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Proximinally Additive Chebychev Spaces in $L^{\phi}(\mu, X)$ and $L^{p}(\mu, X), 1 \le p < \infty$

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Abstract. We prove that the property of being proximinally additive in Banach spaces is enjoyed by G if and only if $L^{\phi}(\mu, G)$ has it in $L^{\phi}(\mu, X)$. Half of this result has been done in [2]. Furthermore, we prove that: With this property assumed, G is a Chebychev subspace of X if and only if $L^{\phi}(\mu, G)$ is Chebyshev in $L^{\phi}(\mu, X)$ if and only if $L^{p}(\mu, G)$ is Chebyshev in $L^{p}(\mu, X)$, $1 \le p < \infty$.

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1. Introduction

For the subset *G* of the normed linear space $(X, \|.\|)$, We define, for $x \in X$, $d(x,G) = \inf \{\|x-g\| : g \in G\}$. If *G* is a subspace of *X*, an element $g_{\circ} \in G$ is called a best approximant of *x* in *G* if $\|x - g_{\circ}\| = d(x,G)$. We shall denote the set of all best approximants of *x* in *G* as *P(x ,G)*. If for each $x \in X$, the set $P(x, G) \neq \phi$, then *G* is said to be proximinal in *X*, and if *P(x, G)* is a singleton for each $x \in X$ than *G* is called a Chebychev subspace.

An increasing function $\phi : [0,\infty) \to [0,\infty)$ is said to be a modulus function if it vanishes at zero, and is subadditive. This means that $\phi(x+y) \le \phi(x) + \phi(y)$ for all *x* and *y* in $[0,\infty)$. Examples of modulus functions are : x^p, $0 < p \le 1$, and $\ln(1+x)$. Furthermore, if φ is a modulus function, then $\varphi(x) = \dfrac{\varphi(x)}{1+\varphi(x)}$ is again modulus. It is also evident that the composition of two modulus functions is a modulus function [5].

Let X be a real Banach space and let (T, μ) be a finite measure space. For a modulus function ϕ, we define the Orlicz space $L^{\phi}(\mu, X)$, as the set

$$\left\{ f : T \to X \,;\, \int_T \phi(\|f(t)\|) d\mu(t) < \infty \right\}.$$

The function d : $L^{\phi}(\mu, X)$ x $L^{\phi}(\mu, X) \rightarrow [0,\infty)$ given by:

d(f,g) = $\int_T \phi(\| f(t) - g(t) \|) d\mu(t)$ turns $L^{\phi}(\mu, X)$ into a complete metric space [5].

For f $\in L^{\phi}(\mu, X)$, we write $\|f\|_{\phi} = \int_T \phi(\| f(t) \|) d\mu(t)$. In what follows, when ϕ is mentioned, it is to be assumed a modulus function. We would also like to mention that in the literature, except for what we partly did in [1], we did not find conditions under which the Chebyshevness of G in X is equivalent to the of Chebyshevness $L^{\phi}(\mu, G)$ in $L^{\phi}(\mu, X)$ and to the Chebyshevness of $L^{p}(\mu, G)$ in $L^{p}(\mu, X)$, $1 \leq p < \infty$. Here we show that the condition of proximinal additivity, again, gives the required equivalence.

In the present time, researchers are working on the extensions of classical results in which they consider Haar subspaces for approximating sets, for reference one may consider [4].

2. Proximinal Additivity

Definition 2.1. A subspace G of a Banach space X is said to proximinally additive if G is closed and $z_1 + z_2 \in P(x_1 + x_2, G)$ whenever $z_1 \in P(x_1, G)$ and $z_2 \in P(x_2, G)$.

Example 2.2. Let $X = R^2$, and let G = $\{(x,0) : x \in R\}$. Then *G* is proximinally additive in *X*, with the Euclidean norm.

Lemma 2.3. Suppose that *G* is a proximinal subspace of a Banach space *X* and that *G* is proximinally additive in *X*. Then *G* is a semi–Chebyshev.

Proof: Let $x \in X \setminus G$ and $z_1, z_2 \in P(x,G)$, then $-z_1, -z_2 \in P(-x,G)$

Since *G* is proximinally additive and $z_1 \in P(x,G)$, $-z_2 \in P(-x,G)$, then

$$z_1 + (-z_2) \in P(x + (-x), G) \Rightarrow z_1 - z_2 \in P(0, G)$$

But $P(0,G) = \{0\}$, since $0 \in G \Rightarrow z_1 - z_2 = 0 \Rightarrow z_1 = z_2$. Therefore, *G* is a semi–Chebyshev subspace of *X*. ■

3. Main results

It is of a great significance to make the following lemma which will be used in the upcoming results. This lemma appeared with its proof, which we give here, in a Master thesis written by Dwaik, the coauthor of both [1] and [2], under the supervision of the first author of this article. That was at An-Najah National University back in the year 2000. It turns out that proximinal additivity is transformed from the Orlicz space $L^{\phi}(\mu, G)$ to the subspsce G of X. Specifically, we have the following:

Theorem 3.1. Let X be a Banach space and G be a closed subspace of X. If $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$, then G is proximinally additive in X.

Proof: Suppose $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$, and let $z_i \in P(x_i, G)$ for i=1, 2; we want to show $z_1 + z_2 \in P(x_1 + x_2, G)$.

Now let $f_i(t) = x_i$ and $g_i(t) = z_i$, $\forall$ t $\in \Omega$ and for i=1,2; and since $\|x\| < \infty$, $\forall$ x$\in$ X (by definition of the norm), then $f_1, f_2, g_1, g_2 \in L^1(\mu, X)$.

Since $L^1(\mu, X) \subset L^{\phi}(\mu, X)$ by [5], then $f_1, f_2, g_1, g_2 \in L^{\phi}(\mu, X)$ such that for $i = 1,2$, $g_i \in L^{\phi}(\mu, G)$.

First, we show that $g_i \in P(f_i, L^{\phi}(\mu, G))$ (i = 1, 2).

Now for i =1, 2, we have

$z_i \in P(x_i, G) \Rightarrow \|x_i - z_i\| \le \|x_i - y\| \quad \forall y \in G.$

$\Rightarrow \|f_i(t) - g_i(t)\| \le \|f_i(t) - y\| \quad \forall y \in G$ and $\forall$ t$\in \Omega$.

$\Rightarrow \|f_i(t) - g_i(t)\| \le \|f_i(t) - h(t)\| \quad \forall$ t$\in \Omega$ and $\forall h \in L^{\phi}(\mu, G)$.

Since ϕ is strictly increasing, then we have

$\phi(\|f_i(t) - g_i(t)\|) \le \phi(\|f_i(t) - h(t)\|)$, $\forall$ t$\in \Omega$ and $\forall h \in L^{\phi}(\mu, G)$.

$\Rightarrow \|f_i - g_i\|_{\phi} \le \|f_i - h\|_{\phi}$, $\forall h \in L^{\phi}(\mu, G) \Rightarrow g_i \in P(f_i, L^{\phi}(\mu, G))$ i =1, 2.

Since $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$,

$g_1 + g_2 \in P(f_1 + f_2, L^{\phi}(\mu, G))$. By the same arguments as in Lemma 2.10 of [8] we have that $(g_1 + g_2)(t) \in P((f_1 + f_2)(t), L^{\phi}(\mu, G)))$ a.e. t.

Hence, $z_1 + z_2 \in P(x_1 + x_2, G)$. Therefore G is proximinally additive in X. ■

Remark : By [(2.3) of [2]), theorem (3.1), now reads:
Let *X* be a Banach space and *G* be a closed subspace of *X*. Then *G* is proximinally additive in *X* if and only if $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$.

Theorem 3.2. Let G be a closed subspace of a Banach space X. Then the following are equivalent :

(i) G is proximinally additive in X

(ii) $L^1(\mu, G)$ is proximinally additive in $L^1(\mu, X)$.

Proof: Suppose (i) holds and let, for i=1,2, $f_i \in L^1(\mu, X)$ and $g_i \in P(f_i, L^1(\mu, G))$.

Our objective is to show that $g_1 + g_2 \in P(f_1 + f_2, L^1(\mu, G))$.

By [Lemma (2.10) of [8]], for i=1,2, $g_i(t) \in P(f_i(t), G)$ *a.e.* $t \in T$. This, by proximinal additivity of G in X, implies that, $(g_1 + g_2)(t) \in P((f_1 + f_2)(t), G)$ *a.e.* $t \in T$.

Hence, $d((f_1 + f_2)(t), G) = \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|$ *a.e.* $t \in T$.

Therefore, we have:

$\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \le \|(f_1 + f_2)(t) - y\|$ *for all* $y \in G$ *a.e.* $\in T$.

It now follows that:

$\|(f_1+f_2)(t)-(g_1+g_2)(t)\| \le \|(f_1+f_2)(t)-h(t)\|$ *a.e.* $\in T$ *and for all* $h \in L^1(\mu,G)$.

Thus, $\|(f_1+f_2)-(g_1+g_2)\|_1 \le \|(f_1+f_2)-h\|_1$ *for all* $h \in L^1(\mu,G)$.

Therefore, $g_1+g_2 \in P(f_1+f_2, L^1(\mu,G)$. Hence $L^1(\mu,G)$ is proximinally additive in $L^1(\mu,X)$.

For the converse, suppose (ii) holds and let for i=1,2, $x_i \in X$ *and* $z_i \in P(x_i,G)$. We will show that $z_1+z_2 \in P(x_1+x_2,G)$. To this end, and for i=1,2, consider the constant function : $f_i(t)=x_i$ *and* $g_i(t)=z_i$ *for all* $t \in T$.

Clearly, $f_1, f_2, g_1, g_2 \in L^1(\mu,G)$.

Now, for i =1, 2, we have:

$$\begin{aligned} \|f_i(t)-g_i(t)\| &= \|x_i-z_i\| \text{ for all } t\in T \\ &\le \|x_i-y\| \text{ for all } y\in G \\ &= \|f_i(t)-y\| \text{ for all } y\in G \text{ and all } t\in T. \end{aligned}$$

Thus, *for all* $i=1,2$, *and all* $h \in L^1(\mu,G)$ *we have*:

$\|f_i(t)-g_i(t)\| \le \|f_i(t)-h(t)\|$ *for all* $t\in T$. This implies that

$\|f_i-g_i\|_1 \le \|f_i-h\|_1$ *for all* $h\in L^1(\mu,G)$ *and all* $i=1,2$.

So $g_i \in P(f_i, L^1(\mu,G))$ *for all* $i=1,2$.

Since $L^1(\mu,G)$ is proximinally additive in $L^1(\mu,X)$, then :

$g_1+g_2 \in P(f_1+f_2, L^1(\mu,G))$, and so, again by [8],

$(g_1+g_2)(t) \in P((f_1+f_2)(t),G)$ *for all* $t\in T$ .Thus $z_1+z_2 \in P(x_1+x_2,G)$.

Therefore, G is proximinally additive. ■

Theorem 3.3. Let G be a closed subspace of a Banach space X. Then the following are equivalent :

(i) G is proximinally additive in X

(ii) $L^p(\mu,G)$ is proximinally additive in $L^p(\mu,X)$, $1<p<\infty$.

Proof : Suppose (i) holds and let, for i=1,2, $f_i \in L^p(\mu,X)$ and $g_i \in P(f_i, L^p(\mu,G))$ $1<p<\infty$. Then for each $h \in L^p(\mu,G)$ we have $\|f_i-g_i\|_p \le \|f_i-h\|_p$.

Using Lemma (2.10) of [8], one gets that $\|f_i(t)-g_i(t)\| \le \|f_i(t)-y\|$ a.e. t, $\forall y \in G$ for i =1, 2. Then we have $g_i(t) \in P(f_i(t),G)$ a.e. t.

Since G is proximinally additive in X, then

$(g_1+g_2)(t) \in P((f_1+f_2)(t),G)$ a.e. t.

Hence, for all y$\in$ G, we have

$\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \leq \|(f_1 + f_2)(t) - y\|$ a.e. t.

Hence $\forall$ h$\in$ $L^p(\mu, G)$ we have

$\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \leq \|(f_1 + f_2)(t) - h(t)\|$ a.e. t.

$\Rightarrow \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|^p \leq \|(f_1 + f_2)(t) - h(t)\|^p$ a.e. t, $1 < p < \infty$.

$\Rightarrow \|f_1 + f_2 - (g_1 + g_2)\|_p^p \leq \|f_1 + f_2 - h\|_p^p$

$\Rightarrow \|f_1 + f_2 - (g_1 + g_2)\|_p \leq \|f_1 + f_2 - h\|_p$ $\forall$ h$\in$ $L^p(\mu, G)$.

Hence, $g_1 + g_2 \in P(f_1 + f_2, L^p(\mu, G))$, $1 < p < \infty$. Therefore $L^p(\mu, G)$ is proximinally additive in $L^p(\mu, X)$, $1 < p < \infty$.

Conversely, let $x_i \in X$ and $z_i \in P(x_i, G)$ for i =1, 2. We want to show that $z_1 + z_2 \in P(x_1 + x_2, G)$. Consider the constant functions $f_i(t) = x_i$ and $g_i(t) = z_i$, for i = 1, 2 and $\forall t \in \Omega$. Clearly $f_i \in L^p(\mu, X)$, $1 < p < \infty$, and $g_i \in L^p(\mu, G)$ for i=1, 2.

We claim that $g_i \in P(f_i, L^p(\mu, G))$ for i =1, 2.

$$\|f_i - g_i\|_p^p = \int_{\Omega} \|f_i(t) - g_i(t)\|^p d\mu(t) = \int_{\Omega} \|x_i - z_i\|^p d\mu(t) \leq \int_{\Omega} \|x_i - y\|^p d\mu(t), \forall y \in G$$

because $z_i \in P(x_i, G)$.

And so for all $h \in L^p(\mu, G)$ and i=1, 2, we get

$$\|f_i - g_i\|_p^p \leq \int_{\Omega} \|x_i - h(t)\|^p d\mu(t) = \int_{\Omega} \|f_i(t) - h(t)\|^p d\mu(t) = \|f_i - h\|_p^p.$$

Then, for all $h \in L^p(\mu, G)$, we have $\|f_i - g_i\|_p \leq \|f_i - h\|_p$, i =1, 2.

Hence $g_i \in P(f_i, L^p(\mu, G))$, i =1, 2.

Since $L^p(\mu, G)$ is proximinally additive in $L^p(\mu, X)$ $1 < p < \infty$, then $g_1 + g_2 \in P(f_1 + f_2, L^p(\mu, G))$.

Thus for all $h \in L^p(\mu, G)$, we have

$\|f_1 + f_2 - (g_1 + g_2)\|_p \leq \|f_1 + f_2 - h\|_p$

And $\|f_1 + f_2 - (g_1 + g_2)\|_p^p \leq \|f_1 + f_2 - h\|_p^p$ $(1 < p < \infty)$.

Now we have

$$\|f_1 + f_2 - (g_1 + g_2)\|_p^p = \int_{\Omega} \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|^p d\mu(t)$$

$$= \int_\Omega \left\| (x_1 + x_2) - (z_1 + z_2) \right\|^p d\mu(t)$$

$$= \left\| (x_1 + x_2) - (z_1 + z_2) \right\|^p \ \mu(\Omega)$$

$$\| f_1 + f_2 - h \|_p^p = \int_\Omega \left\| (f_1 + f_2)(t) - h(t) \right\|^p d\mu(t)$$

$$= \int_\Omega \left\| (x_1 + x_2) - h(t) \right\|^p d\mu(t).$$

So we have that

$$\left\| (x_1 + x_2) - (z_1 + z_2) \right\|^p \ \mu(\Omega) \ \le \int_\Omega \left\| (x_1 + x_2) - h(t) \right\|^p d\mu(t), \ \forall h \in L^p(\mu, G).$$

In particular, for $y \in G$, let $h_y(t) = y$, $\forall$ t $\in \Omega$ be a constant function, and clearly $h_y \in L^p(\mu, G)$, and so we have

$$\left\| (x_1 + x_2) - (z_1 + z_2) \right\|^p \ \mu(\Omega) \ \le \int_\Omega \left\| (x_1 + x_2)(t) - y \right\|^p d\mu(t)$$

$$= \left\| (x_1 + x_2) - y \right\|^p \ \mu(\Omega).$$

Since (μ, Ω) is a finite measure space (i.e. $\mu(\Omega) < \infty$) and assume $\mu(\Omega) > 0$, then

$$\left\| (x_1 + x_2) - (z_1 + z_2) \right\|^p \le \left\| (x_1 + x_2) - y \right\|^p. \qquad (3.34)$$

Since $y \in G$ was arbitrary,

$$\left\| (x_1 + x_2) - (z_1 + z_2) \right\| \le \left\| (x_1 + x_2) - y \right\|, \ \forall y \in G.$$

Hence $z_1 + z_2 \in P(x_1 + x_2, G)$. Therefore G is proximinally additive in X. ■

Theorem 3.4. Let G be a closed subspace of a Banach space X which is proximinally additive in X, then the following are equivalent:
(i) G is a Chebyshev subspace of X
(ii) $L^\phi(\mu, G)$ is a Chebyshev subspace of $L^\phi(\mu, G)$.
(iii) $L^p(\mu, G)$ is a Chebyshev subspace of $L^p(\mu, X)$ such that $1 \le p < \infty$.

Proof: By theorem (3.7) of [1], G is proximinal in X if and only if $L^\phi(\mu, G)$ is proximinal in $L^\phi(\mu, X)$ if and only if $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$. By [[5] p.297] $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$ if and only if $L^p(\mu, G)$ is proximinal in $L^p(\mu, X)$ such that $1 \le p < \infty$. Now invoke Theorems [(3.1),(3.2) and (3.4)] and Lemma 2.3. This completes the proof of the theorem. ■

Theorem 3.5. Let G be a closed subspace of a Banach space X and suppose $L^\infty(\mu, G)$ is a Chebyshev subspace of $L^\infty(\mu, X)$. If G is proximinally additive in X, then $L^\infty(\mu, G)$ is proximinally additive in $L^\infty(\mu, X)$.

Proof: Let $f_i \in$ L$^{\infty}$(μ,X), i=1,2. Since L$^{\infty}$(μ,G) is Chebyshev, then L$^{\infty}$(μ,G) is proximinal in L$^{\infty}$(μ,X), and so G is proximinal in X. Therefore, G is Chebyshev (because G has is proximinally additive in X). Theorem 3.3 and Theorem 3.5 imply that L$^1(\mu, G)$ is Chebyshev and proximinally additive in L$^1(\mu, X)$. Then for i = 1, 2, ∃! $h_i \in L^1(\mu, G)$ such that $h_i \in P(f_i, L^1(\mu, G))$; and since $\|h_i(t)\| \leq 2\|f_i(t)\|$ a.e. t, then $h_i \in L^{\infty}(\mu, G)$.
By using the same arguments as in ([3], Theorem 1.1) we have $h_i \in P(f_i, L^{\infty}(\mu, G))$.

Now since $h_i \in P(f_i, L^1(\mu, G))$, i =1,2, and L$^1(\mu, G)$ is proximinally additive in L$^1(\mu, X)$, then $h_1 + h_2 \in P(f_1 + f_2, L^1(\mu, G))$.

Since $f_1 + f_2 \in$ L$^{\infty}$(μ,G) and $\|h_1(t) + h_2(t)\| \leq 2\|f_1(t) + f_2(t)\|$ a.e. t, then once again; using the same arguments as in ([3], Theorem 1.1) we have $h_1 + h_2 \in P(f_1 + f_2, L^{\infty}(\mu, G))$; and since L$^{\infty}(\mu, G)$ is Chebyshev, then L$^{\infty}$(μ,G) is proximinally additive in L$^{\infty}$(μ,X). ■

Finally, in [2], example (3.8), it was shown that being proximinally additive is not sufficient for subspace G to be proximinal in X.
Specifically, we have the following:

Example 3.6. Let X $= c_0$, the space of null sequences, equipped with the sup. norm. Let $G = \left\{ x \in c_0 : \sum_{n=1}^{\infty} 2^{-n} x_n = 0 \right\}$.

We found it worth proving that this space is not trivial. The following setup shows how, and was suggested by Heavilin during a visit at An-Najah National University back in the year 2007/2008. Here is the construction:

Choose a real sequence $x = (x_n) \in (c_0 \setminus G)$ such that $\sum_{n=1}^{\infty} 2^{-n} x_n < \infty$, and assume that $\alpha = \sum_{n=1}^{\infty} 2^{-n} x_n \neq 0$. Now, consider the sequence $y = (y_n) = \begin{cases} -\alpha & if\ \ n = 1 \\ x_{n-1} & if\ \ n \geq 2 \end{cases}$

It is clear that $y \neq 0$ and we want to show that $y \in G$. To this end;

$$\sum_{n=1}^{\infty} 2^{-n} y_n = -\frac{\alpha}{2} + \sum_{n=2}^{\infty} 2^{-n} y_n = -\frac{\alpha}{2} + \sum_{m=1}^{\infty} 2^{-m-1} x_m$$

$$= -\frac{\alpha}{2} + \frac{1}{2}\sum_{m=1}^{\infty} 2^{-m} x_m \; = -\frac{\alpha}{2} + \frac{\alpha}{2} = 0 .$$

Therefore, $y \in G$ and so G is not trivial.

4. A note on optimization theory

Optimization in mathematics is to search for means by which extreme values of functions are detected within some feasible region. A good optimizing technique is expected to arrive at best solution(s). Particle Swarm Optimization (PSO) is, now, a standard method of advanced optimization technique and has been empirically shown to perform well on many of these optimization problems. It is lucidly and widely used to find the global optimum solution in a complex search space. This, in a sense, is another face of best approximation theory, each in its field of application. The difference is in the fact that, optimal solutions occur as values of functions while proximinal maps have the basic problem of non-being linear. It is routine check [by 3.2 of 2] that the linearity of proximity maps should be maintained under proximinal additivity. Having done this, the scope of invoking such maps in the theory of best approximation will be much wider. For further development, we would like to refer the reader to [1,2,7].

REFERENCES

1. A.A.Hakawati and S.A.Dwaik. On best approximation in $L^1(\mu, X)$ and $L^\phi(\mu, X)$, *Annals of Pure and Applied Mathematics*, 12(1) (2016) 1-8 .
2. A.A.Hakawati and S.A.Dwaik, On best approximation in $L^p(\mu, X)$ and $L^\phi(\mu, X)$, $1 \leq p < \infty$, *Journal of Mathematics and Informatics*, 6 (2016) 31-39.
3. W.Deeb and R.Khalil, Best approximation in *L(X,Y)*, *Math. Proc. Camb. Philos. Soc.* 104 (1988) 527-531.
4. D.Van Dulst, Reflexive and superreflexive banach spaces, *Math. Centre, Tracts*, 102, Amsterdam, 1978.
5. W.Deeb, Multipliers and isometries of orlicz spaces, In "Proceedings of the conference on Mathematical Analysis and its Applications, (Kuwait, 1985), Volume 3 of KFAS Proc. Ser., pp 159-165, Perganon Oxford 1988.
6. D.K.Biswas and S.C.Panja, Advanced optimization technique, *Annals of Pure and Applied Mathematics*, 5(1) (2013) 82-29.
7. I.Singer, Best approximation in normed linear spaces by elements of linear subspaces, Springer-Verlag, New York.
8. W.Light and W.Cheney, Approximation theory in tensor product spaces, Lecture Notes in Math., 1169, Springer–Verlag Berlin, 9–155, 1985.