Progress in Nonlinear Dynamics and Chaos
Vol. 4, No. 1, 2016, 35-43
ISSN: 2321 – 9238 *(online)*
Published on 13 July 2016
www.researchmathsci.org

Progress in **Nonlinear Dynamics and Chaos**

Comparison of Arithmetic Mean, Geometric Mean and Harmonic Mean Derivative-Based Closed Newton Cotes Quadrature

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Received 26 June 2016; accepted 5 July 2016

Abstract. In this paper, the computation of numerical integration using arithmetic mean (AMDCNC), geometric mean (GMDCNC) and harmonic mean (HMDCNC) derivative-based closed Newton cotes quadrature rules are compared with the existing closed Newton cotes quadrature rule (CNC). The comparison shows that, arithmetic mean-based rule gives better solution than the other two rules. This set of quadrature rules which includes the mean value at the function derivative for the computation of numerical integration and the error terms are also obtained by using the concept of precision. Finally, the mathematical relationship between the rules AM > GM > HM are analyzed using numerical examples and the results are compared with the existing methods.

Keyword: Numerical integration, Closed Newton-cotes formula, Arithmetic mean derivative, Geometric mean derivative, Harmonic mean derivative, Numerical examples.

AMS Mathematics Subject Classification (2010): 65D30, 65D32

1. Introduction

Numerical integration is the approximate computation of a definite integral using numerical techniques. Numerical integration procedure provide almost unlimited scope for realistic statistical modelling. Many recent statistical methods are dependent especially on multiple integration, possibly in very high dimensions [9]. In Mathematics, mean has several different definitions depending on the context. There is a Mathematical relationship between arithmetic mean, geometric mean and harmonic mean for any set of values AM > GM > HM. This relationship can be proved with the help of numerical examples.

Definition 1.1. [10] An integration method of the form

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i)$$

is said to be of order P, if it produces exact results ($E_n[f] = 0$) for all polynomials of degree less than or equal to P.

The list of closed Newton cotes formulas that depend on the integer value of n, are given as follows:

When n=1 : Trapezoidal rule

$$\int_a^b f(x)dx = \frac{b-a}{2}\left(f(a)+f(b)\right) - \frac{(b-a)^3}{12} f''(\xi), \quad where\ \xi \in (a,b) \tag{1}$$

When n =2: Simpson's 1/3rd rule

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \quad where\ \xi \in (a,b) \tag{2}$$

When n =3: Simpson's 3/8th rule

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}(\xi), where\ \xi \in (a,b) \tag{3}$$

When n=4 : Boole's rule

$$\int_a^b f(x)dx = \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}(\xi), \quad where\ \xi \in (a,b) \tag{4}$$

It is known that the degree of precision is n+1 for even value of n and n for odd value of n.

The proposed schemes increase two order of precision in arithmetic mean derivative - based rule and increase a single order of precision in geometric and harmonic mean derivative based rule. Dehghan and his companions improved the closed Newton cotes formula [4] by including the location of boundaries of the interval as two additional parameter, and rescaling the original integral to fit the optimal boundary locations. They have applied this technique to open, semi-open, Gauss Legendre and Gauss Chebyshev integration rules [5,6,1,7]. Burg has proposed a derivative based close, open and Midpoint quadrature rules [2,8,3]. In 2013,Weijing Zhao and Hongxing Li [14] took a different approach by introducing a midpoint technique at the computation of derivative. Recently, we proposed [11] midpoint derivative in open Newton cotes quadrature rule and we consider this midpoint technique as an arithmetic mean and propsed [12,13] geometric and harmonic mean derivative - based closed Newton-Cotes quadrature rule.

In this paper, the comparison of arithmetic mean, geometric mean and harmonic mean derivative-based closed Newton cotes quadrature rules are presented. These three mean derivative-based rules gives better solution than the existing rule. The Mathematical relationship between the rules AM > GM >HM are also proved by the numerical examples. If either a=0 or b=0,geometric mean and harmonic mean will be zero. That is, These methods are not applicable in the case of a=0 or b=0. It has shown that the arithmetic mean derivative - based closed Newton-Cotes quadrature rule gives better solution than the other rules.

2. Arithmetic mean (or) midpoint derivative-based closed Newton Cotes quadrature rule with the error terms [14]

The list of arithmetic mean derivative-based closed Newton-Cotes quadrature rule are as follows,

When n=1 : Trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\big(f(a)+f(b)\big) - \frac{(b-a)^3}{12} f''\left(\frac{a+b}{2}\right) - \frac{(b-a)^5}{480} f^{(4)}(\xi), \qquad (5)$$

where $\xi \in (a, b)$.This is fifth order accurate.

When n =2: Simpson's 1/3rd rule

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+b}{2}\right) - \frac{(b-a)^7}{241920} f^{(6)}(\xi) \qquad (6)$$

where $\xi \in (a, b)$. This is seventh order accurate.

When n =3: Simpson's 3/8th rule

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{a+b}{2}\right) - \frac{23(b-a)^7}{9797760} f^{(6)}(\xi), \qquad (7)$$

where $\xi \in (a, b)$. This is seventh order accurate.

When n=4 : Boole's rule

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{a+b}{2}\right) - \frac{17(b-a)^9}{45.2^{11}.8!} f^{(8)}(\xi), \qquad (8)$$

where $\xi \in (a, b)$. This is ninth order accurate.

3. Geometric mean derivative-based closed Newton-Cotes quadrature rule with the error terms [12]

The list of geometric mean derivative -based closed Newton cotes quadrature rule are presented.Here, the geometric mean derivative is zero if either a=0 or b=0.

When n=1 : Trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\big(f(a)+f(b)\big) - \frac{(b-a)^3}{12} f''(\sqrt{ab}) - \frac{(b-a)^3}{24}(\sqrt{b}-\sqrt{a})^2 f^{(3)}(\xi), \qquad (9)$$

where $\xi \in (a, b)$.This is fourth order accurate.

When n =2: Simpson's 1/3rd rule

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\big(\sqrt{ab}\big) - \frac{(b-a)^5}{5760}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi), \qquad (10)$$

where $\xi \in (a, b)$.This is sixth order accurate.

When n =3: Simpson's 3/8th rule

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\big(\sqrt{ab}\big) - \frac{(b-a)^5}{12960}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi), \qquad (11)$$

where $\xi \in (a, b)$.This is sixth order accurate.

When n=4 : Boole's rule

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\big(\sqrt{ab}\big) - \frac{(b-a)^7}{3870720}(\sqrt{b}-\sqrt{a})^2 f^{(7)}(\xi), \qquad (12)$$

where $\xi \in (a, b)$.This is eighth order accurate.

4. Harmonic mean derivative-based closed Newton cotes quadrature rule with the error terms [13]

The list of harmonic mean derivative-based closed Newton-Cotes quadrature rule are presented. Here, the harmonic mean derivative is zero if either a=0 or b=0.

When n=1 : Trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\big(f(a)+f(b)\big) - \frac{(b-a)^3}{12} f''\left(\frac{2ab}{a+b}\right) - \frac{(b-a)^5}{24(a+b)} f^{(4)}(\xi), \qquad (13)$$

where $\xi \in (a, b)$.This is fifth order accurate.

When n =2: Simpson's 1/3rd rule

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{(b-a)^5}{2880}f^{(4)}\left(\frac{2ab}{a+b}\right) -\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi) \quad (14)$$

where $\xi \in (a, b)$.This is seventh order accurate.

When n =3: Simpson's 3/8th rule

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right] -\frac{(b-a)^5}{6480}f^{(4)}\left(\frac{2ab}{a+b}\right)-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi), \quad (15)$$

where $\xi \in (a, b)$.This is seventh order accurate.

When n=4 : Boole's rule

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right] -\frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{2ab}{a+b}\right)-\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi), \quad (16)$$

where $\xi \in (a, b)$.This is ninth order accurate.

The summary of precision, the orders and the error terms for arithmetic, geometric harmonic mean derivative based closed Newton-Cotes quadrature are shown in Table 1.

5. Numerical results

In this section, The values of $\int_1^2 e^x\,dx$ and $\int_1^2 \frac{dx}{1+x}$are estimated using the arithmetic, geometric and harmonic mean derivative-based closed Newton cotes formula and the results are compared with the existing closed Newton-cotes quadrature formula. The comparisons are shown in Table 2 and 3.

We know that

Error = |Exact value - Approximate value|

Example 5.1. Solve $\int_1^2 e^x dx$ and compare the solutions with the CNC, AMDCNC, GMDCNC and HMDCNC rules.

Solution:

Exact value of $\int_1^2 e^x dx$=4.67077427

Example 5.2. Solve $\int_1^2 \frac{dx}{1+x}$ and compare the solutions with the CNC, AMDCNC, GMDCNC and HMDCNC rules.

Solution:

Exact value of $\int_1^2 \frac{dx}{1+x}$=0.405465108

From Table 2 and Table 3,it is proved that (AM > GM > HM) arithmetic mean derivative - based closed Newton cotes quadrature rule gives better solution than the other two rules.

6. Conclusion

In this paper, comparison of arithmetic, geometric and harmonic mean derivative-based closed Newton-Cotes quadrature formulas were presented along with their error terms. This proposed schemes increase an order of the existing formula. The relationship between the arithmetic, geometric and harmonic mean derivative - based rules AM > GM > HM are proved by using the numerical examples. Finally, the numerical example shows that arithmetic mean-based rule gives better solution than the other two rules.

Rules	**AMDCNC**			**GMDCNC**		
	Precision	**Order**	**Error terms**	**Precision**	**Order**	**Error terms**
Trapezoidal rule (n=1)	3	5	$-\frac{(b-a)^5}{480} f^{(4)}(\xi)$	2	4	$-\frac{(b-a)^3}{24}(\sqrt{b}-\sqrt{a})^2 f^{(3)}(\xi)$
Simpson's $1/3^{rd}$ rule (n=2)	5	7	$-\frac{(b-a)^7}{241920} f^{(6)}(\xi)$	4	6	$-\frac{(b-a)^5}{5760}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi)$
Simpson's $3/8^{th}$ rule (n=3)	5	7	$-\frac{23(b-a)^7}{9797760} f^{(6)}(\xi)$	4	6	$-\frac{(b-a)^5}{12960}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi)$
Boole's rule (n=4)	7	9	$-\frac{17(b-a)^9}{45.2^{11}.8!} f^{(8)}(\xi)$	6	8	$-\frac{(b-a)^7}{3870720}(\sqrt{b}-\sqrt{a})^2 f^{(7)}(\xi)$

Rules	HMDCNC		
	Precision	Order	Error terms
Trapezoidal rule (n=1)	2	5	$-\frac{(b-a)^5}{24(a+b)}f^{(4)}(\xi)$
Simpson's 1/3rd rule (n=2)	4	7	$-\frac{(b-a)^7}{5760(a+b)}f^{(6)}(\xi)$
Simpson's 3/8th rule (n=3)	4	7	$-\frac{(b-a)^7}{12960(a+b)}f^{(6)}(\xi).$
Boole's rule (n=4)	6	9	$-\frac{(b-a)^9}{3870720(a+b)}f^{(8)}(\xi)$

Table 1: Comparison of error terms

value of n	CNC		AMDCNC	
	App. value	Error	App. value	Error
n=1	5.053668964	0.382894694	4.680194875	0.009420605
n=2	4.672349035	0.001574765	4.670792893	0.000018623
n=3	4.671476470	0.000702200	4.670784851	0.000010581
n=4	4.670776607	0.000002337	4.670774291	0.000000021

value of n	GMDCNC		HMDCNC	
	App. value	Error	App. value	Error
n=1	4.710898099	0.040123829	4.737529973	0.066755703
n=2	4.670920823	0.000146553	4.671031784	0.000257519
n=3	4.670841709	0.000067439	4.670891027	0.000116757
n=4	4.670774481	0.000000211	4.670774647	0.000000371

Table 2: Comparison of CNC, AMDCNC, GMDCNC and HMDCNC rules

value of n	CNC		AMDCNC	
	App. value	Error	App. value	Error
n=1	0.416666667	0.011201559	0.406000000	0.000534892
n=2	0.405555556	0.000090448	0.405470216	0.000005108
n=3	0.405505952	0.000040844	0.405468026	0.000002918
n=4	0.405465768	0.000000660	0.405465158	0.000000050

value of n	GMDCNC		HMDCNC	
	App. value	Error	App. value	Error
n=1	0.404822031	0.000643077	0.403547132	0.001917976
n=2	0.405453939	0.000011169	0.405435069	0.000030039
n=3	0.405460791	0.000004317	0.405452402	0.000012706
n=4	0.405464989	0.000000119	0.405464780	0.000000328

Table 3: Comparison of CNC, AMDCNC, GMDCNC and HMDCNC rules

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