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Oscillatory Behavior of Solutions of Certain Fourth-Order Nonlinear Neutral Delay Difference Equations

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Abstract. Some new oscillation criteria are obtained for the fourth-order nonlinear neutral delay difference equation of the form $\Delta^3\left(p_n\Delta\left(y_n+q_n y_{n-1}\right)\right)+r_n y_{n-1}=0$. Examples are inserted to illustrate the result.

Keywords: Nonlinear difference equations, oscillation, neutral delay.

1. Introduction

The notion of nonlinear difference equation was studied intensively by R.P.Agarwal[1]. Recently there has been a lot of interest in the study of oscillatory behavior of solutions of nonlinear neutral delay difference equations. B.Selvaraj and I. Mohammed Ali Jaffer[14] considered the fourth order nonlinear neutral delay difference equation of the form $\Delta\left(c_n\Delta^2\left(a_n\Delta\left(y_n+b_n y_{n-\tau}\right)\right)\right)+q_n f\left(y_{n-\sigma}\right)=0$. Motivated by the references cited in [1 - 19], in this paper, we discussed some new oscillation criteria for the forth-order nonlinear neutral delay difference equation of the form

$$\Delta^3\left(p_n\Delta\left(y_n+q_n y_{n-1}\right)\right)+r_n y_{n-1}=0,\ n\geq n_0, \tag{1.1}$$

where $n_0\in N(n_0)=\{n_0,n_0+1,n_0+2,\ldots\}$ and Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, and

$$p_n>0,\ q_n>0,\ r_n\neq 0 \text{ and } \sum_{n=n_0}^{\infty}\frac{1}{p_n}=\sum_{n=n_0}^{\infty}\frac{1}{q_n}=\infty \text{ and } \sum_{n=n_0}^{\infty}\frac{1}{r_n}<\infty,\ \text{ for } n\geq n_0.$$

By a solution of equation (1.1) we mean a real sequence $\{y_n\}$ which satisfies the equation (1.1) for all $n\geq n_0$, where $n_0\geq 0$. We recall that a nontrivial solution of equation (1.1) is said to be oscillatory if for every $M>0$, there exists an integer $n\geq M$ such that $y_n y_{n+1}\leq 0$; otherwise it is said to be nonoscillatory. Thus a nonoscillatory solution is either eventually positive or eventually negative.

2. Main Results

In this section, we concern some sufficient condition for oscillatory behavior of solutions of equation (1.1) and also we deduce some results from the main results.

Lemma 1. If y_n is an eventually positive solution of equation (1.1) and $z_n = y_n + q_n y_{n-1}$, then for all $n>n_0$, there are only two possible cases.

Case (i): $z_n > 0$; $\Delta z_n>0$; $\Delta(p_n\Delta z_n)>0$; $\Delta^2(p_n\Delta z_n)>0$.

Case (ii): $z_n < 0$; $\Delta z_n< 0$; $\Delta(p_n\Delta z_n)>0$; $\Delta^2(p_n\Delta z_n)>0$.

Proof: Let y_n be an eventually positive solution of equation (1.1), then there exists $n_1\geq n_0$ such that $y_{n-1}>0$ for $n\geq n_0$. Then in view of the assumption and definition of z_n, we have $z_n>0$ for all $n\geq n_1$. Thus $\Delta z_n>0$ and z_n are eventually of one sign.

We claim that $\Delta^2(p_n\Delta z_n)>0$ for all large n. (2.1)

We prove the result by contradiction. Suppose that $\Delta^2(p_n\Delta z_n)\leq 0$ for all large n.

It is clear that there is an integer $n_2\geq n_1$ such that $\Delta^2\left(p_n\Delta z_n\right)\leq \Delta^2\left(p_{n_2}\Delta z_{n_2}\right)<0$. (2.2)

Summing the inequality (2.2) from n_2 to $n-1$, we obtain

$$\Delta\left(p_n\Delta z_n\right)\leq -k_1,\ \text{where}\ k_1>0\ \text{is an integer for all}\ n\geq n_3\geq n_2. \tag{2.3}$$

Summing the inequality (2.3) from n_3 to $n-1$, we obtain

$$p_n\Delta z_n\leq -k_2,\ \text{where}\ k_2>0\ \text{is an integer for all}\ n\geq n_4\geq n_3. \tag{2.4}$$

Summing the inequality (2.4) from n_4 to $n-1$, we have

$$z_n\leq -k_2 z_{n_4}\sum_{s=n_4}^{\infty}\frac{1}{p_s}$$

Hence, $z_n\to-\infty$ as $n\to\infty$. Thus there exists an integer $n_5>0$ such that $z_n<-k_3$ for $n\geq n_4$, where $k_3>0$ is a real number. This is a contradiction to the fact that $z_n>0$ for all large $n\geq n_0$. This completes the proof.

Lemma 2. If y_n is an eventually positive solution of equation (1.1), and if case(i) of Lemma1 holds. Then there exists sufficiently large $n_1\geq n_0$ such that $p_{n-1}\Delta z_{n-1}\geq \Delta^2\left(p_n\Delta z_n\right)$, for sufficiently large n.

Proof: From Case(i) of Lemma1 and equation (1.1), we have, for $n\geq n_1$, $z_n > 0$; $\Delta z_n>0$; $\Delta(p_n\Delta z_n)>0$; $\Delta^2(p_n\Delta z_n)>0$ and $\Delta^3(p_n\Delta z_n)<0$.

Since $\sum_{s=n_1}^{n-1}\Delta^2(p_n\Delta z_n)=\Delta(p_n\Delta z_n)-\Delta(p_{n_1}\Delta z_{n_1})$, for $n\ge n_1$, we obtain

$\Delta(p_n\Delta z_n)\ge\Delta^2(p_n\Delta z_n)$.

Summing the above inequality from n_2 to $n-1$, for $n_2\ge n_1$, we have $p_n\Delta z_n\ge p_{n_2}\Delta z_{n_2}+\sum_{s=n_2}^{n-1}\Delta^2(p_n\Delta z_n)$, for $n\ge n_2$.

This implies that $p_n\Delta z_n\ge\Delta^2(p_n\Delta z_n)$, *for* $n\ge n_2$.

Since $\Delta^3(p_n\Delta z_n)<0$, we have $\Delta^2(p_{n-1}\Delta z_{n-1})\ge\Delta^2(p_n\Delta z_n)$, for $n\ge n_2$. It follows that for $n\ge n_2=n_1+1$ sufficiently large, $p_{n-1}\Delta z_{n-1}\ge\Delta^2(p_n\Delta z_n)$. This completes the proof of the Lemma.

Theorem 1. If the condition $\limsup_{n\to\infty}\sum_{s=n_1}^{n-1}\frac{1}{p_s}\sum_{t=s}^{n-1}\sum_{u=t}^{n-1}\sum_{i=u}^{n-1}r_i=\infty$ hold. Then every solution of equation (1.1) is oscillatory.

Proof: Let y_n be a non-oscillatory solution of equation (1.1). Without loss of generality we may assume that $y_n>0$, $y_{n-1}>0$ for $n\ge n_1$, where $n_1\ge n_0$ is chosen so large that Lemma 1 and Lemma 2 holds. From Lemma1, there are two possible cases.

Case (i): $\Delta z_n>0$ for $n\ge n_1\ge n_0$.

In this case, we define the function w_n by $w_n=\frac{\Delta^2(p_n\Delta z_n)}{z_{n-1}}$ for $n\ge n_1$.

Then $\Delta w_n=\frac{\Delta^3(p_n\Delta z_n)}{z_n}-\frac{(\Delta z)^2\Delta^2(p_n\Delta z_n)}{z_n z_{n-1}}$. From equation (1.1) and Lemma 2, we find that $\Delta w_n\le-\frac{r_n y_{n-1}}{z_n}$. Now we can find a real number K such that $\Delta w_n<-K$, for all $n\ge n_2\ge n_1$. Summing the above inequality from n_2 to $n-1$, we obtain

$w_n\to-\infty$ as $n\to\infty$. This is a contradiction to the fact that y_n is a positive solution of equation (1.1).

Case (ii): $\Delta z_n<0$ for $n\ge n_1\ge n_0$.

Since $\Delta z_n<0$, by the definition of z_n, $y_n>0$ and decreasing for all $n\ge n_1$. Summing the equation (1.1) from n_1 to $n-1$, we obtain

$$\Delta^2(p_n\Delta z_n)+\sum_{s=n_1}^{n-1}r_s y_{s-1}\le 0\text{ for all }n\ge n_1. \tag{2.5}$$

Summing the inequality (2.5) from n_2 to n−1, we obtain

$$\Delta\left(p_n \Delta z_n\right)+\sum_{s=n_1}^{n-1}\sum_{t=s}^{n-1} r_t y_{t-1} \leq 0 \text{ for all } n \geq n_2 \geq n_1 . \tag{2.6}$$

Summing the inequality (2.6) from n_3 to n−1, we obtain

$$p_n \Delta z_n + \sum_{s=n_1}^{n-1}\sum_{t=s}^{n-1}\sum_{u=t}^{n-1} r_u y_{u-1} \leq 0 \text{ for all } n \geq n_3 \geq n_2 . \tag{2.7}$$

Summing the inequality (2.7) from n_4 to n−1, we obtain

$$z_n + \sum_{s=n_1}^{n-1}\frac{1}{p_s}\sum_{t=s}^{n-1}\sum_{u=t}^{n-1}\sum_{i=u}^{n-1} r_i y_{i-1} \leq 0 \text{ for all } n \geq n_4 \geq n_3 . \tag{2.8}$$

From the inequality (2.8), we obtain $\sum_{s=n_1}^{n-1}\frac{1}{p_s}\sum_{t=s}^{n-1}\sum_{u=t}^{n-1}\sum_{i=u}^{n-1} r_i \leq q_n$ for all $n \geq n_5 \geq n_4$. This is a contradiction to the assumption $\limsup_{n\to\infty}\sum_{s=n_1}^{n-1}\frac{1}{p_s}\sum_{t=s}^{n-1}\sum_{u=t}^{n-1}\sum_{i=u}^{n-1} r_i = \infty$.

The proof is similar to the case, when $y_n<0$. Thus theorem is completely proved.

Remark 1. The following Example illustrates the result of Theorem1.

Example 1. Consider the difference equation

$$\Delta^3\left(n\Delta\left(y_n + (n+1)y_{n-1}\right)\right) + n(7-n)y_{n-1} = 0 \;,\; n \geq 1 . \tag{E1}$$

Here $p_n = n$, $q_n = n+1$, $r_n = n(7-n)$ and all the conditions of the Theorem1 are satisfied.

Hence all solutions of equation (E1) are oscillatory. In fact, $\{y_n\}= \{(-1)^n\}$ is one such a solution of equation (E1).

Corollary 1. Every solution of equation (1.1) is oscillatory if the condition either $\sum_{n=n_0}^{\infty} p_n < \infty$, or $\sum_{n=n_0}^{\infty} r_n = \infty$ holds, for $n \geq n_0$,

and $p_n > 0$, $q_n > 0$, $r_n \neq 0$, $\sum_{n=n_0}^{\infty}\frac{1}{q_n} = \infty$, for $n \geq n_0$.

Remark 2. In the given assumptions, if the condition either

$\sum_{n=n_0}^{\infty} p_n < \infty$, or $\sum_{n=n_0}^{\infty} r_n = \infty$ holds, for $n \geq n_0$, then the sufficient condition

$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1} \frac{1}{p_s} \sum_{t=s}^{n-1} \sum_{u=t}^{n-1} \sum_{i=u}^{n-1} r_i = \infty$ holds. Hence every solution of equation (1.1) is oscillatory. The following example illustrates the result of the Corollary 1.

Example 2. Consider the difference equation

$$\Delta^3\left(\frac{1}{n^2}\Delta(y_n + ny_{n-1})\right) + \left(\frac{-8n^{10} - 92n^9 - 420n^8 - 908n^7 - 672n^6 + 848n^5 + 1892n^4 + 1016n^3 - 160n^2 - 192n}{(n-1)n^3(n+1)^3(n+2)^3(n+3)}\right) y_{n-1} = 0, \quad n \geq 2. \tag{E2}$$

Here $p_n = \frac{1}{n^2}$; $q_n = n$;

$$r_n = \frac{-8n^{10} - 92n^9 - 420n^8 - 908n^7 - 672n^6 + 848n^5 + 1892n^4 + 1016n^3 - 160n^2 - 192n}{(n-1)n^3(n+1)^3(n+2)^3(n+3)}$$

Hence all solutions of equation ($E2$) are oscillatory. In fact, $\{y_n\} = \{n(-1)^n\}$ is one such a solution of equation ($E2$) .

Corollary 2. Every solution of equation (1.1) is oscillatory if both the conditions $\sum_{n=n_0}^{\infty} p_n = \infty$, and $\sum_{n=n_0}^{\infty} r_n = \infty$ hold, for $n \geq n_0$,

and $p_n > 0$, $q_n > 0$, $r_n \neq 0$, $\sum_{n=n_0}^{\infty} \frac{1}{q_n} = \infty$, for $n \geq n_0$.

Remark 3. In the given assumptions, if both the conditions

$\sum_{n=n_0}^{\infty} p_n = \infty$, and $\sum_{n=n_0}^{\infty} r_n = \infty$, for $n \geq n_0$ hold, then the sufficient condition

$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1} \frac{1}{p_s} \sum_{t=s}^{n-1} \sum_{u=t}^{n-1} \sum_{i=u}^{n-1} r_i = \infty$ holds. Hence every solution of equation (1.1) is oscillatory. The following Example illustrates the result of the Corollary 2.

Example 3. Consider the difference equation

$$\Delta^4\left(y_n + y_{n-1}\right) + \left(\frac{-n^5 + 1103n^4 - 1245n^3 - 463n^2 - 546n + 1152}{(n-1)n(n+1)(n+2)(n+3)(n+4)}\right)y_{n-1} = 0,\ n \geq 2.$$

Here $p_n = 1;\ q_n = 1;\ r_n = \dfrac{-n^5 + 1103n^4 - 1245n^3 - 463n^2 - 546n + 1152}{(n-1)n(n+1)(n+2)(n+3)(n+4)}$ (E3)

Hence all solutions of equation (*E*3) are oscillatory. In fact, $\{y_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$ is one such a solution of equation (E3).

Proposition 1. Every oscillation solution of equation (1.1) is bounded if the sufficient condition $\sum_{n=n_0}^{\infty} \frac{1}{r_n} < \infty$, for $n \geq n_0$ trivially holds.

Proposition 2. Every oscillation solution of equation (1.1) is unbounded if the condition either $\sum_{n=n_0}^{\infty} p_n < \infty$, or $\sum_{n=n_0}^{\infty} r_n = \infty$ holds, for $n \geq n_0$.

Proposition 3. Every oscillation solution of equation (1.1) is asymptotic if both the conditions $\sum_{n=n_0}^{\infty} p_n = \infty$, and $\sum_{n=n_0}^{\infty} r_n = \infty$ hold, for $n \geq n_0$.

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