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Different Types of Product of Fuzzy Graphs

Shovan Dogra

Department of Applied Mathematics with Oceanology ComputerProgramming,
Vidyasagar University, Midnapore - 721102, India
Email: shovansd39@gmail.com

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Abstract. New fuzzy graphs can be obtained from two given fuzzy graphs using different types of fuzzy graph product. In this paper, we define modular, homomorphic, box dot and star fuzzy graph product and determine the degree of vertices of these new fuzzy graphs.

Keywords: fuzzy graph products; modular product of fuzzy graphs, dot product of fuzzy graphs

1. Introduction

Fuzzy graphs are very rich topic of applied mathematics, computer science, social sciences, medical sciences, engineering, etc. Fuzzy graph was introduced by Rosenfield in 1975. Fuzzy graphs can be used in traffic light problem, time table scheduling etc. Lots of works on fuzzy graphs have been done by Samanta, Pal, Rashmanlou, Nagoor Gani [10-32] and many others. The operations union, join, Cartesian product and composition on two fuzzy graphs were defined by Mordeson and Peng [9]. Later, Nirmala and Vijaya [11] determined the degree of vertices in the new fuzzy graphs obtained from two fuzzy graphs using the operations Cartesian, tensor, normal product and composition on two fuzzy graphs. Here, We have defined modular, homomorphic, box dot and star fuzzy graph product and proved some theorems related to each type of fuzzy graph product. Modular and homomorphic graph product were pre-defined in case of crisp graphs. We have defined them in case of fuzzy graphs. Box dot and star fuzzy graph products are newly defined fuzzy graph products. In general, the degree of a vertex in modular, homomorphic, box dot and star graph product of two fuzzy graphs G_1 and G_2cannot be expressed in terms of the degree of vertices of G_1 and G_2. Here, we have determined the degree of the vertices in the new fuzzy graphs in terms of the degree of vertices of the participated fuzzy graphs in the said product operations using some certain conditions.

2. Preliminaries

Definition 2.1. A fuzzy graph $G=(\sigma,\mu)$ is a pair of functions together with underlying vertex set V and underlying edge set E where $\sigma: V\rightarrow [0,1]$ and $\mu: V\times V \rightarrow [0,1]$ such that $\mu(u,v)\leq \sigma(u)\wedge\sigma(v)$ for all $u,v \in V$. Here $\sigma(u)\wedge\sigma(v)$ indicates the minimum among $\sigma(u)$ and $\sigma(v)$.

Definition 2.2. The degree of a vertex u in a fuzzy graph $G=(\sigma,\mu)$ is denoted by d (u) and defined as d (u,v) $=\sum_{v\neq u}\mu(u,v)$.

3. Product of fuzzy graphs

Cartesian product of fuzzy graphs

Let $G_1=(\sigma_1,\mu_1)$ and $G_2=(\sigma_2,\mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then Cartesian product of G_1 and G_2 is a pair of functions $(\sigma_1\times\sigma_2,\mu_1\times\mu_2)$ with underlying vertex set $V_1\times V_2=\{(u_1,v_1): u_1\in V_1 \text{ and } v_1\in V_2\}$ and underlying edge set $E_1\times E_2=\{((u_1,v_1)(u_2,v_2)): u_1=u_2, v_1v_2\in E_2 \text{ or } u_1u_2\in E_1, v_1=v_2\}$ with

$(\sigma_1\times\sigma_2)(u_1,v_1)=\sigma_1(u_1)\wedge\sigma_2(v_1)$, where $u_1\in V_1$ and $v_1\in V_2$.

$(\mu_1\times\mu_2)((u_1,v_1)(u_2,v_2))=\sigma_1(u_1)\wedge\mu_2(v_1v_2)$, if $u_1=u_2$ and $v_1v_2\in E_2$.

$=\mu_1(u_1u_2)\wedge\sigma_2(v_1)$, if $u_1u_2\in E_1$ and $v_1=v_2$.

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Tensor product of fuzzy graphs

Let $G_1=(\sigma_1,\mu_1)$ and $G_2=(\sigma_2,\mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then tensor product of G_1 and G_2 is a pair of functions $(\sigma_1\otimes\sigma_2,\mu_1\otimes\mu_2)$ with underlying vertex set $V_1\otimes V_2=\{(u_1,v_1): u_1\in V_1 \text{ and } v_1\in V_2\}$ and underlying edge set $E_1\otimes E_2=\{((u_1,v_1)(u_2,v_2)): u_1u_2\in E_1, v_1v_2\in E_2\}$ with $(\sigma_1\otimes\sigma_2)(u_1,v_1)=\sigma_1(u_1)\wedge\sigma_2(v_1)$, where $u_1\in V_1$ and $v_1\in V_2$.

$(\mu_1\otimes\mu_2)((u_1,v_1)(u_2,v_2))=\mu_1(u_1u_2)\wedge\mu_2(v_1v_2)$, if $u_1u_2\in E_1$ and $v_1v_2\in E_2$.

Normal product of fuzzy graphs

Let $G_1=(\sigma_1,\mu_1)$ and $G_2=(\sigma_2,\mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then normal product of G_1 and G_2 is a pair of functions $(\sigma_1\circ\sigma_2,\mu_1\circ\mu_2)$ with underlying vertex set $V_1\circ V_2=\{(u_1,v_1): u_1\in V_1 \text{ and } v_1\in V_2\}$ and underlying edge set $E_1\circ E_2=\{((u_1,v_1)(u_2,v_2)): u_1=u_2, v_1v_2\in E_2$ or $u_1u_2\in E_1, v_1=v_2$ or $u_1u_2\in E_1, v_1v_2\in E_2\}$ with

$(\sigma_1\circ\sigma_2)(u_1,v_1)=\sigma_1(u_1)\wedge\sigma_2(v_1)$, where $u_1\in V_1$ and $v_1\in V_2$.

$(\mu_1\circ\mu_2)((u_1,v_1)(u_2,v_2))=\sigma_1(u_1)\wedge\mu_2(v_1v_2)$, if $u_1=u_2$ and $v_1v_2\in E_2$.

$=\mu_1(u_1u_2)\wedge\sigma_2(v_1)$, if $u_1u_2\in E_1$ and $v_1=v_2$.

$=\mu_1(u_1u_2)\wedge\mu_2(v_1v_2)$, if $u_1u_2\in E_1$ and $v_1v_2\in E_2$.

4. New products of fuzzy graphs

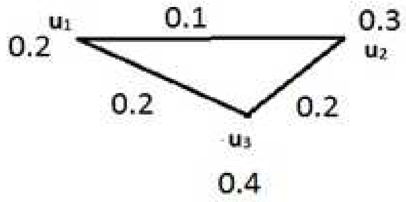
4.1. Modular product of fuzzy graphs

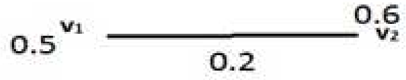
Definition 4.1. Let $G_1=(\sigma_1,\mu_1)$ and $G_2=(\sigma_2,\mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then modular product of G_1 and G_2 is a pair of functions $(\sigma_1\circledcirc\sigma_2,\mu_1\circledcirc\mu_2)$ with underlying vertex set $V_1\circledcirc V_2=\{(u_1,v_1): u_1\in V_1 \text{ and } v_1\in V_2\}$ and underlying edge set $E_1\circledcirc E_2=\{((u_1,v_1)(u_2,v_2)): u_1u_2\in E_1, v_1v_2\in E_2$ or $u_1u_2\notin E_1, v_1v_2\notin E_2\}$ with

$(\sigma_1\circledcirc\sigma_2)(u_1,v_1)=\sigma_1(u_1)\wedge\sigma_2(v_1)$, where $u_1\in V_1$ and $v_1\in V_2$.

$(\mu_1\circledcirc\mu_2)((u_1,v_1)(u_2,v_2))=\mu_1(u_1u_2)\wedge\mu_2(v_1v_2)$, if $u_1u_2\in E_1$ and $v_1v_2\in E_2$.

$=\sigma_1(u_1)\wedge\sigma_1(u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2)$, if $u_1u_2\notin E_1$ and $v_1v_2\notin E_2$.





G_1 G_2

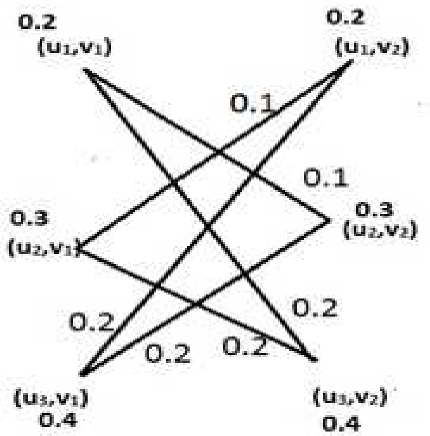


Figure 4.1: Modular fuzzy graph product $G_1 \odot G_2$

Theorem 4.1. The modular product of any two fuzzy graphs is again a fuzzy graph.

Proof: Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. We want to prove that their normal product $G_1 \circ G_2 = (\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ with underlying vertex set $V_1 \odot V_2$ and edge set $E_1 \circ E_2$ is again a fuzzy graph. Let $((u_1,v_1)\ (u_2,v_2)) \in E_1 \circ E_2$. From definition, it follows that

$(\mu_1 \odot \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

$\leq \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_1 and G_2 are fuzzy graphs.

$= (\sigma_1 \odot \sigma_2)(u_1,v_1) \wedge (\sigma_1 \odot \sigma_2)(u_2,v_2)$

$(\mu_1 \odot \mu_2)((u_1,v_1)(u_2,v_2)) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, if $u_1u_2 \notin E_1$ and $v_1v_2 \notin E_2$.

$= (\sigma_1 \odot \sigma_2)(u_1,v_1) \wedge (\sigma_1 \odot \sigma_2)(u_2,v_2)$

Thus, $(\mu_1 \odot \mu_2)((u_1,v_1)\ (u_2,v_2)) \leq (\sigma_1 \odot \sigma_2)\ (u_1,v_1) \wedge (\sigma_1 \odot \sigma_2)\ (u_2,v_2)$

This shows that the modular product of any two fuzzy graphs is again a fuzzy graph.

Theorem 4.2. The modular product of any two strong fuzzy graphs is again a strong fuzzy graph.

Proof: Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two strong fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. We want to prove that their modular product $G_1 \odot G_2 = (\sigma_1 \odot \sigma_2, \mu_1 \odot \mu_2)$ with underlying vertex set $V_1 \odot V_2$ and edge set $E_1 \odot E_2$ is again a strong fuzzy graph. Let $((u_1,v_1)(u_2,v_2)) \in E_1 \odot E_2$. From definition, it follows that

$(\mu_1 \circ \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_1 and G_2 are strong fuzzy graphs.

$=(\sigma_1 \odot \sigma_2)(u_1,v_1) \wedge (\sigma_1 \odot \sigma_2)(u_2,v_2)$

$(\mu_1 \odot \mu_2)((u_1,v_1)(u_2,v_2)) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, if $u_1u_2 \notin E_1$ and $v_1v_2 \notin E_2$ $=(\sigma_1 \odot \sigma_2)(u_1,v_1) \wedge (\sigma_1 \odot \sigma_2)(u_2,v_2)$.

This shows that the modular product of any two strong fuzzy graphs is again a strong fuzzy graph.

4.1.1. Degree of a vertex in Modular fuzzy graph product

Let $G_1= (\sigma_1,\mu_1)$ and $G_2= (\sigma_2,\mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then degree of a vertex (u_1,v_1) in their modular fuzzy graph product $= d_{G_1 \odot G_2}(u_1,v_1)$

$=\sum_{u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2} \mu_1(u_1u_2) \wedge \mu_2(v_1v_2) +$

$\sum_{u_1u_2 \notin E_1 \text{ and } v_1v_2 \notin E_2} \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$

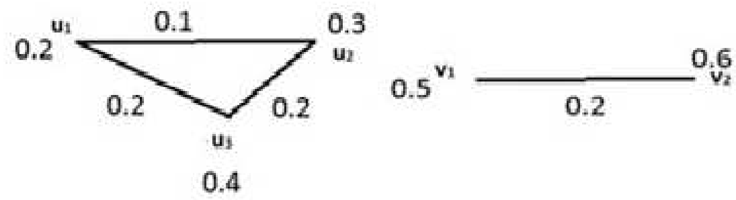
$=\sum_{u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2} \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$, if both fuzzy graphs are complete.

$=\sum_{u_1u_2 \in E_1} \mu_1(u_1u_2)$, if $\mu_1 \leq \mu_2$.

$= d_{G_1}(u_1)$

$d_{G_1 \odot G_2}(u_1,v_1) = \sum_{v_1v_2 \in E_2} \mu_2(v_1v_2)$, ifboth fuzzy graphs are complete and $\mu_2 \leq \mu_1$.

$= d_{G_2}(v_1)$



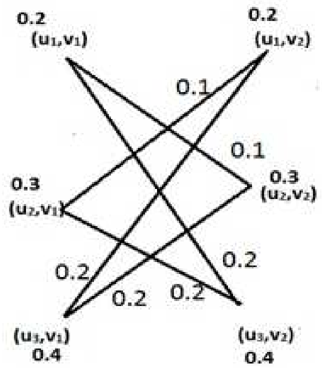
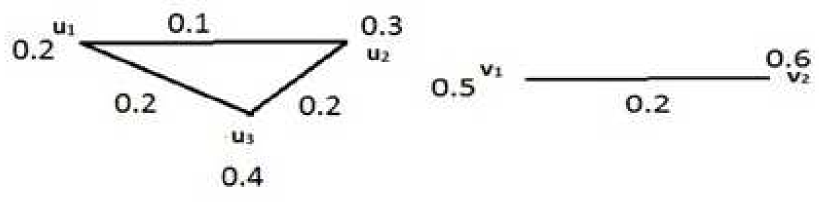


Figure 4.2: Modular fuzzy graph product $G_1 \odot G_2$

Here, $\mu_1 \leq \mu_2$. Then degree of the vertex (u_1,v_1) in the modular fuzzy graph product $G_1 \odot G_2 = d_{G_1 \odot G_2}(u_1,v_1) = 0.1+0.2 = d_{G_1}(u_1)$.

Corollary 4.1. Let $G_1 = (\sigma_1,\mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two complete fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Let $|V_1|=n, |V_2|=m$. Then the vertex set $V_1 \odot V_2$ of the new fuzzy graph produced from modular fuzzy graph product, contains mn vertices. Let $V_1 = \{u_1,u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$. Here, the vertex set $V_1 \odot V_2$ of the new fuzzy graph contains the vertices
$(u_1, v_1), (u_1, v_2), \ldots, (u_1,v_m)$,
$(u_2, v_1), (u_2, v_2), \ldots, (u_2, v_m)$,
…,
$(u_n, v_1), (u_n,v_2), \ldots, (u_n, v_m)$.
If the fuzzy graphs G_1 and G_2 be complete, then the number of possible edges in the modular product $G_1 \odot G_2$ is $\frac{mn(mn-1)}{2}$. Take one vertex (u_1, v_1). It may not be adjacent to $\{(m-1)+(n-1)\}$ vertices and similar incident occurs in case of *mn* vertices. Considering the non-adjacency between any two vertices for only one time, the total number of edges in $G_1 \odot G_2 = \frac{mn(mn-1)-\{(m-1)+(n-1)\}mn}{2}$.



G_1 G_2

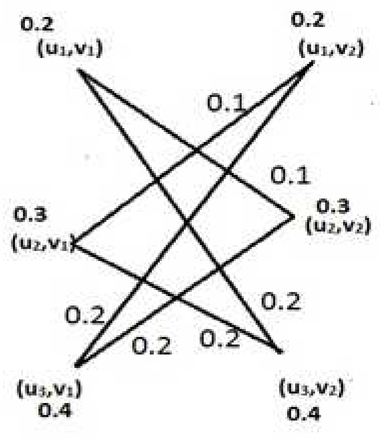


Figure 4.3: Modular fuzzy graph product $G_1 \odot G_2$

Here, $|V_1|=n=3, |V_2|=m=2$. Then the number of edges in the modular fuzzy graph product $G_1 \odot G_2 = \frac{mn(mn-1)-\{(m-1)+(n-1)\}mn}{2} = \frac{6(6-1)-\{(3-1)+(2-1)\times 6}{2} = \frac{30-18}{2} = 6$.

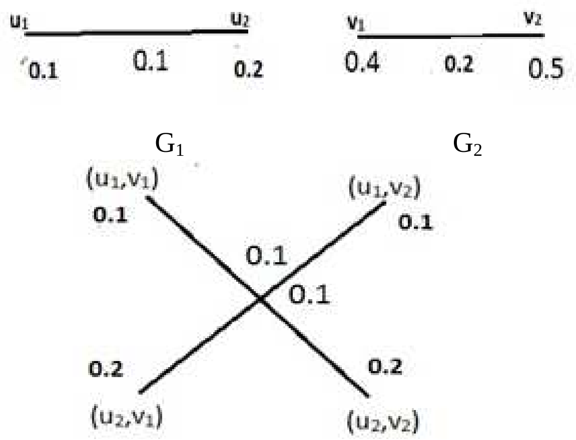


Figure 4.4: Modular fuzzy graph product $G_1 \odot G_2$

Here,$|V_1|$=n=2,$|V_2|$=m=2. The number of edges in the modular fuzzy graph product $G_1 \odot G_2 = \frac{mn(mn-1)-\{(m-1)+(n-1)\}mn}{2} = \frac{4(4-1)-\{(2-1)+(2-1)\}\times 4}{2} = \frac{12-8}{2} = 2$.

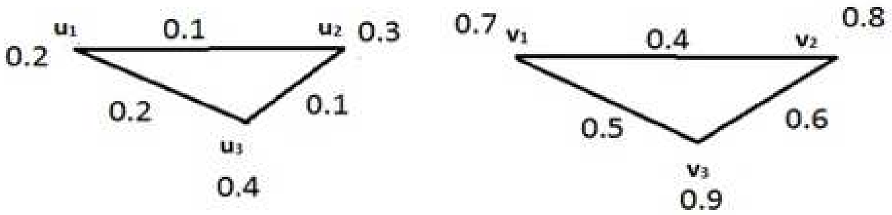
4.2. Homomorphic product of fuzzy graphs

Definition 4.2. Let $G_1 = (\sigma_1,\mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then homomorphic product of G_1 and G_2 is a pair of functions $(\sigma_1 \diamond \sigma_2, \mu_1 \diamond \mu_2)$ with underlying vertex set $V_1 \diamond V_2 = \{(u_1, v_1) : u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and underlying edge set $E_1 \diamond E_2 = \{((u_1,v_1)(u_2,v_2)) : u_1 = u_2, v_1v_2 \in E_2 \text{ or } u_1u_2 \in E_1, v_1v_2 \notin E_2\}$ with

$(\sigma_1 \diamond \sigma_2)(u_1,v_1) = \sigma_1(u_1) \wedge \sigma_2(v_1)$, where $u_1 \in V_1$ and $v_1 \in V_2$.

$(\mu_1 \diamond \mu_2)((u_1,v_1)(u_2,v_2)) = \sigma_1(u_1) \wedge \mu_2(v_1v_2)$, if $u_1 = u_2$ and $v_1v_2 \in E_2$.

$= \mu_1(u_1u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \notin E_2$.



G_1 G_2

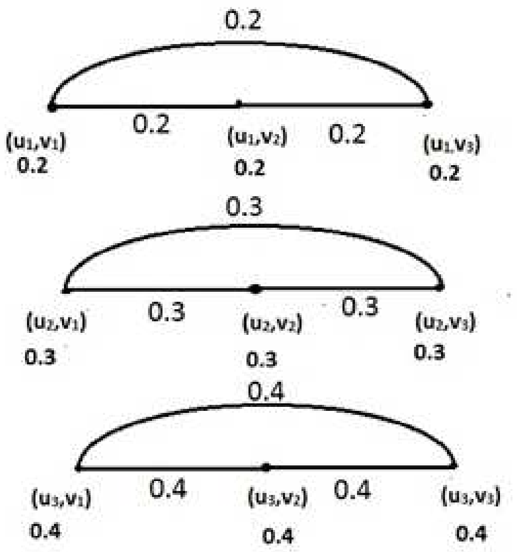


Figure 4.5: Homomorphic fuzzy graph product $G_1 \diamond G_2$

Theorem 4.3. The homomorphic product of any two fuzzy graphs is again a fuzzy graph.
Proof: Let$G_1= (\sigma_1,\mu_1)$ and $G_2= (\sigma_2,\mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets $E_1$and E_2 respectively. We want to prove that their homomorphic product $G_1 \diamond G_2= (\sigma_1 \diamond \sigma_2,\mu_1 \diamond \mu_2)$ with underlying vertex set $V_1 \diamond V_2$ and edge set $E_1 \diamond E_2$is again a fuzzygraph.Let $((u_1,v_1)\ (u_2,v_2)) \in E_1 \diamond E_2$. From definition,
$(\mu_1 \diamond \mu_2)((u_1,v_1)(u_2,v_2))= \sigma_1(u_1) \wedge \mu_2(v_1v_2)$, if$u_1=u_2$ and $v_1v_2 \in E_2$.
$\leq \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_2 is a fuzzy graph and $u_1=u_2$.
$=(\sigma_1 \diamond \sigma_2)(u_1,v_1) \wedge (\sigma_1 \diamond \sigma_2)(u_2,v_2)$
$(\mu_1 \diamond \mu_2)((u_1,v_1)(u_2,v_2))= \mu_1(u_1u_2)\wedge \sigma_2(v_1)\wedge \sigma_2(v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \notin E_2$.
$\leq \sigma_1(u_1)\wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_1 is a fuzzy graph.
$=(\sigma_1 \diamond \sigma_2)(u_1,v_1)\wedge(\sigma_1 \diamond \sigma_2)(u_2,v_2)$
Thus, $(\mu_1 \diamond \mu_2)((u_1,v_1)\ (u_2,v_2)) \leq (\sigma_1 \diamond \sigma_2)\ (u_1,v_1) \wedge (\sigma_1 \diamond \sigma_2)\ (u_2,v_2)$
This shows that the homomorphic product of any two fuzzy graphs is again a fuzzy graph.

Theorem 4.4. The homomorphic product of any two strong fuzzy graphs is again a strong fuzzy graph.
Proof: Let $G_1= (\sigma_1,\mu_1)$ and $G_2= (\sigma_2,\mu_2)$ be two strong fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets $E_1$and E_2 respectively. We want to prove that their homomorphic product $G_1 \diamond G_2= (\sigma_1 \diamond \sigma_2,\mu_1 \diamond \mu_2)$ with underlying vertex set $V_1 \diamond V_2$ and edge set $E_1 \diamond E_2$ is again a strong fuzzy graph. Let $((u_1,v_1)\ (u_2,v_2)) \in E_1 \diamond E_2$. From definition, it follows that
$(\mu_1 \diamond \mu_2)((u_1,v_1)(u_2,v_2))= \sigma_1(u_1) \wedge \mu_2(v_1v_2)$, if $u_1=u_2$ and $v_1v_2 \in E_2$.
$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_2 is a strong fuzzy graph.

$=(\sigma_1 \diamond \sigma_2)(u_1,v_1) \wedge (\sigma_1 \diamond \sigma_2)(u_2,v_2)$

$(\mu_1 \diamond \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, if $u_1u_2 \in E_1$and $v_1v_2 \notin E_2$.

$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_1 is a strong fuzzy graph.

$=(\sigma_1 \diamond \sigma_2)(u_1,v_1) \wedge (\sigma_1 \diamond \sigma_2)(u_2,v_2)$

This shows that the homomorphic product of any two strong fuzzy graphs is again a strongfuzzy graph.

4.2.1. Degree of a vertex inhomomorphic fuzzy graph product

Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then the degree of a vertex (u_1,v_1)in homomorphic fuzzy graph product $G_1 \diamond G_2 = d_{G_1 \diamond G_2}(u_1,v_1)$

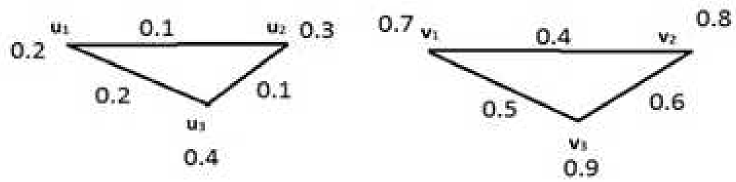
$=\sum_{u_1=u_2 \text{ and } v_1v_2 \in E_2} \sigma_1(u_1) \wedge \mu_2(v_1v_2) +$

$\sum_{u_1u_2 \in E_1 \text{ and } v_1v_2 \notin E_2} \mu_1(u_1u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$

$=\sum_{u_1=u_2 \text{ and } v_1v_2 \in E_2} \sigma_1(u_1) \wedge \mu_2(v_1v_2)$,if G_2be complete.

$=\sum_{u_1=u_2} \sigma_1(u_1)$, if$\sigma_1 \leq \mu_2$.

Here, summation is taken over $\sigma_1(u_1)$ when $v_1v_2 \in E_2$. If G_2be complete, then the term $\sigma_1(u_1)$ occurs here $(|V_2|-1)$ times. Therefore, $d_{G_1 \diamond G_2} == (|V_2|-1)\sigma_1(u_1)$.



G_1 G_2

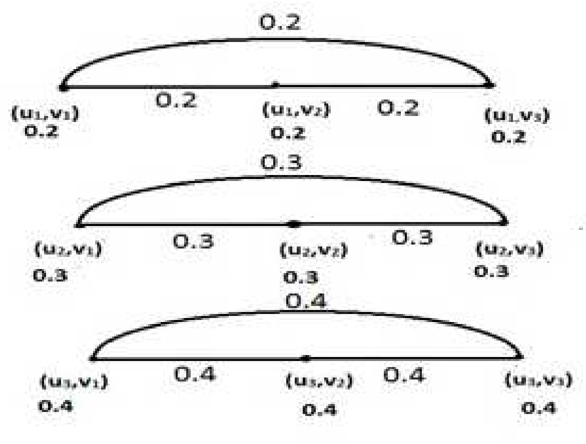


Figure 4.6: Homomorphic fuzzy graph product $G_1 \diamond G_2$

Here, the fuzzy graph G_2 is complete. Now, the degree of the vertex (u_1,v_1) in the homomorphic fuzzy graph product $G_1 \diamond G_2 = 0.2+0.2 = (3-1)\times 0.2 = (|V_2|-1)\sigma_1(u_1)$.

Corollary 4.2. Let$G_1 = (\sigma_1,\mu_1)$ and $G_2 = (\sigma_2,\mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets $E_1$and E_2respectively. Let $|V_1|=m, |V_2|=n$.Then the vertex set $V_1 \diamond V_2$of new fuzzy graph produced from homomorphic fuzzy graph product, contains mn vertices. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Here, the vertex set$V_1 \diamond V_2$ of the new fuzzy graph contains the vertices
(u_1, v_1), (u_1, v_2),…..,(u_1, v_n),
(u_2, v_1),(u_2, v_2),…,(u_2, v_n),
…..,
(u_m, v_1),(u_n, v_2),…….,(u_m, v_n).
If G_2 be complete, then from the first row, any two vertices can be chosen in $\frac{n(n-1)}{2}$ ways. Similar incident occurs for m rows. Therefore, the total number of edges in $G_1 \diamond G_2 = m \times \frac{n(n-1)}{2} = \frac{mn(n-1)}{2}$.

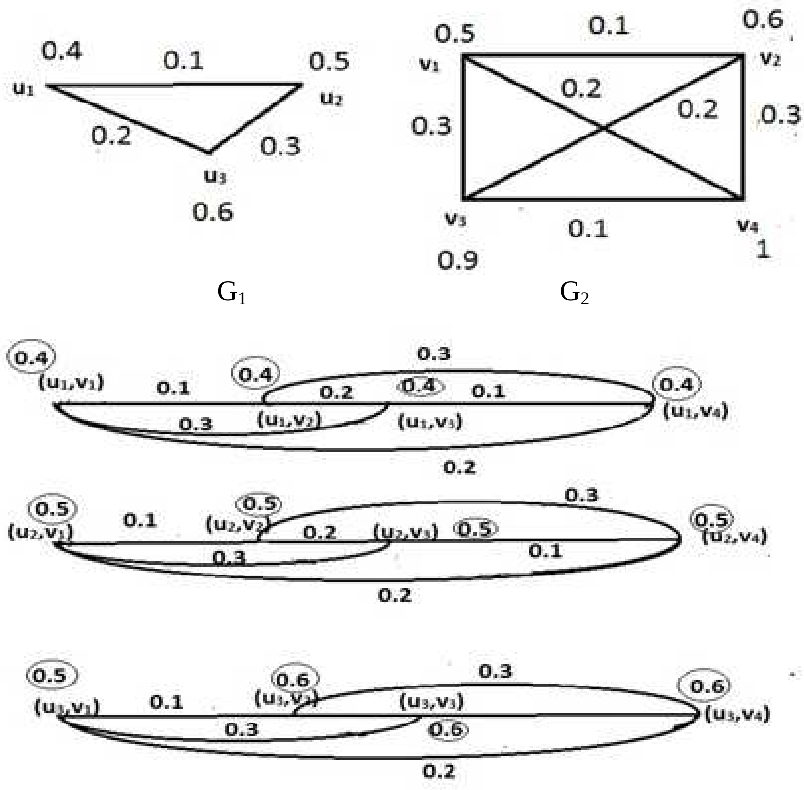
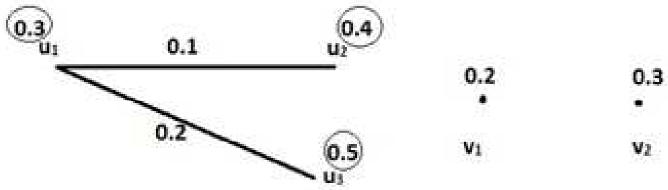


Figure 4.7: Homomorphic fuzzy graph product $G_1 \diamond G_2$

Here, the fuzzy graph G_2 is complete. $|V_1|=3=m$ and $|V_2|=4=n$. Then the total number of edges in the homomorphic fuzzy graph product
$G_1 \diamond G_2 = \frac{mn(n-1)}{2} = \frac{3\times4(4-1)}{2} = \frac{3\times4\times3}{2} = 18$.

4.3. Boxdot fuzzy graph product

Definition 4.3. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then box dot graph product of G_1 and G_2 is a pair of functions $(\sigma_1 \boxdot \sigma_2, \mu_1 \boxdot \mu_2)$ with underlying vertex set $V_1 \boxdot V_2 = \{(u_1, v_1): u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and underlying edge set $E_1 \boxdot E_2 = \{((u_1,v_1)\ (u_2,v_2)): u_1 = u_2, v_1v_2 \notin E_2 \text{ or } u_1u_2 \in E_1, v_1v_2 \notin E_2\}$ with
$(\sigma_1 \boxdot \sigma_2)\ (u_1,v_1) = \sigma_1(u_1) \wedge \sigma_2(v_1)$, where $u_1 \in V_1$ and $v_1 \in V_2$.
$(\mu_1 \boxdot \mu_2)((u_1,v_1)(u_2,v_2)) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_1)$, if $u_1 = u_2$ and $v_1v_2 \notin E_2$.
$(\mu_1 \boxdot \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \notin E_2$.



G_1 G_2

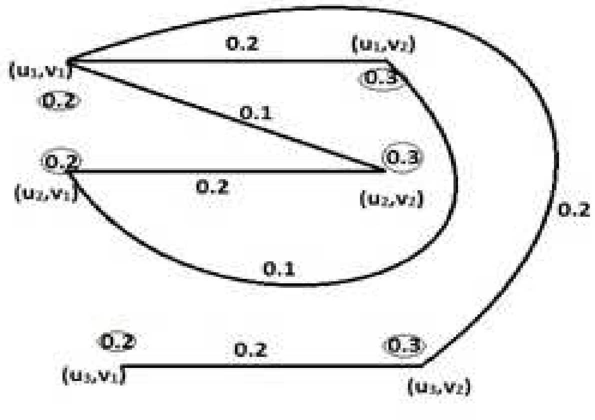


Figure 4.8: Box dotfuzzy graph product $G_1 \boxdot G_2$

Theorem 4.5. The box dot graph product of any two fuzzy graphs is again a fuzzy graph.

Proof: Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. We want to prove that their box dotgraph product $G_1 \boxdot G_2 = (\sigma_1 \boxdot \sigma_2, \mu_1 \boxdot \mu_2)$ with underlying vertex set $V_1 \boxdot V_2$ and edge set $E_1 \boxdot E_2$ is again a fuzzy graph. Let $((u_1,v_1)\ (u_2,v_2)) \in E_1 \boxdot E_2$. From definition, it follows that
$(\mu_1 \boxdot \mu_2)((u_1,v_1)(u_2,v_2)) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_1)$, if $u_1 = u_2$ and $v_1v_2 \notin E_2$.
$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$
$= (\sigma_1 \boxdot \sigma_2)(u_1,v_1) \wedge (\sigma_1 \boxdot \sigma_2)(u_2,v_2)$
$(\mu_1 \boxdot \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \notin E_2$.
$\leq \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_2 is a fuzzy graph.
$= (\sigma_1 \boxdot \sigma_2)(u_1,v_1) \wedge (\sigma_1 \boxdot \sigma_2)(u_2,v_2)$
Thus, $(\mu_1 \boxdot \mu_2)((u_1,v_1)\ (u_2,v_2)) \leq (\sigma_1 \boxdot \sigma_2)\ (u_1,v_1) \wedge (\sigma_1 \boxdot \sigma_2)\ (u_2,v_2)$

This shows that box dot graph product of any two fuzzy graphs is again a fuzzy graph.

Theorem 4.6. The box dot graph product of any two strong fuzzy graphs is again a strong fuzzy graph.

Proof: Let G_1= (σ_1,μ_1) and G_2= (σ_2,μ_2) be two strong fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2respectively. We want to prove that their box dot graph product $G_1 \boxdot G_2$= $(\sigma_1 \boxdot \sigma_2,\mu_1 \boxdot \mu_2)$ with underlying vertex set $V_1 \boxdot V_2$ and edge set $E_1 \boxdot E_2$ is again a strong fuzzy graph. Let $((u_1,v_1)\ (u_2,v_2)) \in E_1 \boxdot E_2$. From definition, it follows that

$(\mu_1 \boxdot \mu_2)((u_1,v_1)(u_2,v_2))= \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_1)$, if$u_1=u_2andv_1v_2 \notin E_2$.

$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$

$=(\sigma_1 \boxdot \sigma_2)(u_1,v_1) \wedge (\sigma_1 \boxdot \sigma_2)(u_2,v_2)$

$(\mu_1 \boxdot \mu_2)((u_1,v_1)(u_2,v_2))= \mu_1(u_1u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, if$u_1u_2 \in E_1andv_1v_2 \notin E_2$.

$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_2 is a strong fuzzy graph.

$=(\sigma_1 \boxdot \sigma_2)(u_1,v_1) \wedge (\sigma_1 \boxdot \sigma_2)(u_2,v_2)$

This shows that the box dot graph product of any two strong fuzzy graphs is again a strong fuzzy graph.

4.3.1. Degree of vertex in box dot graph product

Let G_1= (σ_1, μ_1) and G_2= (σ_2, μ_2) be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then degree of a vertex (u_1,v_1) in box dot fuzzy graph product $G_1 \boxdot G_2$= $d_{G_1 \boxdot G_2}(u_1,v_1)$

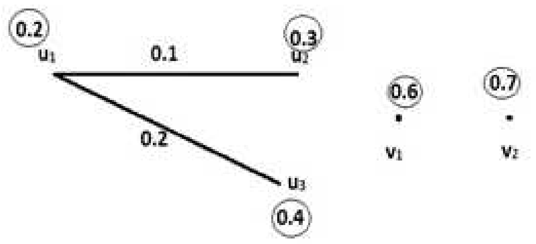
$=\sum_{u_1=u_2 \text{ and } v_1v_2 \notin E_2} \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_1)+$

$\sum_{u_1u_2 \in E_1 \text{ and } v_1v_2 \notin E_2} \mu_1(u_1u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$

$=\sum_{u_1=u_2} \sigma_1(u_1)+\sum_{u_1u_2 \in E_1} \mu_1(u_1u_2)$, if$\mu_1 \leq \sigma_2$ and $\sigma_1 \leq \sigma_2$

Here, summation is taken over$\sigma_1(u_1)$ when$v_1v_2 \notin E_2$. If the membership values of all edges of G_2 are zero, then $\sigma_1(u_1)$ occurs $(|V_2|-1)$ times. Therefore, $d_{G_1 \boxdot G_2}(u_1,v_1)$

$= (|V_2|-1)\sigma_1(u_1) + d_{G_1}(u_1)$.



G_1 G_2

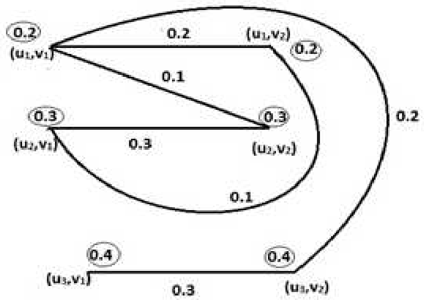


Figure 4.9: Box dot fuzzy graph product $G_1 \boxdot G_2$

From figure, we see that $\mu_1 \le \sigma_2$ and $\sigma_1 \le \sigma_2$. The degree of the vertex (u_1,v_1) in the box dot fuzzy graph product $G_1 \boxdot G_2 = d_{G_1 \boxdot G_2}(u_1,v_1) = 0.2+0.2+0.1 = (2-1) \times 0.2 + (0.2+0.1) = (|V_2|-1)\,\sigma_1(u_1) + d_{G_1}(u_1)$.

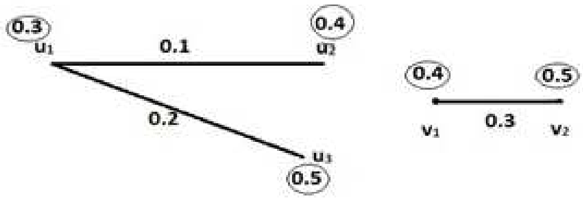
4.4. Star fuzzy graph product

Definition 4.4. Let $G_1 = (\sigma_1,\mu_1)$ and $G_2 = (\sigma_2,\mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then star graph product of G_1 and G_2 is a pair of functions $(\sigma_1 * \sigma_2, \mu_1 * \mu_2)$ underlying vertex set $V_1 * V_2 = \{(u_1,v_1): u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and underlying edge set $E_1 * E_2 = \{((u_1,v_1)(u_2,v_2)): u_1 = u_2, v_1v_2 \notin E_2 \text{ or } u_1u_2 \in E_1, v_1v_2 \in E_2\}$ with

$(\sigma_1 * \sigma_2)(u_1,v_1) = \sigma_1(u_1) \wedge \sigma_2(v_1)$, where $u_1 \in V_1$ and $v_1 \in V_2$.

$(\mu_1 * \mu_2)((u_1,v_1)(u_2,v_2)) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_1)$, if $u_1 = u_2$ and $v_1v_2 \notin E_2$.

$= \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.



G_1 G_2

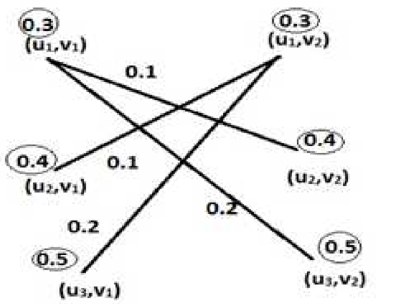


Figure 4.10: Star fuzzy graph product $G_1 * G_2$

Theorem 4.7. The star graph product of any two fuzzy graphs is again a fuzzy graph.
Proof: Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively .We want to prove that their star graph product $G_1 * G_2 = (\sigma_1 * \sigma_2, \mu_1 * \mu_2)$ with underlying vertex set $V_1 * V_2$ and edge set $E_1 * E_2$ is again a fuzzy graph. Let $((u_1,v_1)(u_2,v_2)) \in E_1 * E_2$. Then from definition,
$(\mu_1 * \mu_2)((u_1,v_1)(u_2,v_2)) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_1)$, if $u_1 = u_2$ and $v_1v_2 \notin E_2$.
$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$
$= (\sigma_1 * \sigma_2)(u_1,v_1) \wedge (\sigma_1 * \sigma_2)(u_2,v_2)$
$(\mu_1 * \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.
$\leq \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_1 and G_2 are fuzzy graphs.
$= (\sigma_1 * \sigma_2)(u_1,v_1) \wedge (\sigma_1 * \sigma_2)(u_2,v_2)$
Thus, $(\mu_1 * \mu_2)((u_1,v_1)(u_2,v_2)) \leq (\sigma_1 * \sigma_2)(u_1,v_1) \wedge (\sigma_1 * \sigma_2)(u_2,v_2)$
This shows that the star graph product of any two fuzzy graphs is again a fuzzy graph.

Theorem 4.8. The star graph product of any two strong fuzzy graphs is again a strong fuzzy graph.
Proof: Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two strong fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. We want to prove that their star graph product $G_1 * G_2 = (\sigma_1 * \sigma_2, \mu_1 * \mu_2)$ with underlying vertex set $V_1 * V_2$ and edge set $E_1 * E_2$ is again a strong fuzzy graph. Let $((u_1,v_1)(u_2,v_2)) \in E_1 * E_2$. Then from definition, it follows that
$(\mu_1 * \mu_2)((u_1,v_1)(u_2,v_2)) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_1)$, if $u_1 = u_2$ and $v_1v_2 \notin E_2$.
$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$
$= (\sigma_1 * \sigma_2)(u_1,v_1) \wedge (\sigma_1 * \sigma_2)(u_2,v_2)$
$(\mu_1 * \mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$, if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.
$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$, as G_1 and G_2 are strong fuzzy graphs.
$= (\sigma_1 * \sigma_2)(u_1,v_1) \wedge (\sigma_1 * \sigma_2)(u_2,v_2)$
This shows that the star graph product of any two strong fuzzy graphs is again a strong fuzzy graph.

4.4.1. Degree of a vertex in star fuzzy graph product
Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then degree of a vertex (u_1,v_1) in their star graph product= $d_{G_1 * G_2}(u_1,v_1)$

$= \sum_{u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2} \mu_1(u_1u_2) \wedge \mu_2(v_1v_2) +$
$\sum_{u_1 = u_2 \text{ and } v_1v_2 \notin E_2} \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_1)$
$= d_{G_1 * G_2}(u_1,v_1)$, .
$= \sum_{u_1u_2 \in E_1} \mu_1(u_1u_2)$, if G_2 be complete and $\mu_1 \leq \mu_2$.
$= d_{G_1}(u_1)$
$d_{G_1 * G_2}(u_1, v_1) = \sum_{v_1v_2 \in E_2} \mu_2(v_1v_2)$, if G_2 be complete and $\mu_2 \leq \mu_1$.
$= d_{G_2}(v_1)$

G_1 G_2

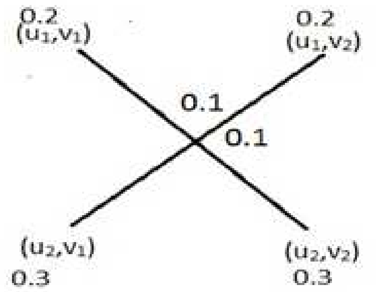


Figure 4.11: Star fuzzy graph product G_1*G_2

Here, both fuzzy graphs are complete and $\mu_1 \leq \mu_2$.Then degree of the vertex (u_1,v_1) in the star fuzzy graph product $G_1*G_2 = d_{G_1*G_2}(u_1,v_1)=0.1= d_{G_1}(u_1)$.

5. Conclusion

In this paper, we have determined the degree of the vertices in $G_1 \odot G_2$, $G_1 \diamond G_2$ and $G_1 \boxdot G_2$ and G_1*G_2 in terms of the degree of vertices of G_1 and G_2 under some certain conditions and illustrated them with examples and figures. We have proved some theorems related to modular, homomorphic, box dot and star fuzzy graph product. In case of modular and homomorphic fuzzy graph product, we have developed some formulas under some certain conditions to determine the number of edges in the new graphs in terms of the number of vertices of the participated fuzzy graphs in the said product operations. These formulas will play important role when the fuzzy graphs are very large. All these will be helpful in studying various new properties of modular, homomorphic, box dot and star graph product of two fuzzy graphs in future.

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