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Common Fixed Point Theorem for Weakly Compatible Mappings in Metric Space

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Abstract. In this paper we prove a common fixed point theorem for pairs of weakly compatible mappings in a metric space which generalizes the existing result.

Keywords: Fixed point, self maps, weakly compatible mappings, associated sequence.

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1. Introduction

In 1976, Jungck proved some common fixed point theorems for commuting maps which generalize the Banach contraction principle. Further this result was generalized and extended in various ways by several authors. On the other hand Sessa [5] introduced the concept of weak commutativity and proved a common fixed point theorem for weakly commuting maps. In 1986, G.Jungck[1] introduced the concept of compatible maps which is more general than that of weakly commuting maps. Afterwards Jungck and Rhoades [4] introduced the notion of weakly compatible and proved that compatible maps are weakly compatible but not conversely.

The purpose of this paper is to prove a common fixed point theorem for four self maps in metric space using weaker conditions such as weakly compatible mappings and associated sequence related to four self maps.

2. Definitions and Preliminaries

Definition 2.1. [1] Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.2. [4] Two self maps S and T of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence point. i.e., if $Su = Tu$ for some $u \in X$ then $STu = TSu$.

Definition 2.3. [9,11] Suppose P, Q, S and T are self maps of a metric space (X,d) such that $S(X)\subset Q(X)$ *and* $T(X)\subset P(X)$. Now for any arbitrary $x_0\in X$, we have $Sx_0\in S(X)\subset Q(X)$ so that there is a $x_1\in X$ such that $Sx_0=Qx_1$ and for this x_1, there is a point $x_2\in$ X such that $Tx_1=Px_2$ and so on. Repeating this process to obtain a sequence $\{y_n\}$ in X such that $y_{2n}=Px_{2n}=Tx_{2n-1}$ *and* $y_{2n+1}=Qx_{2n+1}=Sx_{2n}$ for $n\geq 0$.

We shall call this sequence an associated sequence of x_0 relative to the four self maps P,Q,S and T.

Lemma 2.4. Let P, Q, S and T be self mappings of a metric space (X,d) satisfying

$$S(X)\subset Q(X) \text{ and } T(X)\subset P(X) \tag{2.4.1}$$

and $$d(Sx,Ty)\leq\left[\alpha+\beta\frac{d(\mathrm{S}x,\mathrm{P}x)}{1+d(Px,Qy)}\right]d(\mathrm{Ty},Qy) \tag{2.4.2}$$

for all x,y in X, where α,β ≥ 0,α+β<1.

Further if X is complete,then for any $x_0\in$ X and for any of its associated sequence $\{y_n\}=\{Sx_0,Tx_1,Sx_2,Tx_3,.....Sx_{2n},Tx_{2n+1},.....\}$ relative to four self maps, converges to some point in X.

Proof: From (2.3) and (2.4.2) , we have

$$\begin{aligned}
d(y_{2n},y_{2n+1}) &= d(Tx_{2n-1},Sx_{2n})\\
&= d(Sx_{2n},Tx_{2n-1})\\
&\leq\left[\alpha+\beta\frac{d(\mathrm{S}x_{2n},\mathrm{P}x_{2n})}{1+d(Px_{2n},Qx_{2n-1})}\right]d(\mathrm{Tx}_{2n-1},Qx_{2n-1})\\
&=\left[\alpha+\beta\frac{d(\mathrm{y}_{2n+1},\mathrm{y}_{2n})}{1+d(\mathrm{y}_{2n},\mathrm{y}_{2n-1})}\right]d(\mathrm{y}_{2n},\mathrm{y}_{2n-1})\\
&\leq\alpha d(\mathrm{y}_{2n},\mathrm{y}_{2n-1})+\beta d(\mathrm{y}_{2n+1},\mathrm{y}_{2n}) \text{ implies}
\end{aligned}$$

$(1-\beta)d(y_{2n},y_{2n+1})\leq\alpha d(y_{2n-1},y_{2n})$ so that

$$d(y_{2n},y_{2n+1})\leq\frac{\alpha}{(1-\beta)}d(y_{2n-1},y_{2n}) = \mathrm{h}\,d(y_{2n-1},y_{2n}),\text{where } h=\frac{\alpha}{1-\beta}.$$

That is, $d(y_{2n},y_{2n+1})\leq h\,d(y_{2n-1},y_{2n})$. (2.4.4)

Similarly, we can prove that $d(y_{2n+1},y_{2n+2})\leq h\,d(y_{2n},y_{2n+1})$. (2.4.4)

Hence, from (2.6.3) and (2.6.4), we get

$$d(y_n,y_{n+1})\leq h\,d(y_{n-1},y_n)\leq h^2d(y_{n-2},y_{n-1})\leq.......\leq h^nd(y_0,y_1)\ . \tag{2.4.5}$$

Now , for any positive integer p we have

$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots\ldots.. + d(y_{n+p-1}, y_{n+p})$$

$$\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + \ldots\ldots.. + h^{n+p-1} d(y_0, y_1)$$

$$= (h^n + h^{n+1} + \ldots\ldots... + h^{n+p-1}) d(y_0, y_1)$$

$$= h^n (1 + h + h^2 + \ldots\ldots. + h^{p-1}) d(y_0, y_1)$$

$$< \frac{h^n}{1-h} d(y_0, y_1) \to 0 \quad \text{as } n \to \infty, \text{since } h<1.$$

Thus the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete , the sequence $\{y_n\}$ converges to some point z in X.

Remark 2.5. The converse of the above Lemma is not true. That is, if P,Q,S and T are self maps of a metric space (X, d) satisfying (2.4.1) , (2.4.2) and even if for any x_0 in X and for any of its associated sequence converges, then the metric space (X, d) need not be complete.

Example 2.6. Let $X = (0,1)$ with $d(x, y) = |x - y|$ for $x, y \in X$. Define the self maps S,T,P and Q on X by

$$Sx = Tx = \begin{cases} \frac{1}{2} & if\ 0 < x \leq \frac{1}{2} \\ \frac{2}{3} & if\ \frac{1}{2} < x \leq 1 \end{cases} ,$$

$$Px = Qx = \begin{cases} 1-x & if\ 0 < x \leq \frac{1}{2} \\ \frac{1}{3} & if\ \frac{1}{2} < x \leq 1 \end{cases} .$$

Then $S(X) = T(X) = \left\{\frac{1}{2}, \frac{2}{3}\right\}$ while $P(X) = Q(X) = \left[\frac{1}{2}, 1\right) \cup \left\{\frac{1}{3}\right\}$.

Clearly $S(X) \subset Q(X)$ *and* $T(X) \subset P(X)$. It is also easy to see that the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \ldots..Sx_{2n}, Tx_{2n+1}, \ldots..$ converges to $\frac{1}{2}$. Also the inequality (2.6.2) holds for $\alpha, \beta \geq 0, \alpha + \beta < 1$. Note that (X, d) is not complete.

Now we generalize the result of Sharma et al. in the following section.

3. Main result

Theorem 3.1. Let P,Q,S and T be self maps of a metric space (X, d) such that

$$S(X) \subset Q(X) \text{ and } T(X) \subset P(X) \tag{3.1.1}$$

$$d(Sx, Ty) \leq \left[\alpha + \beta \frac{d(Sx, Px)}{1 + d(Px, Qy)}\right] d(Ty, Qy) \tag{3.1.2}$$

for all x,y in X where $\alpha,\beta \geq 0, \alpha+\beta<1$.
one of S(X) and T(X) is closed subset of X (3.1.3)
and the pairs (S,P) and (T,Q) are weakly compatible (3.1.4)

Further if there is point $x_0 \in X$ and an associated sequence
$Sx_0, Tx_1, Sx_2, Tx_3 ......Sx_{2n}, Tx_{2n+1}...$ of x_0 relative to four self maps P, Q, S and T
converges to some point $z \in X$, (3.1.5)
then z is a unique common fixed point of P,Q,S and T.

Proof: From (3.1.5), we have
$Sx_{2n} \to z$ and $Tx_{2n+1} \to z$ as $n \to \infty$. (3.1.6)

Let S(X) be a closed subset of X. Then $z \in S(X)$.
Now, since $S(X) \subset Q(X)$, there exists a $u \in X$ such that z = Qu .

So by (3.1.2), we have

$$d(Sx_{2n}, Tu) \leq \left[\alpha + \beta \frac{d(Sx_{2n}, Px_{2n})}{1 + d(Px_{2n}, Qu)} \right] d(Tu, Qu)$$

Letting $n \to \infty$, we obtain

$$d(z, Tu) \leq [\alpha + 0] d(Tu, z)$$

$$= \alpha \, d(z, Tu) \leq d(z, Tu)$$, a contradiction since $\alpha < 1$.

Thus we have Tu = z.

Hence Qu = Tu = z .
Since the pair (T,Q) is weakly compatible mappings, we have TQu = QTu and so that Tz = Qz.

Again from (3.1.2), we get

$$d(Sx_{2n}, Tz) \leq \left[\alpha + \beta \frac{d(Sx_{2n}, Px_{2n})}{1 + d(Px_{2n}, Qz)} \right] d(Tz, Qz)$$

Letting $n \to \infty$ and using Tz = Qz , we obtain

$d(z, Tz) \leq 0$, a contradiction .

Thus we have Tz = z.

Since $T(X) \subset P(X)$, there exists a v$\in X$ such that z = Pv .

Hence from (3.1.2) ,we get

$$d(Sv,Tz) \leq \left[\alpha + \beta \frac{d(\mathrm{S}v,\mathrm{P}v)}{1+d(Pv,Qz)} \right] d(\mathrm{Tz},Qz)$$

Using Tz = z , we obtain

$d(\mathrm{S}v,z) \leq 0$, a contradiction .

Thus we have Sv = z .

Hence Sv = Pv = z.

Since the pair (P,S) is weakly compatible mappings, we have SPv = PPv and so that Sz = Pz.

Now from (3.1.2), we get

$$\begin{aligned} d(Sz,Tz) &= d(Sz,z) \\ &\leq \left[\alpha + \beta \frac{d(\mathrm{Sz},\mathrm{Pz})}{1+d(Pz,Qz)} \right] d(\mathrm{z},Qz) \\ &\leq 0 \text{ , a contradiction.} \end{aligned}$$

Thus we have Sz = z.

Therefore $Sz = Pz = Qz = Tz = z$, showing that z is a common fixed point of P,Q,S and T.

Uniqueness: Let z and w be two common fixed points of P,Q,S and T. Then we have $z = Sz = Pz = Qz = Tz$ and $w = Sw = Pw = Qw = Tw$.

Using (3.1.2) , we get

$$d(Sz,Tw) \leq \left[\alpha + \beta \frac{d(\mathrm{Sz},\mathrm{Pz})}{1+d(Pz,Qw)} \right] d(\mathrm{Tw},Qw) \text{ implies}$$

d(z,w) ≤ 0 , a contradiction.

Thus we have $d(z,w) = 0$ and so that $z = w$.

Hence z is a unique common fixed point of P,Q,S and T.

Remark 3.2. It is easy to verify that the self mappings P,Q,S and T defined in the example (2.6) satisfy all the conditions of the Theorem (3.1) . It may be noted that ‘ $\frac{1}{2}$ ’ is the unique common fixed point of P, Q, S and T.

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