Progress in Nonlinear Dynamics and Chaos
Vol. 1, 2013, 34-38
ISSN: 2321 – 9238 *(online)*
Published on 31 May 2013
www.researchmathsci.org

Oscillation of Solutions of Certain Nonlinear Difference Equations

B. Selvaraj **and** ***S. Kaleeswari***

Department of Science and Humanities,
Nehru Institute of Engineering and Technology,
Coimbatore, Tamil Nadu, India – 641 105.
email: professorselvaraj@gmail.com, kaleesdesika@gmail.com

Received 15 May 2013; accepted 24 May 2013

Abstract. In this paper, some sufficient conditions for the oscillation of all solutions of the nonlinear difference equation of the form

$$\Delta^2 x_n + f(x_n) = 0, \qquad n = 0,1,2,.......$$

are obtained. Examples are given to illustrate the results.

Keywords: Difference equations, oscillation, nonlinear.

1. Introduction

The notion of nonlinear difference equation was studied intensively by Agarwal [1] and oscillatory properties were discussed by Agarwal et al. [2]. Recently there has been a lot of interest in the study of oscillatory behavior of solutions of nonlinear difference equations. We can see this in [3]-[9].

Szafranski and Szmanda [10] considered the second order nonlinear difference equation of the form

$$\Delta\left(r_n \Delta x_n\right) + q_n f\left(x_{n-\tau_n}\right) = 0, \; n = 0,1,2,.....$$

and gave the sufficient conditions for the oscillation of solutions of the equation considered. Motivated by this article, we consider the nonlinear difference equation of the form

$$\Delta^2 x_n + f(x_n) = 0, \qquad n = 0,1,2,....... \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $f : R \to R$ is continuous with $xf(x) > 0$ for $x \neq 0$. Our aim in this paper is to obtain some new oscillation criteria for the solution of equation (1).

By a solution of equation (1), we mean a real sequence $\{x_n\}$ which satisfies equation (1) for all large n. A solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory.

2. Main results

In this section, we present some sufficient conditions for the oscillation of all the solutions of the equation (1).

Theorem 2.1. Every solution of equation (1) is oscillatory, if the sufficient condition $\lim_{n\to\infty}\inf \mathrm{f}\left(x\right) > 0$ holds.

Proof: Assume that equation (1) has non-oscillatory solution $\{x_n\}$ and we assume that $\{x_n\}$ is eventually positive.

Then there exists a positive integer n_0 such that

$$x_n > 0 \text{ for } n \geq n_0. \tag{2}$$

From (1), $\Delta^2 x_n = -f\left(x_n\right) \leq 0, \qquad n \geq n_0$

Therefore Δx_n is an eventually non-increasing sequence.

We first show that $\Delta x_n \geq 0$ for $n \geq n_0$.

In fact, if there is an $n_1 \geq n_0$ such that $\Delta x_{n_1} = c < 0$ and $\Delta x_n \leq c$ for $n \geq n_1$.

Summing the last inequality from n_1 to $n-1$, we get

$$\sum_{k=n_1}^{n-1} \Delta x_n \leq \sum_{k=n_1}^{n-1} c,$$

which implies, $x_n - x_{n_1} \leq \left(n - n_1\right)c$.

Therefore, $x_n \leq x_{n_1} + \left(n - n_1\right)c \to -\infty$ as $n \to \infty$,

which is a contradiction to the fact that $x_n > 0$ for $n \geq n_0$.

Hence $\Delta x_n \geq 0$ for $n \geq n_0$.

Therefore we have $x_n > 0$, $\Delta x_n \geq 0$, $\Delta^2 x_n \leq 0$ for $n \geq n_0$.

Let $L = \lim_{n\to\infty} x_n$.

Then $L > 0$ is finite or infinite.

Case (i): $L > 0$ is finite.

Since f is continuous function, we have

$$\lim_{n\to\infty} f\left(x_n\right) = f\left(L\right) > 0. \tag{A}$$

This implies we can choose a positive integer $n_2 \geq n_0$ such that

$$f\left(x_n\right) > \frac{1}{2} f\left(L\right), \qquad n \geq n_2 \tag{3}$$

Substituting (3) in (1), we get

$$\Delta^2 x_n = -f(x_n)$$
$$< -\frac{1}{2} f(L),$$

which implies,

$$\Delta^2 x_n + \frac{1}{2} f(L) \leq 0, \qquad n \geq n_2. \tag{4}$$

Summing up both sides of (4) from n_2 to n, we obtain

$$\sum_{i=n_2}^{n} \Delta(\Delta x_i) + \frac{1}{2} f(L) \sum_{i=n_2}^{n} 1 \leq 0.$$

This implies,

$$\Delta x_{n+1} - \Delta x_{n_3} + \frac{1}{2} f(L)(n+1-n_2) \leq 0.$$

Hence $$\frac{1}{2} f(L) \leq \frac{1}{(n+1-n_2)} \Delta x_{n_2}, \qquad n \geq n_2.$$

That is, $$f(L) \leq 0 \qquad as \qquad n \to \infty,$$

which is a contradiction to (A).

Case (ii): $L = \infty$

Since $\liminf_{n\to\infty} \mathrm{f}(x_n) > 0$, we may choose a positive constant c and a positive integer n_3 such that

$$f(x_n) \geq c \text{ for } n \geq n_3. \tag{5}$$

Substituting (5) in (1), we get

$$\Delta^2 x_n = -f(x_n) \leq -c.$$

This implies,

$$\Delta^2 x_n + c \leq 0, \qquad n \geq n_3.$$

Summing the last inequality from n_3 to n, we get

$$\Delta x_{n+1} - \Delta x_{n_4} + c(n+1-n_3) \leq 0.$$

So, $$\Delta x_{n+1} - \Delta x_{n_4} \leq -c(n+1-n_3).$$

Therefore, $$\Delta x_n \to -\infty \qquad as \qquad n \to \infty,$$

which is a contradiction to the fact that $\Delta x_n > 0$ and hence the proof.

Corollary 2.2. With the assumption of the above theorem, every bounded solution of equation (1) is oscillatory.

Proof: Proceeding as in the proof of Theorem 2.1 with assumption that $\{x_n\}$ is bounded non-oscillatory solution of (1), we get the inequality (4)

$$\Delta^2 x_n + \frac{1}{2} f\left(L\right) \le 0, \qquad n \ge n_2 \;.$$

Now, $\Delta^2 x_n \ge \Delta^2 x_n - \Delta x_n \,.$ (6)

From (4) and (6),

$$\Delta^2 x_n - \Delta x_n + \frac{1}{2} f\left(L\right) \le 0 \,.$$

Summing the last inequality from n_2 to n, we get

$$\sum_{i=n_2}^{n} \Delta^2 x_i - \sum_{i=n_2}^{n} \Delta x_i + \frac{1}{2} f\left(L\right) \sum_{i=n_2}^{n} 1 \le 0 \,.$$

That is,

$$\Delta x_{n+1} - \Delta x_{n_2} - x_{n+1} + x_{n_2} + \frac{1}{2} f\left(L\right)\left(n+1-n_2\right) \le 0 \,.$$

This implies,

$$\frac{1}{2} f\left(L\right)\left(n+1-n_2\right) \le x_{n+1} + \Delta x_{n_2} - x_{n_2} \,.$$

Since $\{x_n\}$ is bounded, we may choose a positive constant c such that

$$\frac{1}{2} f\left(L\right)\left(n+1-n_2\right) \le c \,, \tag{7}$$

which implies, $\frac{1}{2} f\left(L\right) \le \frac{1}{\left(n+1-n_2\right)} c \,.$

So, $f\left(L\right) \le 0 \qquad as \qquad n \to \infty \,,$

which contradicts (A). Hence the proof is complete.

Example 2.1. Consider the difference equation $\Delta^2 x_n + 4x_{n+1} = 0$. The condition of Theorem 2.1 is satisfied. Hence all solutions of equation (1) are oscillatory. One of the solutions is $\left(-1\right)^n$ which is oscillatory. The condition of corollary is also satisfied and the above solution is bounded one.

REFERENCES

1. R.P. Agarwal, *Difference Equation and Inequalities: Theory, Methods and Applications*, 2nd edition, Marcel–Dekker, Inc., New York, 2000.
2. R.P. Agarwal, M. Bohner, S. R. Grace and D.O'Regan, *Discrete Oscillation Theory*, CMIA Book Series, Vol. 1. ISBN: 977-5945-19-4, New York, 2005.
3. S.R. Grace, R.O. Agarwal and J.R. Greaf, Oscillation criteria for certain third order nonlinear difference equations, *Appl. Anal. Discrete. Math.*, 3 (2009) 27-28.
4. Sh. Salem and K.R. Raslan, Oscillation of some second order damped difference equations, *IJNS*, 5(3) (2008) 246-254.

5. E.Thandapani and B.Selvaraj, Existence and asymptotic behavior of non oscillatory solutions of certain nonlinear difference equations, *Far East Journal of Mathematical Sciences*, 14 (1) (2004) 9–25.
6. B. Selvaraj and I. Mohammed Ali Jaffer, Oscillation behavior of certain nonlinear difference equations, *Far East Journal of Mathematical Sciences*, 40 (2) (2009) 169–178.
7. B.Selvaraj and G.G.Jawahar, Non-oscillation of second order neutral delay difference equations, *J. Comp. and Math. Sci*., 1(5) (2010) 566–571.
8. B.Selvaraj and J.D.L. Lovenia, Oscillatory properties of certain first and second order difference equations, *J. Comp. and Math. Sci.*, 2(3) (2011) 567–571.
9. B. Selvaraj and S. Kaleeswari, Oscillation of solutions of certain fifth order difference equations, *J. Comp. and Math. Sci*., 3 (6) (2012) 653-663.
10. Zdzislaw Szafranski and Blazej Szmanda, Oscillation of solutions of some nonlinear difference equations, *Publicacions Matematiques*, 40 (1996) 127-133.