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The Weak (Monophonic) Convexity Number of a Graph

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Anstract. In a connected graph G a subset S of vertices of G is said to be a weak convex set if for any two vertices u, v of S, S contains all the vertices of a $u-v$ shortest path in G . Maximum cardinality of a proper weak convex set of G is the weak convexity number of G denoted by $wcon(G)$. Let the set $J[u,v]$ consists of all those vertices lying on a $u-v$ induced path in G. A subset S of vertices of G is said to be a monophonic convex set (in short m - convex set) if for any two vertices u, v of S, S contains all the vertices of every $u-v$ induced path in G. The *m*-covexity number $mcon(G)$ of G is the maximum cardinality of a proper m - convex set of G. The clique number $\omega(G)$ is the maximum cardinality of a clique in G. Every m - convex set is a convex set. If G is a connected graph of order n which is not complete , then $n \geq 3$ and $2 \leq \omega(G) \leq mcon(G) \leq con(G) \leq wcon(G) \leq n-1$. In this paper it is shown that for every quardruple k_1, k, k', n of integers with $n \geq 3$ and $2 \leq k_1 \leq k \leq k' \leq n-1$, there exists a non-complete connected graph G of order n with $mcon(G) = k_1$, $con(G) = k$ and $wcon(G) = k'$. Also for every triple l, k_1, n of integers with $n \geq 3$ and $2 \leq l \leq k_1 \leq n-1$, there exists a non-complete connected graph G of order n with $\omega(G) = l$ and $mcon(G) = k_1$. Similar construction is given for weak convexity number. Other interesting results on these numbers are presented.

Keywords: weak convex set , weak convexity number, *m*-convex set

1. Introduction

By a Graph we mean undirected graph without loops or multiple edges.For terminology and notation not given here,the reader may refer to [6]. The Graphs considered here are connected. For two vertices u and v in a connected graph G, the distance d(u, v) is the length of a shortest $u-v$ path in G. Shortest $u-v$ path is referred to as a $u-v$ geodesic. Convexity in Graphs was studied in [8]. A subset S of vertices of G is said to be a weak convex set if for any two vertices u, v of S , S contains all the vertices of a $u-v$ shortest path in G . The weak convexity number $wcon(G)$ of G is the maximum cardinality of a proper weak convex set of G, $wcon(G) = max\{|S| \ / \ S$ is a weak convex set of G and $S \neq V(G)\}$. These type of sets are already called isometric sets. We prefer to use the term weak convex sets since the discussions are related to the convexity and the results there in. Also the condition of convexity is relaxed and hence we use the word weak convex. If S is a weak convex set in a connected graph G, then the subgraph $\langle S \rangle$ induced by S is connected. Weak convex set S in G with $|S| = wcon(G)$ is called a maximum weak convex set . If G is a connected graph of order $n \geq 3$ then $2 \leq wcon(G) \leq n-1$. If G is a non-complete graph containing a complete subgraph H, then the vertex set $V(H)$ is convex in G thus $V(H)$ is weak convex and so $wcon(G) \geq |V(H)|$. The clique number $\omega(G)$ of a graph is the maximum order of a complete subgraph in G . Thus if $n \geq 3$, then $\omega(K_n) = n$ while $wcon(K_n) = n-1$. But if G is non-complete then $2 \leq \omega(G) \leq wcon(G) \leq n-1$.

A vertex v in a graph G is called a weak-complete vertex if for any two vertices $\{x, y\}$ in $N(v)$ either x, y are adjacent or there exists a $u \neq v$ such that $x-u-y$ is a geodesic. A vertex v in a graph G is called a complete vertex if any two vertices $\{x, y\}$ in $N(v)$ are adjacent .

For two vertices u and v in a connected graph G , the induced path $u-v$ is one in which $v_i v_j$ is an edge if and only if $j = i+1$. A subset S of vertices of G is said to be a m - convex set if for any two vertices u , v of S , S contains all the vertices of every $u-v$ induced path in G . The m - convexity number $mcon(G)$ of G is the maximum cardinality of a proper m - convex set of G . If G is a connected graph of order $n \geq 3$ then $2 \leq mcon(G) \leq n-1$. If G is a non - complete graph containing a complete subgraph H , then the vertex set $V(H)$ is convex in G thus $V(H)$ is m - convex and so $mcon(G) \geq |V(H)|$. The clique number $\omega(G)$ of a graph is the maximum order of a complete subgraph in G . Thus if $n \geq 3$, then $\omega(K_n) = n$ whil $mcon(K_n) = n-1$. But if G is non - complete then $2 \leq \omega(G) \leq mcon(G) \leq n-1$.

Example 1.1.

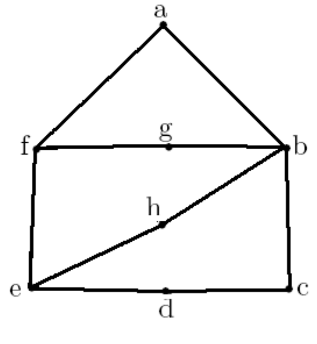

Figure 1:

$con(G) = 6$, $wcon(G) = 7$, $mcon(G) = 2$. In figure 1 $\{a,b,c,f,g,h\}$, $\{b,c,d,e,f,g,h\}$, $\{a,b\}$ (say) form convex , weak - convex , m - convex set respectively and $'a'$ is a weak complete vertex.

Observation 1.1. *If a connected graph G of order n has an end-vertex v , then $V(G)-\{v\}$ is both a weak convex set and a m - convex set. In particular , if the minimum degree of G is 1 , then $wcon(G) = n-1 = mcon(G)$.*

Corollary 1.2. *For every tree T of order $n \geq 2$,*

$$wcon(T) = n-1 = mcon(G) \ .$$

Theorem 1.3. *Let G be a non-complete connected graph of order n . Then $wcon(G) = n-1$ iff G contains a weak complete vertex.*

Proof: Suppose v is a weakly complete vertex . Let $\{x, y\}$ in $V(G)-\{v\}$. Consider a geodesic I between x and y . If v in I then there exists

$\{u,w\}$ in $I \bigcap N(v)$. Clearly u and w are non-adjacent . Since v is weakly complete there exists $t \neq v \in V(G)$ such that $u-t-w$ exists . This gives a geodesic I_1 not containing v . Therefore $V(G)-\{v\}$ is weak convex . Hence $wcon(G) = n-1$. Conversely suppose $wcon(G) = n-1$ then there exists v in $V(G)$ such that $V(G)-\{v\}$ is weak convex . Let $x, y \in N(v)$. If x, y are adjacent we are through . Suppose x, y are non - adjacent and there exists no $u \neq v$ such that $x-u-y$ is a geodesic then $x-v-y$ is the only geodesic and $\{x,v,y\} \subseteq V(G)-\{v\}$ which is a contradiction .Therefore v is a weakly complete vertex.

Theorem 1.4. *Let G be a non-complete connected graph of order n . Then $mcon(G) = n-1$ iff G contains a complete vertex.*

Observation 1.5. *For $n \geq 3$, $wcon(C_n) = \{\left\lceil \frac{n}{2} \right\rceil$ is odd $.5cm \left\lceil \frac{n}{2} \right\rceil + 1 if$ n is even .*

Observation 1.6. $mcon(C_n) = 2$.

Theorem 1.7. *For integers $k, n_1, n_2, n_3, \cdots, n_k \geq 2$,*

$$wcon(K_{n_1,n_2,n_3,\cdots,n_k}) = n_1 + n_2 + n_3 + \cdots + n_k - 1.$$

Proof: Let $G = K_{n_1,n_2,n_3,\cdots,n_k}$ whose partite sets are $V_i \, for 1 \leq i \leq k$. Clearly G is a non - complete connected graph of order $n_1 + n_2 + n_3 + \cdots + n_k$ and any vertex u in $V(G)$ is a weak complete vertex .

Hence by 1.3 $wcon(K_{n_1,n_2,n_3,\cdots,n_k}) = n_1 + n_2 + n_3 + \cdots + n_k - 1$.

2. Graphs with prescribed clique number , weak convexity number and order

If G is a non-complete connected graph of order n such that $\omega(G) = l$ and $wcon(G) = k'$, then G is called an (l, k', n) graph . If G is an (l, k', n) graph, then $n \geq 3$ and $2 \leq l \leq k \leq k' \leq n-1$. Now we show that $(2,3,5)$ is unique. If G is a non-complete connected graph of order n such that $\omega(G) = l$ and $mcon(G) = k_1$, then G is called an (l, k_1, n) graph .If G is an (l, k_1, n) graph , then $n \geq 3$ and $2 \leq l \leq k_1 \leq n-1$. Now we show that $(2,5,7)$ is a graph with only three structures.

Theorem 2.1 *The* $(2,3,5)$ *graph is unique where weak convexity number* $k' = 3$.

Proof: Let G be a connected graph of order 5 with $\omega(G) = 2$ and $wcon(G) = 3$. Let $S = \{u, v, w\}$ be a maximum weak convex set in G and let $u - v - w$ be a path of length 2 . From hypothesis we observe the following .

(i) G has no triangles , since $\omega(G) = 2$.

(ii) There is no pendant since $wcon(G) = 3 \neq n-1$. Therefore $deg(u) \geq 2$ for all $u \in V(G)$.

(iii) $N(u) \bigcap N(w) = \{v\}$. Suppose there exists v_1 as shown in following figures then v, u or w is a weak complete vertex . Hence we get $wcon(G) = 4$ which is a contradiction . Therefore v is the only vertex adjacent to u and w. Clearly G has no four cycle. Also $deg(u) = 2$ for all $u \in V(G)$. Thus $G = C_5$.

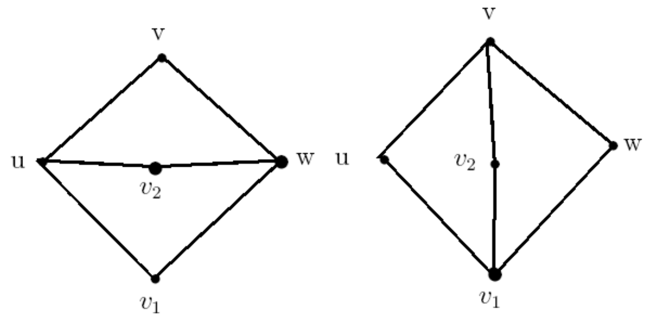

Figure 2:

Theorem 2.2. *The* $(2,5,7)$ *graph is with only three structures where* m - convexity number $k_1 = 5$.

Proof: Let G be a connected graph of order 7 with $\omega(G) = 2$ and $mcon(G) = 5$. Let $S = \{u, v, w, s, t\}$ be a maximum m - convex set in G . From hypothesis we observe the following .

(i) G has no triangles , since $\omega(G) = 2$.

(ii) There is no pendant since $mcon(G) = 5 \neq n-1$. Therefore $deg(u) \geq 2$ for all

$u \in V(G)$.

(iii) G has no complete vertex . Suppose G has a six cycle then $mcon(G) = 2$ as the left vertex also forms a cycle. Let remaining vertices be t , p , q . If G has a C_5 then the remaining two vertices p , q (say) can be of the following types. p, q are adjacent and joined to two adjacent vertices of C_5 .Here neither p nor q can be in m -convex set. If G has a C_4 then let $\{u, v, w, s\}$ form a cycle. One of t, p, q can be adjacent to $\{u, w\}$ or $\{v, s\}$. Rest of two vertices of t , p , q can be adjacent to adjacent vertices of $\{u, v, w, s, t\}$ and themselves adjacent or t , p , q can form a P_3 and t , q can be adjacent to a single vertex of C_4. Therefore the possible figures are shown in Figure 3.

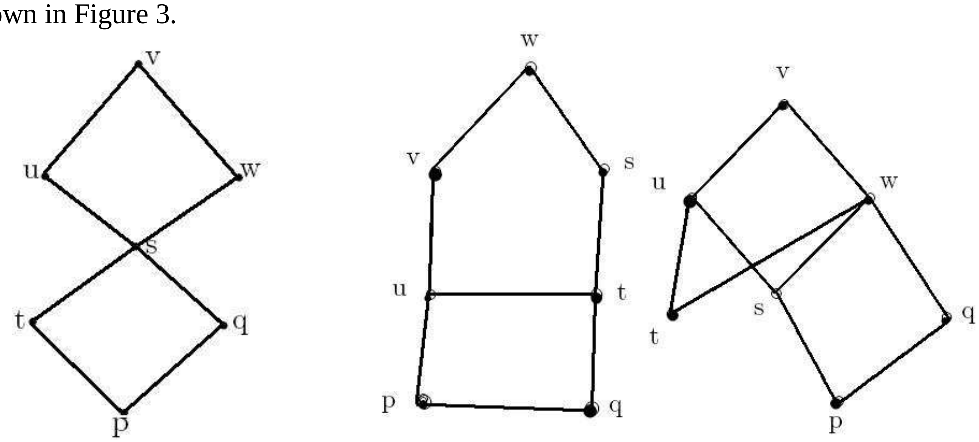

Figure 3:

3. Realization theorems

Lemma 3.1. *For every pair k', n of integers with $n \geq 3$, $2 \leq k' \leq n-1$ there exists a non-complete connected graph such that $\omega(G) = 2$, $wcon(G) = k'$.*

Proof: For $k' = 2$ the graph $K_{1,2}$ satisfies the purpose . For $k' = 3$ the graph $K_{1,3}$ and for $k' = 4$, $K_{1,4}$ satisfies the purpose . For $k' = n-1$, $K_{2,n-2}$ is the required graph.So $5 \leq k' \leq n-2$. Consider $C_{k'}$. Form a path with $n-k'$ vertices . Join one of the end vertex u (say) of the path to one of the vertex v (say)on $C_{k'}$. Now join the vertex say u' adjacent to u from the path to a vertex say v' next to the vertex adjacent to v in $C_{k'}$. Repeat the process for all the vertices of the path .

Case (i) k' even and $n-k' \geq k'$

By construction $((k'/2)+1)th$ vertex of the path and the starting end vertex of the path are at distance two via a vertex on $C_{k'}$ but at distance $k'/2$ via vertices on path. Since k' is even and $k' \geq 5$ we have this distance greater than three. Therefore

$wcon(G)$ has only vertices on $C_{k'}$ or $(k'/2)$ vertices of $C_{k'}$ and $(k'/2)$ vertices of $P_{n-k'}$

Case (ii) k' odd and $n-k' > k'$

By construction $(k'+1)th$ vertex of the path and the starting end vertex of the path are at distance two via a vertex on $C_{k'}$ but at distance k' via vertices on path. since k' is odd and $k' \geq 5$ we have this distance greater than five. Therefore $wcon(G)$ has only vertices on $C_{k'}$ or $\lceil k'/2 \rceil$ vertices of $C_{k'}$ and $\lfloor k'/2 \rfloor$ vertices of $P_{n-k'}$.

Case (iii) k' odd and $n-k' = k'$

By construction either vertices of $C_{k'}$ or vertices of the path can form a weak convex set but no vertex of $C_{k'}$ or $P_{n-k'}$ can be included in a weak convex set containing vertices of $P_{n-k'}$ or $C_{k'}$ respectively.Vertices of $C_{k'}$ and $P_{n-k'}$ that add to k' vertices can be chosen to form a maximum weak convex set.

Case (iv) k' odd or even and $n-k' < k'$

Clearly $C_{k'}$ forms the maximum weak convex set and no vertex of the path can be included.

Lemma 3.2. *For any given integer* $l \geq 3$*,there exists a non-complete* connected graph G of order $n = (l+m)$ where $m \geq 5$ and $k' = l + \lceil m/2 \rceil$, $\omega(G) = l$.

Proof: Consider a clique of order l and a cycle of order m .Join a vertex of the clique and cycle. Also join consecutive vertices of the clique to a vertex next to the consecutive vertex on the cycle . Repeat the process till all vertices of the clique exhaust.Clearly if C_m is included in $wcon(G)$ set then no vertex of the clique belong to $wcon(G)$ set. Thus $wcon(G)$ set is the set of vertices on the clique and $\lceil m/2 \rceil$ vertices of C_m . Therefore $k' = l + \lceil m/2 \rceil$.

Example 3.2.

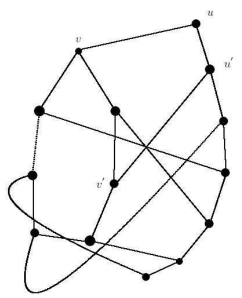

Figure 4:

Lemma 3.3. *For any given integer* $l \geq 4$*,there exists a non-complete* connected graph G of order $n = (l+3)$ or $(l+4)$ with $k^{'} = l+1$ or $l+2$ respectively, $\omega(G) = l$.

Proof: Consider a clique of order l.Attach a C_5 with one of its edge on K_l.Thus end vertices of a P_3 is joined to a pair of adjacent vertices of K_l. Therefore order of G is $l+3$. Now vertices of K_l not on C_5 are joined to exactly one of the non-adjacent vertex of C_5 not on K_l. Thus no vertex is a weak complete vertex of G. Therefore $wcon(G) = l+1$. The same procedure is repeated for $wcon(G) = l+2$ with a C_6 instead of C_5.Thus for a clique number $l \geq 4$ there exists a non-complete connected graph G of order $n = l+3$ or $l+4$ with $k^{'} = n-2$.

Theorem 3.4. *For every triple* $l, k^{'}, n$ *with* $2 \leq l \leq k^{'} \leq n-1$ *there exists a non-complete connected graph of order* n *having clique* l *, weak convexity* $k^{'}$.

Proof: If $\omega(G) = wcon(G) = k^{'}$ then order of the graph is $k^{'}+1$. Therefore let $l < k^{'}$. If $l = 2$ by 3.1 we are through. Let $l > 2$. For $l = 3$ consider a clique of order 3. Attach a C_4 with one edge on K_3 . Here $wcon(G) = 4$. For $wcon(G) = 5$, attach a C_5 instead of C_4.Join the third vertex of K_3 to a vertex of C_5 which is adjacent to K_3 . Using 3.2 and 3.3 we get the result for other values of l , $k^{'}$.

Lemma 3.5. *For every pair* k_1, n *of integers with* $n \geq 3$, $2 \leq k_1 \leq n-1$ *there exists a non-complete connected graph such that* $\omega(G) = mcon(G) = k_1$.

Proof: Construction same as in [2] where construction for $\omega(G) = con(G) = k$ is given .

Theorem 3.6. *For every triple* l, k_1, n *with* $2 \leq l \leq k_1 \leq n-1$ *there exists a non - complete connected graph of order* n *having clique* l *, m - convexity* k_1.

Proof: If $\omega(G) = mcon(G) = k_1$ then from 3.5 the result follows .

Assume $l < k_1$. Let $F = \overline{K}_2 + (K_{l-1} \bigcup \overline{K}_{k_1-l-1})$, where $V(\overline{K}_2) = \{u_1, u_2\}$, $V(K_{l-1}) = \{v_1, v_2, \cdots, v_{l-1}\}$ and $V(\overline{K}_{k_1-l-1}) = \{w_1, w_2, \cdots, w_{k_1-l-1}\}$. Clearly order of F is k_1 . If $n = k_1+1$ then F_1 is obtained from F by adding a pendant edge a u_1.

If $n = k_1+2$ then F_2 is obtained from F by adding u , v and edges uu_1 , uv , vv_1. If $n = k_1+3$ then F_3 is obtained by adding u , v , w and uu_1 , uv , vv_1 , vw , wu_2 , wv_2.. In all the above values of n , $\omega(G) = l$ and $mcon(G) = k_1$.

If $n \geq k_1+4$ then G is obtained as follows. Consider F .Rest of $n-k_1$ should be a path such that y_1u_1 , y_2v_1 , y_3v_2 , $\cdots$, y_lv_{l-1} , $y_{l+1}u_1$, $y_{l+2}v_1$ $\cdots$, $y_{n-k_1}v_j$ are edges. Here none of he verices from P_{n-k_1} are included in monophonic convex set of G.

Corollary 3.7. *For every two integers* $k^{'}, N$ *such that* $2 \le k^{'}$ *and* $N \ge 2$ *there exists a connected graph* G *with* $\omega(G) = l$, $wcon(G) = k^{'}$ *whose vertices can be partitioned into* N *maximum weak convex sets* .

Proof: Take N copies of $C_{k^{'}}$. For $N = 2$, consider two copies of $C_{k^{'}}$. Let $V(C_{k^{'}}) = \{u_1^{'}, u_2^{'}, \cdots, u_{k^{'}}^{'}\}$, $V(C_{k^{'}}) = \{u_1^{''}, u_2^{''}, \cdots, u_{k^{'}}^{''}\}$ be the vertices of first and second $C_{k^{'}}$ respectively. For odd $k^{'}$, join $u_1^{'}u_1^{''}$. Join $u_2^{''}$ to $u_3^{'}$, $u_3^{''}$ to $u_5^{'}$ untill all $u_i^{''}$ exhaust. For even $k^{'}$, repeat as in the case of odd $k^{'}$ untill $u_{(k^{'}/2)}^{''}$ is joined to $u_k^{'}$. Now $u_{(k^{'}/2)+1}^{''}$ is joined to $u_2^{'}$, $u_{(k^{'}/2)+2}^{''}$ is joined to $u_4^{'}$ and the process is continued untill all $u_i^{''}$ exhaust. By construction it is clear that each $C_{k^{'}}$ forms a maximum weak convex set.

For $N = 3$, consider three $C_{k^{'}}$. Repeat the same construction as in $N = 2$ for first and second $C_{k^{'}}$, second and third $C_{k^{'}}$. Now join $u_1^{'''}$ to $u_{(k^{'}-1)}^{'}$ and $u_1^{'}$ to $u_{k^{'}}^{''}$, $u_{k^{'}}^{'''}$

For $N = 4$, consider four $C_{k^{'}}$. Consider the construction as in $N = 3$. Repeat as in $N = 2$ for third and fourth $C_{k^{'}}$. Now join $u_1^{''''}$ to $u_{k^{'}}^{''}$, and $u_1^{'}$ to $u_{k^{'}}^{''''}$. This construction can be extended to any N .

Corollary 3.8. *For every three integers* l, k_1, N *such that* $2 \le l \le k_1$ *and* $N \ge 2$ *there does not exist a connected graph* G *with* $\omega(G) = l$, $mcon(G) = k_1$ *whose vertices can be partitioned into* N *maximum monophonic convex sets.*

Corollary 3.9. *For every three integers* $k_1, k, k^{'}$ *such that* $2 \le k_1 \le k \le k^{'} \le n-1$ *there exists a connected graph* G *of order* n *with* $mcon(G) = k_1$, $con(G) = k$ *and* $wcon(G) = k^{'}$.

Proof: Let $G_1 = C_{k_1}$ where $V(G_1) = \{u_1, u_2, \cdots, u_{k_1}\}$ and let $G_2 = K_{k-k_1}$ where $V(G_2) = \{v_1, v_2, \cdots, v_{k-k_1}\}$. Fix an edge say u_1u_2 in C_{k_1}. Let $G_3 = K_{k-k_1} + u_1u_2$. Let $G_4 = P_{k^{'}-k}$ where $V(G_4) = \{w_1, w_2, \cdots, w_{k^{'}-k}\}$. Form $w_1u_{k_1}, {}_{k^{'}-k}u_{k_1}, w_2v_1$ edges. Now consider $P_{k^{'}-k}$. Form $w_2w_4, w_2w_5, \cdots, w_2w_{k^{'}-k-1}$ edges. Also $w_3v_2, w_4v_2, \cdots, w_{k^{'}-k-1}v_2$ are new edges formed. Let the resulting graph be called as G_5 and its order is $k^{'}$. Remaining $n-k^{'}$ vertices are formed as a path with $V(P_{n-k^{'}}) = \{x_1, x_2, \cdots, x_{n-k^{'}}\}$. Clearly from the construction $w_3, w_4, \cdots, w_{k^{'}-k}, u_{k_1}, u_1, v_1, w_2$ form a cycle. Join x_1 to

w_3 , x_2 to w_5 , x_3 to w_7 , $\cdots$, $x_{n-k'}$ to w_j for some j in the above cycle untill all vertices in $P_{n-k'}$ exhaust.

4. **Conclusion**

In this paper, we have shown that for every quardruple k_1, k, k', n of integers with $n \geq 3$ and $2 \leq k_1 \leq k \leq k' \leq n-1$, there exists a non-complete connected graph G of order n with $mcon(G) = k_1$, $con(G) = k$ and $wcon(G) = k'$. Also for every triple l, k_1, n of integers with $n \geq 3$ and $2 \leq l \leq k_1 \leq n-1$, there exists a non-complete connected graph G of order n with $\omega(G) = l$ and $mcon(G) = k_1$. Similar construction is given for weak convexity number. I shall explore the above parameters on product graphs as a part of my future work.

REFERENCES

1. F.Buckley and F.Harary, *Distance in graphs*, Redwood City, Addison-Wesley , 1990.
2. G.Chartrand,C.E.Wall and P.Zhang, The convexity number of a graph, *Graphs and Combinatorics,* 18 (2002) 209-217.
3. D.B .West, *Introduction to Graph Theory,* Second Edition, Prentice-Hall, 2001.
4. P.Duchet, Convex sets in graphs II minimal path convexity, *J. Comb. Theory Ser-B*, 44 (1988).
5. J.Gimbel, Some remarks on the convexity number of a graph. *Graphs and Combinatorics,* 19 (2003) 351-361.
6. F.Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.
7. J.Manora and P.Jayasimman, Neighborhood sets and neighborhood polynomial of cycles, *Annals of Pure and Applied Mathematics*, 7(2) (2014) 45-51.
8. J.Nieminen and F.Harary, Convexity in graphs, *J.Differ, Geom.*,16 (1981) 185-190.
9. M.Van de Vel, Theory of convex structures, North-Holland, Amsterdam, M, 1993.