-metric S-connection on a generalized contact metric structure manifold, *International Journal of Advanced Research*, 5(6) (2017) 1685-1690.
7. S.Singh and S.D.Singh, A semi-symmetric non-metric connexion on an Unified structure manifold, *Adv. Theor. Appl. Mech.*, 3 (5) (2010) 203-210.