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On Coefficients Inequalities of Functions Related to *q*-Derivative and Conic Regions

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Abstract. In this article using the concept of q-derivatives and conic domains we introduce classes generalizing the classes studied by Noor, Malik, Kanas, Wisniowska and deduce some interesting coefficient inequalities.

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AMS Mathematics Subject Classification (2010): 30C45

1. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. The class of functions with positive real part P plays a significant role because of the fact that many simple subclasses of the class of univalent functions can be completely characterised in terms of analytic conditions by using this concept. This motivated Janowski to define the class $P(A,B)$.

Definition 1.1. *[4] Let $P(A,B)$, where $-1 \le B < A \le 1$, denote the class of analytic functions p defined on $\mathcal{U}$ with $p(0)=1$ and $p(z) \prec \frac{1+Az}{1+Bz}$, where $\prec$ denote subordination.*

The linear transformation $\frac{1+Az}{1+Bz}$ maps the circle $|z| = r$ onto the circle on the real line segment $\left(\frac{1-Ar}{1-Br}, \frac{1+Ar}{1+Br}\right)$ as diameter. Also a function $p(z) \in P(A,B)$ maps the open disc $\mathsf{U}_r\,(0 < r \le 1)$ univalently onto the open disc

$$\Omega_r(A,B) = \left\{\omega : \left|\omega - \frac{1-ABr^2}{1-B^2r^2}\right| < \frac{(A-B)r}{1-B^2r^2}\right\}.$$

Kanas and Wisniowska [5,6] generalized the parabolic domain $\Omega = \{\omega : Re\{\omega\} > |\,\omega - 1|\}$ and introduced the conic domain Ω_k, $k \ge 0$ and studied it comprehensively. This domain is defined as

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}.$$

The functions which play the role of extremal functions for conic regions are given by

$$P_k(z) = \begin{cases} \dfrac{1+z}{1-z}, \quad k = 0, \\ 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, k = 1, \\ 1 + \dfrac{2}{1-k^2}\sinh^2\left[(\dfrac{2}{\pi}\arccos k)\arctan h\sqrt{z}\right], 0 < k < 1, \\ \dfrac{1}{k^2-1}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{t}}}\dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\,dx\right) + \dfrac{1}{k^2-1}, \ k > 1, \end{cases} \tag{1.2}$$

where $u(z) = \dfrac{z-\sqrt{t}}{1-\sqrt{tz}}$ $, t \in (0,1)\ z \in \mathsf{U}$ and z is chosen such that $k = \cosh\left(\dfrac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, [5, 6].

If $p_k(z) = 1 + \delta_k z + \ldots$, then it is shown in [6] that from (1.2) one can have

$$\delta_k = \begin{cases} \dfrac{8(arcosk)^2}{\pi^2(1-k^2)}, & 0 \le k < 1, \\ \dfrac{8}{\pi^2}, & k = 1, \\ \dfrac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)}, & k > 1. \end{cases} \tag{1.3}$$

Noor and Malik [8] introduced the class of functions $p(z) \in k - P[A,B]$ which take all values from the domain $\Omega_k[A,B]$, $-1 \le B < A \le 1$, $k \ge 0$ where

$$\Omega[A,B] = \left\{\omega : \left|\omega - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\right\}.$$

Definition 1.2. *A function* $p(z)$ *is said to be in the class* $k - P[A,B]$*, if and only if,*

$$p(z) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad k \geq 0,$$

where $p_k(z)$ is defined by (1.2) and $1 \leq B < A \leq 1$.

Jackson[3] at the beginning of the twentieth century studied consequences. The key concept is the q-derivative operator defined as follows:

Definition 1.3.

$$\partial_q f(z) = \begin{cases} \dfrac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), \quad z = 0, & 0 < q < 1. \end{cases} \tag{1.4}$$

Equivalently (1.4), may be written as $\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$ where

$[n]_q = \dfrac{1-q^n}{1-q}$. Note that as $q \to 1$, $[n]_q \to n$.

using the concept of q-derivative with conic domains we define the following :

Definition 1.4. *A function* $f(z) \in \mathrm{A}$ *is said to be in the class* $k-ST[A,B,q]$, $k \geq 0$, $-1 \leq B < A \leq 1$, $0 < q < 1$, *if and only if,*

$$\Re\left(\frac{(B-1)\dfrac{z\partial_q f(z)}{f(z)}-(A-1)}{(B+1)\dfrac{z\partial_q f(z)}{f(z)}-(A+1)}\right) > k\left|\frac{(B-1)\dfrac{z\partial_q f(z)}{f(z)}-(A-1)}{(B+1)\dfrac{z\partial_q f(z)}{f(z)}-(A+1)}-1\right|,$$

or equivalently,

$$\frac{z\partial_q f(z)}{f(z)} \in k-P[A,B]. \tag{1.5}$$

Definition 1.5. *A function* $f(z) \in \mathrm{A}$ *is said to be in the class* $k-UCV[A,B,q]$, $k \geq 0$, $-1 \leq B < A \leq 1$, $0 < q < 1$, *if and only if,*

$$\Re\left(\frac{(B-1)\dfrac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)}-(A-1)}{(B+1)\dfrac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)}-(A+1)}\right) > k\left|\frac{(B-1)\dfrac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)}-(A-1)}{(B+1)\dfrac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)}-(A+1)}-1\right|,$$

or equivalently,

$$\frac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)} \in k-P[A,B]. \tag{1.6}$$

It can be easily seen that

$$f(z) \in k-UCV[A,B,q] \Leftrightarrow z\partial_q f(z) \in k-ST[A,B,q]. \tag{1.7}$$

We need the following lemma to prove our main results.

Lemma 1.1. *[8] Let* $g(z)=1+\sum_{n=1}^{\infty}c_n z^n \in P[A,B]$. *Then*

$$|c_n|\le \frac{(A-B)|\delta_k|}{2}, \quad n\ge 1,$$

where δ_k is defined by (1.3).

2. Main results

Theorem 2.1. *A function* $f\in \mathcal{A}$ *and of the form (1.1) is in the class* $k-ST[A,B,q]$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty}\left\{2(k+1)([n]_q-1)+|[n]_q(B+1)-(A+1)|\right\}|a_n|<|B-A|, \tag{2.1}$$

where $-1\le B<A\le 1$, $0<q<1$ and $k\ge 0$.

Proof: Assuming that (2.1) holds, then it suffices to show that

$$k\left|\frac{(B-1)\frac{z\partial_q f(z)}{f(z)}-(A-1)}{(B+1)\frac{z\partial_q f(z)}{f(z)}-(A+1)}-1\right|-\Re\left(\frac{(B-1)\frac{z\partial_q f(z)}{f(z)}-(A-1)}{(B+1)\frac{z\partial_q f(z)}{f(z)}-(A+1)}\right)<1.$$

We have

$$k\left|\frac{(B-1)\frac{z\partial_q f(z)}{f(z)}-(A-1)}{(B+1)\frac{z\partial_q f(z)}{f(z)}-(A+1)}-1\right|-\Re\left(\frac{(B-1)\frac{z\partial_q f(z)}{f(z)}-(A-1)}{(B+1)\frac{z\partial_q f(z)}{f(z)}-(A+1)}\right)$$

$$\le (k+1)\left|\frac{(B-1)z\partial_q f(z)-(A-1)f(z)}{(B+1)z\partial_q f(z)-(A+1)f(z)}-1\right|$$

$$=2(k+1)\left|\frac{f(z)-z\partial_q f(z)}{(B+1)z\partial_q f(z)-(A+1)f(z)}\right|$$

$$=2(k+1)\left|\frac{\sum_{n=2}^{\infty}(1-[n]_q)a_n z^n}{(B-A)z+\sum_{n=2}^{\infty}\left\{[n]_q(B+1)-(A+1)\right\}a_n z^n}\right|$$

$$\le 2(k+1)\frac{\sum_{n=2}^{\infty}|1-[n]_q||a_n|}{|B-A|-\sum_{n=2}^{\infty}|[n]_q(B+1)-(A+1)||a_n|}.$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\left\{2(k+1)([n]_q-1)+|[n]_q(B+1)-(A+1)|\right\}|a_n|<|B-A|$$

and this completes the proof.

As $q\to 1$, we have following result proved by Khalida Inyat Noor and Sarfraz Nawaz Malik [8].

Corollary 2.1. *A function* $f\in \mathcal{A}$ *and of the form (1.1) is in the class* $k-ST[A,B]$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty}\left\{2(k+1)(n-1)+|n(B+1)-(A+1)|\right\}|a_n|<|B-A|, \tag{2.2}$$

where $-1\le B<A\le 1$ and $k\ge 0$.

As $q\to 1$, $A=1$, $B=-1$, we have following result, proved by Kanas and Wisniowska [6].

Corollary 2.2. *A function* $f\in \mathcal{A}$ *and of the form (1.1) is in the class* $k-ST$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty}\left\{n+k(n-1)\right\}|a_n|<1,\quad k\ge 0. \tag{2.3}$$

As $q\to 1$, for the parametric values $A=1-2\alpha$, $B=-1$ $0\le\alpha<1$, we have the following result, proved by Shams et al. [10].

Corollary 2.3. *A function* $f\in \mathcal{A}$ *and of the form (1.1) is in the class* $SD(k,\alpha)$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty}\left\{n(k+1)-(k+\alpha)\right\}|a_n|<1-\alpha,\quad k\ge 0. \tag{2.4}$$

Special choices, $A=1-2\alpha$, $B=-1$ and $k=0$, as $q\to 1$ yield the following result, proved by Silverman [11].

Corollary 2.4. *A function* $f\in \mathcal{A}$ *and of the form (1.1) is in the class* $S^*(\alpha)$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|<1-\alpha,\quad 0\le\alpha<1. \tag{2.5}$$

Theorem 2.2. *A function* $f\in \mathcal{A}$ *and of the form (1.1) is in the class* $k-UCV[A,B,q]$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty}[n]_q\left\{2(k+1)([n]_q-1)+|[n]_q(B+1)-(A+1)|\right\}|a_n|<|B-A|, \tag{2.6}$$

where $-1\le B<A\le 1$, $0<q<1$ and $k\ge 0$.

The proof follows by using Theorem 2.1 and (1.7).

Theorem 2.3. *Let* $f(z)\in k-ST[A,B,q]$ *and of the form (1.1). Then, for* $n\ge 2$,

$$|a_n| \le \prod_{j=1}^{n-1} \frac{\left|\delta_k (A-B) - 2([j]_q - 1)B\right|}{2([j+1]_q - 1)}, \tag{2.7}$$

where δ_k is defined by (1.3) and $0 < q < 1$.

Proof: By definition (1.4) for $f(z) \in k - ST[A, B, q]$, we have

$$\frac{z\partial_q f(z)}{f(z)} = p(z), \text{ where } p(z) \in [A, B],$$

then we have

$$z\partial_q f(z) = f(z)p(z),$$

which implies

$$z + \sum_{n=2}^{\infty} [n]_q a_n z^n = \left(z + \sum_{n=2}^{\infty} a_n z^n\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

Equating coefficients of z^n on both sides, we have

$$([n]_q - 1)a_n = \sum_{j=1}^{n-1} a_{n-j} c_j, \qquad a_1 = 1.$$

This implies that

$$|a_n| \le \frac{1}{[n]_q - 1} \sum_{j=1}^{n-1} |a_{n-j}||c_j|, \qquad a_1 = 1.$$

Using Lemma (1.1), we get

$$|a_n| \le \frac{|\delta_k|(A-B)}{2([n]_q - 1)} \sum_{j=1}^{n-1} |a_j|, \quad a_1 = 1. \tag{2.8}$$

Now we prove that

$$\frac{|\delta_k|(A-B)}{2([n]_q - 1)} \sum_{j=1}^{n-1} |a_j| \le \prod_{j=1}^{n-1} \frac{\left|\delta_k (A-B) - 2([j]_q - 1)B\right|}{2([j+1]_q - 1)}. \tag{2.9}$$

We proof (2.9) by the induction method.
For n=2, from (2.8), we have

$$|a_2| \le \frac{|\delta_k|(A-B)}{2([2]_q - 1)}.$$

From (2.7), we get

$$|a_2| \le \frac{|\delta_k|(A-B)}{2([2]_q - 1)}.$$

For n=3, from (2.8), we get

$$|a_3| \le \frac{|\delta_k|(A-B)}{2([3]_q - 1)}(a_1 + a_2) \le \frac{|\delta_k|(A-B)}{2([3]_q - 1)}\left(1 + \frac{|\delta_k|(A-B)}{2([2]_q - 1)}\right).$$

Also from (2.7), we derive

$$|a_3| \le \frac{|\delta_k|(A-B)}{2([2]_q-1)} \frac{|\delta_k(A-B)-2([2]_q-1)B|}{2([3]_q-1)}$$

$$\le \frac{|\delta_k|(A-B)}{2([2]_q-1)} \frac{|\delta_k|(A-B)+2([2]_q-1)|B|}{2([3]_q-1)} \quad \le \frac{|\delta_k|(A-B)}{2([3]_q-1)}\left(\frac{|\delta_k|(A-B)}{2([2]_q-1)}+1\right).$$

Let the hypothesis be true for $n=m$.

From (2.8), we have

$$|a_m| \le \frac{|\delta_k|(A-B)}{2([m]_q-1)} \sum_{j=1}^{m-1} |a_j|, \quad a_1 = 1. \tag{2.10}$$

from (2.7), we have

$$|a_m| \le \prod_{j=1}^{m-1} \frac{\left|\delta_k(A-B)-2([j]_q-1)B\right|}{2([j+1]_q-1)} \qquad \le \prod_{j=1}^{m-1} \frac{\left|\delta_k(A-B)+2([j]_q-1)\right|}{2([j+1]_q-1)}$$

By the induction hypothesis, we have

$$\frac{|\delta_k|(A-B)}{2([m]_q-1)} \sum_{j=1}^{m-1} |a_j| \le \prod_{j=1}^{m-1} \frac{\left|\delta_k(A-B)+2([j]_q-1)\right|}{2([j+1]_q-1)}.$$

Multiplying both sides by $\dfrac{|\delta_k|(A-B)+2([m]_q-1)}{2([m+1]_q-1)}$, we get

$$\prod_{j=1}^{m} \frac{\left|\delta_k(A-B)+2([j]_q-1)\right|}{2([j+1]_q-1)} \ge \frac{|\delta_k|(A-B)}{2([m]_q-1)} \frac{|\delta_k|(A-B)+2([m]_q-1)}{2([m+1]_q-1)} \sum_{j=1}^{m-1} |a_j|$$

$$= \frac{|\delta_k|(A-B)}{2([m+1]_q-1)} \left\{ \frac{|\delta_k|(A-B)}{2([m]_q-1)} \sum_{j=1}^{m-1} |a_j| + \sum_{j=1}^{m-1} |a_j| \right\}$$

$$\ge \frac{|\delta_k|(A-B)}{2([m+1]_q-1)} \left\{ |a_m| + \sum_{j=1}^{m-1} |a_j| \right\} = \frac{|\delta_k|(A-B)}{2([m+1]_q-1)} \sum_{j=1}^{m} |a_j|.$$

Hence

$$\prod_{j=1}^{m} \frac{\left|\delta_k(A-B)+2([j]_q-1)\right|}{2([j+1]_q-1)} \ge \frac{|\delta_k|(A-B)}{2([m+1]_q-1)} \sum_{j=1}^{m} |a_j|.$$

which shows that the inequality (2.9) is true for $n=m+1$, and the result is true.

As $q \to 1$, we have following result proved by Khalida Inyat Noor and Sarfraz Nawaz Malik [8].

Corollary 2.5. *Let* $f(z) \in k-ST[A,B]$ *and of the form (1.1). Then, for* $n \ge 2$,

$$|a_n| \le \prod_{j=0}^{n-2} \frac{\left|\delta_k(A-B)-2jB\right|}{2(j+1)}, \tag{2.11}$$

where δ_k is defined by (0.3).

As $q \to 1$ and for $A=1$, $B=-1$, we have the following result,

Corollary 2.6. *A function* $f \in \mathcal{A}$ *and of the form (1.1) is in the class* $k-ST$ *, if it satisfies the condition*

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|\delta_k + j|}{j+1}, \quad n \ge 2, \tag{2.12}$$

which is the coefficient inequality of the class $k-ST$ introduced by Kanas and Wisniowska [6].

As $q \to 1$ and for $A = 1-2\alpha$, $B = -1$ with $0 \le \alpha < 1$, we get the following result,

Corollary 2.7. *A function* $f \in \mathcal{A}$ *and of the form (1.1) is in the class* $SD(k,\alpha)$, *if it satisfies the condition*

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|\delta_k(1-\alpha) + j|}{j+1}, \quad n \ge 2, \tag{2.13}$$

which is the coefficient inequality of the class $SD(k,\alpha)$, introduced by Shams et al. [10].

As $q \to 1$, and $k = 0$, then $\delta_k = 2$ and we get the following result, proved by Janowski [4].

Corollary 2.8. *A function* $f \in \mathcal{A}$ *and of the form (1.1) is in the class* $S^*[A,B]$, *if it satisfies the condition*

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|(A-B) - jB|}{j+1}, \quad n \ge 2, \quad -1 \le B < A < 1. \tag{2.14}$$

For $A = 1-2\alpha$, $B = -1$ with $0 \le \alpha < 1$, $k = 0$ and as $q \to 1$, we get the following result.

Corollary 2.9. *A function* $f \in \mathcal{A}$ *and of the form (1.1) is in the class* $S^*(\alpha)$, *if it satisfies the condition*

$$|a_n| \le \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!}, \quad n \ge 2. \tag{2.15}$$

Theorem 2.4. *Let* $f(z) \in k-UCV[A,B,q]$ *and of the form (1.1). Then, for* $n \ge 2$,

$$|a_n| \le \frac{1}{[n]_q} \prod_{j=1}^{n-1} \frac{|\delta_k(A-B) - 2([j]_q - 1)B|}{2([j+1]_q - 1)}, \tag{2.16}$$

where δ_k is defined by (1.3) and $0 < q < 1$.

Proof: The proof follows immediately by using Theorem 2.1 and (1.7).

As $q \to 1$, we have following result proved by Noor and Malik [8].

Corollary 2.10 *Let* $f(z) \in k-UCV[A,B]$ *and of the form (1.1). Then, for* $n \ge 2$,

$$|a_n| \leq \frac{1}{n}\prod_{j=0}^{n-2}\frac{\left|\delta_k(A-B)-2jB\right|}{2(j+1)}, \tag{2.17}$$

where δ_k is defined by (1.3) and $0 < q < 1$.

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